RMM - Cyclic Inequalities Marathon 101- 200

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101. **Let a, b, c be positive real numbers with $a+b+c=1$.**

Prove that

$$\frac{a^5}{(b+1)(c+1)}+\frac{b^5}{(c+1)(a+1)}+\frac{c^5}{(a+1)(b+1)}\geq\frac{1}{144}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS=\sum a^3\cdot\frac{a^2}{(b+1)(c+1)}$$

WLOG, we may assume $a\geq b\geq c$

$$\frac{a^2}{(b+1)(c+1)}\geq\frac{b^2}{(c+1)(a+1)}\Leftrightarrow a^3+a^2\geq b^3+b^2$$

$$\Leftrightarrow (a-b)(a^2+ab+b^2+a+b)\geq 0\rightarrow \textbf{true}, \because a\geq b$$

$\therefore \frac{a^2}{(b+1)(c+1)}\geq\frac{b^2}{(c+1)(a+1)}$. ***Similarly,*** $\frac{b^2}{(c+1)(a+1)}\geq\frac{c^2}{(a+1)(b+1)}$

$\therefore$ ***applying Chebyshev's inequality,*** $LHS=\sum a^3\cdot\frac{a^2}{(b+1)(c+1)}$

$$\underset{(1)}{\geq}\frac{1}{3}\left(\sum a^3\right)\sum\frac{a^2}{(b+1)(c+1)}$$

$$\sum a^3\overset{Chebyshev}{\geq}\frac{1}{3}\sum a\cdot\sum a^2=\frac{1}{3}\sum a^2\underset{(2)}{\overset{Chebyshev}{\geq}}\frac{1}{9}\left(\sum a\right)^2=\frac{1}{9}$$

$$\sum\frac{a^2}{(b+1)(c+1)}\overset{Bergstrom}{\geq}\frac{(\sum a)^2}{\sum ab+2\sum a+3}=\frac{1}{\sum ab+5}$$

$$\underset{(3)}{\geq}\frac{1}{\frac{1}{3}(\sum a)^2+5}\quad\left(\because\sum ab\leq\frac{1}{3}\left(\sum a\right)^2\right)=\frac{3}{16}$$

Using (1), (2), (3), LHS $\geq\frac{1}{3}\cdot\frac{1}{9}\cdot\frac{3}{16}=\frac{1}{144}$

(Proved)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS=\sum a\cdot\frac{a^4}{(b+1)(c+1)}$$

WLOG, we may assume $a\ge b\ge c$

$$\frac{a^4}{(b+1)(c+1)}\ge\frac{b^4}{(c+1)(a+1)}\Leftrightarrow a^5+a^4\ge b^5+b^4$$

$$\Leftrightarrow (a-b)(a^4+a^3b+a^2b^2+ab^3+b^4+a^3+a^2b+ab^2+b^3)\ge 0$$

$\rightarrow$ **true,** $\because a\ge b$

$\therefore \frac{a^4}{(b+1)(c+1)}\ge\frac{b^4}{(c+1)(a+1)}$**. Similarly,** $\frac{b^4}{(c+1)(a+1)}\ge\frac{c^4}{(b+1)(a+1)}$

$$\therefore LHS\underbrace{\overset{Chebysev}{\ge}}_{(1)}\frac{1}{3}\sum a\cdot\sum\frac{a^4}{(b+1)(c+1)}$$

$$\sum\frac{a^4}{(b+1)(c+1)}\overset{Bergstrom}{\ge}\frac{\left(\sum a^2\right)^2}{\sum ab+2\sum a+3}\overset{Chebyshev}{\ge}\frac{\frac{1}{9}(\sum a)^4}{\sum ab+5}=\frac{1}{9(\sum ab+5)}$$

$$\underset{(2)}{\ge}\frac{1}{9\cdot\frac{16}{3}}\left(\because\sum ab\le\frac{1}{3}\left(\sum a\right)^2\right)=\frac{1}{48}$$

using (1), (2), $LHS\ge\frac{1}{3}\cdot(1)\cdot\frac{1}{48}=\frac{1}{144}$

(Proved)

Solution 3 by Soumava Chakraborty-Kolkata-India

WLOG, $a\ge b\ge c$**. Then,** $\frac{1}{(b+1)(c+1)}\ge\frac{1}{(c+1)(a+1)}\ge\frac{1}{(a+1)(b+1)}$

$$\therefore LHS\underbrace{\overset{Chebysev}{\ge}}_{(1)}\frac{1}{3}\left(\sum a^5\right)\sum\frac{1}{(b+1)(c+1)}$$

$$\sum a^5\underbrace{\overset{Chebysev}{\ge}}_{(2)}\frac{1}{3^4}\left(\sum a\right)^5=\frac{1}{81}$$

$$\sum \frac{1}{(b+1)(c+1)} \overset{Bergstrom}{\geq} \frac{9}{\sum ab + 2\sum a + 3} = \frac{9}{\sum ab + 5}$$

$$\underset{(3)}{\geq} \frac{9}{\frac{16}{3}}\left(\because \sum ab \leq \frac{1}{3}\left(\sum a\right)^2\right) = \frac{27}{16}$$

using (1), (2), (3), $LHS \geq \frac{1}{3} \cdot \frac{1}{81} \cdot \frac{27}{16} = \frac{1}{144}$

(Proved)

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\left(\sum_{cyc} \frac{a^5}{(b+1)(c+1)}\right)\left(\sum_{cyc}(b+1)(c+1)\right)(1+1+1)(1+1+1)(1+1+1)$$

$$\overset{HOLDER'S\ INEQUALITY}{\geq} (a+b+c)^5 = 1$$

$$\Rightarrow \left(\sum_{cyc} \frac{a^5}{(b+1)(c+1)}\right)(ab+bc+ca+2+3)3^3 \geq 1$$

$$\Rightarrow \left(\frac{(a+b+c)^2}{3}+5\right)\left(\sum_{cyc} \frac{a^5}{(b+1)(c+1)}\right)3^3 \geq \left(\sum_{cyc} \frac{a^5}{(b+1)(c+1)}\right)\left(\sum_{cyc} ab + 5\right)3^3 \geq 1$$

$$\Rightarrow \left(\sum_{cyc} \frac{a^5}{(b+1)(c+1)}\right)\left(5+\frac{1}{3}\right)3^3 \geq 1 \Rightarrow \sum_{cyc} \frac{a^5}{(b+1)(c+1)} \geq \frac{1}{144}$$

(Proved)

equality at $a = b = c = \frac{1}{3}$

102. Let a, b, c be nonnegative numbers such that $a + b + c = 3$. Prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} - 3 \geq \frac{4(3-\sqrt{6})}{3}(ab + bc + ca - 3)$$

Proposed by Richdad Phuc-Hanoi-Vietnam

Solution by Richdad Phuc-Hanoi-Vietnam

$$\text{Let } s=a+b;\, p^2=ab$$

We have

$$\sqrt{a}+\sqrt{b}+\sqrt{c}=\sqrt{c}+\sqrt{b+a+2\sqrt{ab}}=\sqrt{3-s}+\sqrt{s+2p}$$

$$f(p)=\sqrt{3-s}+\sqrt{s+2p}-3-k(p^2+s(3-s)-3),$$

$$p\in\left[0;\frac{s}{2}\right], k=\frac{4(3-\sqrt{6})}{3}$$

$$f'(p)=\frac{1}{\sqrt{s+2p}}-2kp;\, f''(p)=-\frac{1}{\left(\sqrt{s+2p}\right)^3}-2k<0$$

$$f \text{ is concave on } \left(0;\frac{s}{2}\right)\Rightarrow f(p)\geq \min\left\{f(0);f\left(\frac{s}{2}\right)\right\}$$

$$*\, f(0)=\sqrt{3-s}+\sqrt{s}-3-k(s(3-s)-3)$$

$$t=\sqrt{3-s}+\sqrt{s}\Rightarrow t^2=3+2\sqrt{3-s}\sqrt{s}\leq 6 \ (AM-GM)\Rightarrow t\in\left[\sqrt{3};\sqrt{6}\right]$$

$$f(0)=t-3-\frac{12-4\sqrt{6}}{3}\left[\frac{(t^2-3)^2}{4}-3\right]\geq 0$$

$$\Leftrightarrow (t-\sqrt{6})(t^3+\sqrt{6}t^2-\sqrt{6}-3)\leq 0 \text{ is true with } t\in\left[\sqrt{3};\sqrt{6}\right]$$

$$\text{Equality hold if } t=\sqrt{6}\Leftrightarrow s=\frac{3}{2}\Leftrightarrow a=c=\frac{3}{2}, b=0 \text{ or perumutations}$$

$$*\, f\left(\frac{s}{2}\right)=\sqrt{3-s}+\sqrt{2s}-3+\frac{3k}{4}(s-2)^2=$$

$$=(s-2)\left[\frac{-1}{\sqrt{3-s}+1}+\frac{2}{\sqrt{2s}+2}\right]+\frac{3k}{4}(s-2)^2$$

$$f\left(\frac{s}{2}\right)=-\frac{6(s-2)^2}{\left(\sqrt{3-s}+1\right)\left(\sqrt{2s}+2\right)\left(2\sqrt{3-s}+\sqrt{2s}\right)}+\frac{3k}{4}(s-2)^2$$

$$f\left(\frac{s}{2}\right)=(s-2)^2\left[\frac{3k}{4}-\frac{6}{\left(\sqrt{3-s}+1\right)\left(\sqrt{2s}+2\right)\left(2\sqrt{3-s}+\sqrt{2s}\right)}\right]$$

By Cauchy – Schwarz

$$\left(2\sqrt{3-s}+\sqrt{2s}\right)^2 \le 6(2-s+s)=18 \Rightarrow 2\sqrt{3-s}+\sqrt{2s}\le 3\sqrt{2}$$

By AM – GM

$$2\left(\sqrt{3-s}+1\right)\left(\sqrt{2s}+2\right)\left(2\sqrt{3-s}+\sqrt{2s}\right)\le$$

$$\le \frac{\left(2\sqrt{3-s}+\sqrt{2s}+4\right)^2}{3}\left(2\sqrt{3-s}+\sqrt{2s}\right)\le \frac{72+51\sqrt{2}}{2}$$

$$f\left(\frac{s}{2}\right)\ge (s-2)^2\left[3-\sqrt{6}-\frac{6}{\frac{72+51\sqrt{2}}{4}}\right]\ge 0$$

Equality holds if $s=2 \Leftrightarrow a=b=c=1$

103. For: $a,b,c>0 \wedge a+b+c=3$**. Prove:**

$$ln\left(e^{\sqrt{1+a^2}}+e^{\sqrt{1+b^2}}+e^{e^{\sqrt{1+c^2}}}\right)\ge ln\,3+\sqrt{2}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$f(x)=e^{\sqrt{1+x^2}} \Rightarrow f'(x)=\frac{x\cdot e^{\sqrt{1+x^2}}}{\sqrt{1+x^2}};\ f''=\frac{\left(x\cdot e^{\sqrt{1+x^2}}\right)'\cdot\sqrt{1+x^2}-x\cdot e^{\sqrt{1+x^2}}\left(\sqrt{1+x^2}\right)'}{1+x^2}$$

$$=\frac{\left(e^{\sqrt{1+x^2}}+x\cdot e^{\sqrt{1+x^2}}\cdot\frac{2x}{2\sqrt{1+x^2}}\right)\cdot\sqrt{1+x^2}-x\cdot e^{\sqrt{1+x^2}}\cdot\frac{2x}{2\sqrt{1+x^2}}}{(1+x^2)}=$$

$$=\frac{e^{\sqrt{1+x^2}}}{1+x^2}\cdot\left(\left(1+\frac{x^2}{\sqrt{1+x^2}}\right)\cdot\sqrt{1+x^2}-\frac{x^2}{\sqrt{1+x^2}}\right)=\frac{e^{\sqrt{1+x^2}}}{1+x^2}\left(\sqrt{1+x^2}+x^2-\frac{x^2}{\sqrt{1+x^2}}\right)$$

$$=\frac{e^{\sqrt{1+x^2}}}{(1+x^2)\sqrt{1+x^2}}\left(1+x^2+x^2\sqrt{1+x^2}-x^2\right)=\frac{\left(1+x^2\sqrt{1+x^2}\right)e^{\sqrt{1+x^2}}}{(1+x^2)\sqrt{1+x^2}}>0$$

$$f''(x) > 0 \Leftrightarrow f(x) - \textit{concave} \Rightarrow \textit{Jensen: } \ln\left(\sum e^{\sqrt{1+a^2}}\right) \ge \ln 3e^{\sqrt{1+\left(\frac{a+b+c}{3}\right)^2}} = \ln 3e^{\sqrt{2}} =$$

$$= \ln 3 + \sqrt{2} \quad (a = b = c = 1)$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\ln\left(\sum_{cyc} e^{\sqrt{1+a^2}}\right) \ge \ln\left(\sum_{cyc} e^{\frac{1+a}{\sqrt{2}}}\right) \left[\because \sqrt{1+x^2} \ge \frac{1+x}{\sqrt{2}}\right]$$

$$\overset{AM\ge GM}{\ge} \ln\left(3\sqrt[3]{e^{\frac{3+a+b+c}{\sqrt{2}}}}\right) = \ln\left(3e^{\frac{6}{3\sqrt{2}}}\right) = \ln\left(3e^{\sqrt{2}}\right) = \ln 3 + \sqrt{2} \textit{ (prove).}$$

Equality at $a = b = c = 1$

104. Prove that for all positive real numbers a, b, c the inequality holds

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{ab+bc+ca}{2(a^2+b^2+c^2)} \ge 2$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los numerous R^+ la siguiente desigualdad

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{ab+bc+ca}{2(a^2+b^2+c^2)} \ge 2$$

Como $a, b, c > 0 \Leftrightarrow ab + bc + ca > 0$

Por $MA \ge MG$

$$\frac{a^2+b^2+c^2}{2(ab+bc+ca)} + \frac{ab+bc+ca}{2(a^2+b^2+c^2)} \ge 1$$

Aplicando la desigualdad de Cauchy en la desigualdad propuesta

$$\sum \frac{a}{b+c} + \frac{ab+bc+ca}{2(a^2+b^2+c^2)} \ge \frac{(a+b+c)^2}{2(ab+bc+ca)} + \frac{ab+bc+ca}{2(a^2+b^2+c^2)} =$$

$$= \frac{a^2 + b^2 + c^2}{2(ab + bc + ca)} + \frac{ab + bc + ca}{2(a^2 + b^2 + c^2)} + 1 \geq 2$$

105. **Let x, y, z be positive real numbers such that $xyz = 1$. Prove that**

$$\frac{1}{(x+1)^2 + y^2 + 1} + \frac{1}{(y+1)^2 + z^2 + 1} + \frac{1}{(z+1)^2 + x^2 + 1} \leq \frac{1}{2}$$

24th Pan African Mathematics Olympiad

Solution by Daniel Sitaru – Romania

$$x = \frac{b}{a}, y = \frac{c}{b}, z = \frac{a}{c}$$

$$\sum \frac{1}{(x+1)^2 + y^2 + 1} = \sum \frac{1}{x^2 + y^2 + 2x + 2} \leq$$

$$\leq \frac{1}{2}\sum \frac{1}{xy + x + y} = \frac{1}{2}\sum \frac{1}{\frac{c}{a} + \frac{b}{a} + 1} =$$

$$= \frac{1}{2}\sum \frac{a}{a+b+c} = \frac{1}{2} \cdot \frac{a+b+c}{a+b+c} = \frac{1}{2}$$

106. ***From the book: "Math Phenomenon"***

If $a, b, c \in (0, \infty), \sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 3$ then:

$$3\sum (a^3 + b^3)\, c \geq 4abc(a+b+c) + 6abc$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si $a, b, c \in <0, \infty>$, además $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 3$. Probar

$$3\sum (a^3 + b^3)c \geq 4abc(a+b+c) + 6abc$$

Dado que es $a, b, c > 0 \Leftrightarrow abc > 0$, ***dividiendo*** $(\div abc)$ ***la desigualdad***

$$\Leftrightarrow 3\sum\left(\frac{a^2}{b}+\frac{b^2}{a}\right) \geq 4(a+b+c)+6$$

Por la desigualdad de Cauchy

$$3\sum\left(\frac{a^2}{b}+\frac{b^2}{a}\right) \geq 3\sum(a+b) = 6\sum a \geq 4\sum a + 2\sum a \geq$$

$$\geq 4(a+b+c)+2\sum\sqrt{ab} = 4(a+b+c)+6$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$a^3+b^3 \geq ab(a+b) \Rightarrow c(a^3+b^3) \geq abc(a+b)$$

$$\therefore 3\sum c(a^3+b^3) \overset{(1)}{\geq} 3abc\left(\sum(a+b)\right) = 6abc(a+b+c)$$

(1) $\Rightarrow$ ***it suffices to prove:*** $6abc(\sum a) \geq 4abc(\sum a) + 6abc$

$\Leftrightarrow \sum a \geq 3$ ***(2)***

Now, $\sum a = (\sqrt{a})^2 + (\sqrt{b})^2 + (\sqrt{c})^2$

$$\geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca}\ \left(\because \sum x^2 \geq \sum xy\right)$$

$= 3 \Rightarrow$ ***(2) is true (Proved)***

Solution 3 by Nirapada Pal-Jhargram-India

Given, $\sum\sqrt{ab} = 3$

Now, $4abc(a+b+c)+6abc = 2abc[2\sum a + 3]$

$$= 2abc\left[2\sum a + \sum\sqrt{ab}\right] \overset{CBS}{\leq} 2abc\left[2\sum a + \sum a\right] = 6abc\sum a$$

$$\overset{AGM}{\leq} 2\sum a^3 \sum a = \left(\sum a^3 + b^3\right)\sum a \leq 3\sum(a^3+b^3)c$$

107. If $a,b,c,x,y>0$ then:

$$\frac{a^2+bc}{a^2(bx+cy)}+\frac{b^2+ca}{b^2(cx+ay)}+\frac{c^2+ab}{c^2(ax+by)}\ge\frac{18}{(x+y)(a+b+c)}$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{a^2+bc}{a^2(bx+cy)}=\sum_{cyc}\frac{1}{bx+cy}+\sum_{cyc}\frac{bc}{a^2(bx+cy)}$$

$$\overset{A.M.\ge G.M.\,and\,BERGSTROM}{\ge}\frac{9}{\sum(bx+cy)}+\frac{3}{\sqrt[3]{(ax+by)(bx+cy)(cx+ay)}}$$

$$\overset{REVERSE\,A.M\ge G.M.}{\ge}\frac{9}{(x+y)(a+b+c)}+\frac{3}{\frac{\sum(ax+by)}{3}}=\frac{18}{(x+y)(a+b+c)}$$

(Proved)

Solution 2 by Uche Eliezer Okeke-Anambra-Nigerie

$$\sum\frac{a^2+bc}{a^2(bx+cy)}=\underbrace{\sum\frac{1}{(bx+cy)}}_{I}+\underbrace{\sum\frac{bc}{a^2(bx+cy)}}_{II}$$

$$I\ge\frac{(1+1+1)^2}{\sum(bx+cy)}=\frac{9}{(x+y)(a+b+c)}$$

$$II\in\sum\frac{\left(\frac{1}{a}\right)^2}{\left(\frac{bx+cy}{bc}\right)}\ge\frac{\left(\sum\frac{1}{a}\right)^2}{(x+y)\left(\sum\frac{1}{a}\right)}=\frac{1}{(x+y)}\sum\frac{1}{a}$$

$$\overset{AM-GM}{\ge}\frac{1}{(x+y)}\cdot\frac{9}{3\sqrt[3]{abc}}$$

$$\overset{GM-AM}{\ge}\frac{1}{(x+y)}\cdot\frac{9}{(a+b+c)}=I$$

$$LHS = I + II \geq \frac{9 \times 2}{(x+y)(a+b+c)} = \frac{18}{(x+y)(a+b+c)}$$

(Proved)

108. If $a, b, c > 0$ then:

$$(a+b+c)^2 \geq 3(ab+bc+ca) + \frac{1}{4}\sum \frac{(a^2-b^2)^2}{a^2+b^2}$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $a, b, c > 0$. Probar que

$$(a+b+c)^2 \geq 3(ab+bc+ca) + \frac{1}{4}\sum \frac{(a^2-b^2)^2}{a^2+b^2}$$

$$(a+b+c)^2 \geq 3(ab+bc+ca) + \frac{1}{4}\sum \left(a^2+b^2-\frac{4a^2b^2}{a^2+b^2}\right)$$

$$a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca) + \frac{a^2+b^2+c^2}{2} - \sum \frac{a^2b^2}{a^2+b^2}$$

$$\Leftrightarrow a^2+b^2+c^2+2\sum \frac{a^2b^2}{a^2+b^2} \geq 2ab+2bc+2ca$$

Aplicando MA ≥ MG

$$\sum \frac{a^2+b^2}{2} + 2\sum \frac{a^2b^2}{a^2+b^2} \geq 2\sum ab = 2(ab+bc+ca) \quad \textbf{(LQQD)}$$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$RHS = 3(ab+bc+ca) + \frac{1}{4}\sum \frac{(a^2-b^2)^2}{a^2+b^2}$$

$$\frac{1}{4}\sum \frac{(a^2-b^2)^2}{a^2+b^2} \leq \frac{1}{2}\sum (a-b)^2$$

$$RHS \leq 3(ab+bc+ca)+a^2+b^2+c^2-ab-bc-ca=$$
$$=(a+b+c)^2$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$(a+b+c)^2 \overset{(1)}{\geq} 3(ab+bc+ca)+\frac{1}{4}\sum\frac{(a^2-b^2)^2}{a^2+b^2}$$

$$\textbf{(1)} \Leftrightarrow a^2+b^2+c^2-ab-bc-ca$$

$$\geq \frac{1}{4}\sum\frac{(a^2-b^2)^2}{a^2+b^2} \quad \textbf{(2)}$$

We shall prove that: $\frac{a^2+b^2}{2}-ab \geq \frac{1}{4}\cdot\frac{(a^2-b^2)^2}{(a^2+b^2)}$

$$\Leftrightarrow \frac{(a-b)^2}{2}-\frac{(a-b)^2(a+b)^2}{4(a^2+b^2)} \geq 0$$

$$\Leftrightarrow \frac{(a-b)^2}{2}\left\{1-\frac{(a+b)^2}{2(a^2+b^2)}\right\} \geq 0$$

$$\Leftrightarrow (a-b)^2\left\{\frac{(a+b)^2+(a-b)^2-(a+b)^2}{2(a^2+b^2)}\right\} \geq 0$$

$$\Leftrightarrow \frac{(a-b)^4}{(a^2+b^2)} \geq 0 \rightarrow \textit{true}$$

$$\therefore \frac{a^2+b^2}{2}-ab \geq \frac{1}{4}\cdot\frac{(a^2-b^2)^2}{(a^2+b^2)} \quad \textit{(i)}$$

Similarly, $\frac{b^2+c^2}{2}-bc \overset{(ii)}{\geq} \frac{1}{4}\cdot\frac{(b^2-c^2)^2}{(b^2+c^2)}$***, and,***

$$\frac{c^2+a^2}{2}-ca \overset{(iii)}{\geq} \frac{1}{4}\cdot\frac{(c^2-a^2)^2}{(c^2+a^2)}$$

(i) + (ii) + (iii) $\Rightarrow$ ***(2) is true (Proved)***

109. **If $x, y, z \geq 0$ then:**

$$2\sqrt{2}(xy + yz + zx) \geq \sqrt{2xyz}(\sqrt{x} + \sqrt{y} + \sqrt{z}) + \sum \sqrt{x^2z^2 + y^2z^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palcios – Huarmey – Peru

Siendo $x, y, z \geq 0$. Probar la siguiente desigualdad

$$2\sqrt{2}(xy + yz + zx) \geq \sqrt{2xyz}(\sqrt{x} + \sqrt{y} + \sqrt{z}) + \sum \sqrt{x^2(z^2 + y^2)}$$

Realizamos los siguientes cambios de variables

$$x = a^2 \geq 0, y = b^2 \geq 0, z = c^2 \geq 0$$

La desigualdad es equivalente

$$2\sqrt{2}(a^2b^2 + b^2c^2 + c^2a^2) \geq \sqrt{2}abc(a + b + c) + \sum a^2\sqrt{(c^4 + b^4)}$$

Probaremos que $c^4 + b^4 \leq 2(b^2 - bc + c^2)^2$

$$\Leftrightarrow c^4 + b^4 \leq 2(b^2 + c^2)^2 + 2b^2c^2 - 4bc(b^2 + c^2)$$

$$\Leftrightarrow (b^4 + c^4 + 2b^2c^2) + 4b^2c^2 - 4bc(b^2 + c^2) =$$

$$= (b^2 + c^2)^2 - 4bc(b^2 + c^2) + 4b^2c^2 = (b - c)^4 \geq 0$$

Por lo tanto

$$\sum a^2\sqrt{(c^4 + b^4)} \leq \sum \sqrt{2}a^2(b^2 - bc + c^2) =$$

$$= 2\sqrt{2}(a^2b^2 + b^2c^2 + c^2a^2) - \sqrt{2}abc(a + b + c)$$

$$\Leftrightarrow 2\sqrt{2}(a^2b^2 + b^2c^2 + c^2a^2) \geq \sqrt{2}abc(a + b + c) + \sum a^2\sqrt{(c^4 + b^4)}$$

(LQQD)

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$2\sqrt{2} \cdot (xy + yz + zx) = \sqrt{2}((xy + zx) + (yz + xy) + (zx + yz)) =$$

$$= \sqrt{2}\cdot\sum x(y+z) = \sum x\cdot\sqrt{2(y+z)^2} = \sum x\sqrt{2(y^2+z^2+2yz)} =$$

$$= \sum x\sqrt{(1^2+1^2)\cdot\left(\left(\sqrt{y^2+z^2}\right)^2+\left(\sqrt{2yz}\right)^2\right)} \overset{CBS}{\geq}$$

$$\geq \sum x\left(\sqrt{y^2+z^2}+\sqrt{2yz}\right) = \sum\sqrt{x^2y^2+x^2z^2}+\sum\sqrt{2xyz}\cdot\sqrt{x}$$

$$= \sqrt{2xyz}\cdot\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)+\sum\sqrt{x^2y^2+y^2z^2}$$

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\underbrace{2\sqrt{2}(xy+yz+zx)}_{a} \geq \underbrace{\sqrt{2xyz}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)+\sum\sqrt{x^2z^2+y^2z^2}}_{b}$$

$$\sum 2\sqrt{2}xy \geq \sum x\sqrt{2yz}+\sum z\sqrt{x^2+y^2}$$

1) $\sum\sqrt{2}\,xy \overset{?}{\geq} \sum x\sqrt{2yz}$ ***(AM-GM)***

$$\sqrt{2}(xy+zx) \geq 2x\sqrt{2yz}$$

$$\sqrt{2}(xz+zy) \geq 2z\sqrt{2xy}$$

$$\sqrt{2}(xy+yz) \geq 2y\sqrt{xz}$$

2) $\sum\sqrt{2}xy \overset{?}{\geq} \sum z\sqrt{x^2+y^2}$

$$\sqrt{2}(xy+yz) \geq y\sqrt{x^2+y^2}$$

$$2y^2(x+z)^2 \geq y^2(x^2+z^2)$$

$$2x^2+2z^2+4xz \geq x^2+z^2$$

$$(x+z)^2+2xz \geq 0 \Rightarrow x,y,z \geq 0$$

(1) + ***(2)*** $\geq$ ***RHS (Proved)***

Solution 4 by Uche Eliezer Okeke-Ananbra-Nigerie

$$RHS = \sqrt{2xyz}(\sqrt{x}+\sqrt{y}+\sqrt{z}) + \sum \sqrt{x^2z^2+z^2y^2}$$

$$= \sqrt{2}\sum(x\sqrt{yz}) + \sum\left(x\sqrt{y^2+z^2}\right)$$

$$\overset{CBS}{\leq} \sqrt{2}\sqrt{\sum x^2\cdot\sum yz} + \sqrt{2}\sqrt{\sum x^2\cdot\sum x^2}$$

$$\overset{CBS}{\leq} \sqrt{2}\sqrt{\sum x^2}\left\{\sqrt{\sum x^2}+\sqrt{\sum x^2}\right\} = 2\sqrt{2}\sum x^2$$

$$\Rightarrow 2\sqrt{2}\sum x^2 \overset{CBS}{\leq} 2\sqrt{2}\sum xy = LHS$$

(Proved)

110. If $x, y, z, a > 0$ then:

$$\frac{1}{2a}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq \frac{x}{x^2+a^2yz}+\frac{y}{y^2+a^2zx}+\frac{z}{z^2+a^2xy}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Nirapada Pal-Jhargram-India

$$\sum\frac{x}{x^2+a^2yz} \overset{AM-GM}{\leq} \sum\frac{x}{2ax\sqrt{yz}}$$

$$= \frac{1}{2a}\sum\frac{1}{\sqrt{xy}} \leq \frac{1}{2a}\sum\frac{1}{x} \text{ since } \sum AB \leq \sum A^2$$

Solution 2 by Uche Eliezer Okeke-Anambra-Nigerie

If $x, y, z, a > 0$

$$\frac{1}{2a}\sum\left(\frac{1}{x}\right) \geq \sum\frac{x}{x^2+a^2yz}$$

$$LHS=\frac{1}{2a}\sum\left(\frac{1}{x}\right)=\frac{1}{4a}\sum\left(\frac{1}{y}+\frac{1}{z}\right)\overset{AM-GM}{\geq}\frac{1}{2a}\sum\frac{1}{\sqrt{yz}}$$

$$=\sum\frac{x}{2\sqrt{x^2(a^2yz)}}\overset{GM-AM}{\geq}\sum\frac{x}{x^2+a^2yz}\ (RHS)$$

(Proved)

111. **If $a,b,c>0$ then:**

$$3(a^2+b^2+c^2)^2\geq 8abc(a+b+c)+\sum(a^2+b^2-c^2)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Aziz Abdul-Semarang-Indonesia

a Fact: $(ab-bc)^2\geq 0$

$$a^2b^2+b^2c^2\geq 2ab^2c$$

Analogoue,

$$b^2c^2+c^2a^2\geq 2abc^2$$

$$a^2c^2+a^2b^2\geq 2a^2bc\quad +$$

$$-----------$$

$$a^2b^2+b^2c^2+c^2a^2\geq abc(a+b+c)$$

$$\Leftrightarrow 8(a^2b^2+b^2c^2+c^2a^2)\geq 8abc(a+b+c)$$

$$3(a^4+b^4+c^4)+6(a^2b^2+b^2c^2+c^2a^2)\geq$$

$$\geq 3(a^4+b^4+c^4)-2(a^2b^2+b^2c^2+c^2a^2)+8abc(a+b+c)$$

$$3(a^2+b^2+c^2)^2\geq(a^4+b^4+c^4+2a^2b^2-2b^2c^2-2a^2c^2)$$

$$+(a^4+b^4+c^4+2a^2c^2-2a^2b^2-2b^2c^2)$$

$$+(a^4+b^4+c^4+2b^2c^2-2a^2b^2-2a^2c^2)$$

$$+8abc(a+b+c)$$

$$3(a^2+b^2+c^2)^2 \geq (a^2+b^2-c^2)^2+(a^2+c^2-b^2)+(b^2+c^2-a^2)^2$$

$$+8abc(a+b+c)$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Si $a, b, c \in R$***. Probar la siguiente desigualdad***

$$3(a^2+b^2+c^2)^2 \geq 8abc(a+b+c)+\sum(a^2+b^2-c^2)^2$$

$$\Leftrightarrow 3\sum a^4+6\sum a^2b^2 \geq 8abc(a+b+c)+3\sum a^4-2\sum a^2b^2$$

$$\Leftrightarrow 8\sum a^2b^2 \geq a^2b^2 \geq 8abc(a+b+c) \Leftrightarrow \sum a^2b^2 \geq abc(a+b+c)$$

Si $x, y, z \in R$***, se cumple la siguiente desigualdad***

$$x^2+y^2+z^2 \geq xy+yz+zx$$

Siendo $x=ab, y=bc, z=ca$

$$\Rightarrow a^2b^2+b^2c^2+c^2a^2 \geq abc(a+b+c)$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$3(a^2+b^2+c^2)^2-(a^2+b^2-c^2)^2-(b^2+c^2-a^2)^2-(a^2+b^2-c^2)^2$$

$$=8(a^2b^2+b^2c^2+c^2a^2) \geq 8abc(a+b+c)$$

$(\because x^2+y^2+z^2 \geq xy+yz+zx)$ ***where*** $x=ab, y=bc, z=ca$***)***

Solution 4 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$a=\sqrt{x}$$

$$b=\sqrt{y}$$

$$c=\sqrt{z}$$

$$3(x+y+z)^2 \geq 8\sqrt{xyz}(\sqrt{x}+\sqrt{y}+\sqrt{z})+\sum(x+y-z)^2$$

$$3(x+y+z)^2-(x+y-z)^2-(x+z-y)^2-(y+z-x)^2=$$

$$= 3x^2 + 3y^2 + 3z^2 + 6xy + 6xz + 6yz - x^2 - y^2 - z^2 - 2xy + 2xz + 2yz -$$
$$-x^2 - z^2 - y^2 - 2xz + 2xy + 2zy - y^2 - z^2 - x^2 - 2yz + 2xy + 2xz =$$
$$= 8xy + 8xz + 8yz$$

$$\begin{cases} xy + xz \ge 2x\sqrt{yz} \\ xy + yz \ge 2y\sqrt{xz} \\ xz + yz \ge 2z\sqrt{xy} \end{cases} \Rightarrow AM - GM$$

112. If $x, y, z \ge 0$ then:

$$\sum \left(\sqrt[3]{x} + \sqrt[3]{4(y+z)}\right) \le 3\sqrt[3]{9(x+y+z)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $x, y, z \ge 0$. Probar la siguiente desigualdad

$$\sum \left(\sqrt[3]{x} + \sqrt[3]{4(y+z)}\right) \le 3\sqrt[3]{9(x+y+z)}$$

Como $x, y, z \ge 0$

Por la desigualdad de Holder

1) $\sqrt[3]{(x+y)} + \sqrt[3]{(y+z)} + \sqrt[3]{(z+x)} \le$

$$\le \sqrt[3]{\big((x+y) + (y+z) + (z+x)\big)(1+1+1)(1+1+1)}$$

$$\Leftrightarrow \sqrt[3]{4(x+y)} + \sqrt[3]{4(y+z)} + \sqrt[3]{4(z+x)} \le 2\sqrt[3]{9(x+y+z)} \quad \textit{(A)}$$

2) $\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} \le \sqrt[3]{(x+y+z)(1+1+1)(1+1+1)}$

$$\Leftrightarrow \sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} \le \sqrt[3]{9(x+y+z)} \quad \textit{(B)}$$

Sumando (A) + (B)

$$\sum\left(\sqrt[3]{x} + \sqrt[3]{4(y+z)}\right) \le 3\sqrt[3]{9(x+y+z)} \quad \textit{(LQQD)}$$

Solution 2 by Ravi Prakash-New Delhi-India

There is nothing to prove if $x = y = z = 0$.

Assume at least one of $x, y, z > 0$.

For $0 \le t \le 1$, ***let***

$$f(t) = t^{\frac{1}{3}} + \left(4(1-t)\right)^{\frac{1}{3}}$$

$$f'(t) = \frac{1}{3}\cdot\frac{1}{t^{\frac{2}{3}}} - \frac{2^{\frac{2}{3}}}{3(1-t)^{\frac{2}{3}}}, 0 < t < 1 = \frac{1}{3}\left[\frac{(1-t)^{\frac{2}{3}} - 2^{\frac{2}{3}}t^{\frac{2}{3}}}{t^{\frac{2}{3}}(1-t)^{\frac{2}{3}}}\right]$$

$$= \frac{1}{3}\left[\frac{\left((1-t)^{\frac{1}{3}} - (2t)^{\frac{1}{3}}\right)\left((1-t)^{\frac{1}{3}} + (2t)^{\frac{1}{3}}\right)}{t^{\frac{2}{3}}(1-t)^{\frac{2}{3}}}\right]$$

$$f'(t) > 0 \ \textit{if} \ 0 < t < \frac{1}{3}$$

$$= 0 \ \textit{if} \ t = \frac{1}{3} < 0 \ \textit{if} \ \frac{1}{3} < t < 1$$

$$\therefore f(t) \textit{ is maximum at } t = \frac{1}{3}$$

$$\textit{Thus, } f(t) \le f\left(\frac{1}{3}\right), 0 \le t \le 1$$

$$\Rightarrow f(t) \le 3^{\frac{2}{3}}, 0 \le t \le 1$$

Now,

$$\left[\frac{x^{\frac{1}{3}} + \left(4(y+z)\right)^{\frac{1}{3}}}{(x+y+z)^{\frac{1}{3}}}\right] = f\left(\frac{x}{x+y+z}\right) \le 3^{\frac{2}{3}}$$

$$\Rightarrow \sum\left[x^{\frac{1}{3}} + \left(4(y+z)\right)^{\frac{1}{3}}\right] \le \sum 9^{\frac{1}{3}}(x+y+z)^{\frac{1}{3}} = 3\left(9(x+y+z)\right)^{\frac{1}{3}}$$

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113. **Prove that the following inequalities hold for all positive real numbers a, b, c**

(a) $\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\ge\frac{3(a^3+b^3+c^3)}{a+b+c}$

(b) $\frac{b^5}{a^3}+\frac{c^5}{b^3}+\frac{a^5}{c^3}\ge\frac{3(a^3+b^3+c^3)}{a+b+c}$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los numerous R^+ "a, b, c":

$$\textbf{a)}\ 3(a^3+b^3+c^3)\le(a+b+c)\left(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\right)$$

$$\Rightarrow\left(\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\right)+\left(\frac{ab^3}{c}+\frac{bc^3}{a}+\frac{ca^3}{b}\right)+(a^2b+b^2c+c^2a)\ge$$

$$\ge 2(a^3+b^3+c^3)+(a^2b+b^2c+c^2a)$$

Desdes que: $a, b, c>0$. ***Por: MA ≥ MG***

$$\frac{ab^3}{c}+\frac{bc^3}{a}\ge 2b^2c,\ \frac{bc^3}{a}+\frac{ca^3}{b}\ge 2c^2a,\ \frac{ca^3}{b}+\frac{ab^3}{c}\ge a^2b$$

Sumando obtenemos:

$$2\left(\frac{ab^3}{c}+\frac{bc^3}{a}+\frac{ca^3}{b}\right)\ge 2(a^2b+b^2c+c^2a)\rightarrow$$

$$\rightarrow\frac{ab^3}{c}+\frac{bc^3}{a}+\frac{ca^3}{b}\ge a^2b+b^2c+c^2a\ \dots\ \textbf{(A)}$$

$$\frac{a^4}{b}+a^2b\ge 2a^3,\ \frac{b^4}{c}+b^2c\ge 2b^3,\ \frac{c^4}{a}+c^2a\ge 2c^3$$

$$\Rightarrow\left(\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\right)+(a^2b+b^2c+c^2a)\ge 2(a^3+b^3+c^3)\ \dots\ \textbf{(B)}$$

Sumando: (A) + (B)

$$\Rightarrow \left(\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\right)+\left(\frac{ab^3}{c}+\frac{bc^3}{a}+\frac{ca^3}{b}\right)+(a^2b+b^2c+c^2a)\geq$$

$$\geq 2(a^3+b^3+c^3)+(a^2b+b^2c+c^2a)$$

b) $3(a^3+b^3+c^3)\leq (a+b+c)\left(\frac{b^5}{a^3}+\frac{c^5}{b^3}+\frac{a^5}{c^3}\right)$

$$\Rightarrow 3(a^3+b^3+c^3)\leq\left(\frac{b^5}{a^2}+\frac{c^5}{b^2}+\frac{a^5}{c^2}\right)+\left(\frac{ac^5}{b^3}+\frac{ba^5}{c^3}+\frac{cb^5}{a^3}\right)+\left(\frac{a^6}{c^3}+\frac{b^6}{a^3}+\frac{c^6}{b^3}\right)$$

Siendo: $a,b,c>0$***. Por la desigualdad de Cauchy:***

$$\Rightarrow \frac{a^6}{c^3}+\frac{b^6}{a^3}+\frac{c^6}{b^3}\geq\frac{(a^3+b^3+c^3)^2}{c^3+a^3+b^3}=a^3+b^3+c^3\ \dots\ \textit{(A)}$$

Por MA ≥ MG

$$\Rightarrow \frac{ac^5}{b^3}+\frac{ba^5}{c^3}\geq\frac{2a^3c}{b},\frac{ba^5}{c^3}+\frac{cb^5}{a^3}\geq\frac{2b^3a}{c},\frac{ac^5}{b^3}+\frac{cb^5}{a^3}\geq\frac{2c^3b}{a}\quad \textit{(B)}$$

$$\Rightarrow \frac{ab^3}{c}+\frac{ca^3}{b}\geq 2a^2b,\frac{bc^3}{a}+\frac{ab^3}{c}\geq 2b^2c,\frac{ca^3}{b}+\frac{bc^3}{a}\geq 2c^2a\quad \textit{(C)}$$

$$\Rightarrow \frac{ac^5}{b^3}+\frac{ba^5}{c^3}+\frac{cb^5}{a^3}\geq\frac{ab^3}{c}+\frac{bc^3}{a}+\frac{ca^3}{b}\geq a^2b+b^2c+c^2a$$

Por transitividad:

$$\Rightarrow\left(\frac{b^5}{a^2}+\frac{c^5}{b^2}+\frac{a^5}{c^2}\right)+\left(\frac{ac^5}{b^3}+\frac{ba^5}{c^3}+\frac{cb^5}{a^3}\right)\geq\left(\frac{b^5}{a^2}+a^2b\right)+\left(\frac{c^5}{b^2}+b^2c\right)+\left(\frac{a^5}{c^2}+c^2a\right)\geq$$

$$\geq 2(a^3+b^3+c^3)$$

$$\Rightarrow\left(\frac{b^5}{a^2}+\frac{c^5}{b^2}+\frac{a^5}{c^2}\right)+\left(\frac{ac^5}{b^3}+\frac{ba^5}{c^3}+\frac{cb^5}{a^3}\right)+\left(\frac{a^6}{c^3}+\frac{b^6}{a^3}+\frac{c^6}{b^3}\right)\geq$$

$$\geq 2(a^3+b^3+c^3)+a^3+b^3+c^3=3(a^3+b^3+c^3)$$

(LQQD)

Solution 2 Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{a^{n+2}}{b^n}+\frac{b^{n+2}}{c^n}+\frac{c^{n+2}}{a^n}\geq\frac{3\cdot(a^3+b^3+c^3)}{a+b+c}$$

ASSURE

a) b) – same

$$(a^{2n+2}\cdot c^n+b^{2n+2}\cdot a^n+c^{2n+2}\cdot b^n)\cdot(a+b+c)\ge$$

$$\ge 3(a^3+b^3+c^3)\cdot a^n\cdot b^n\cdot c^n\Leftrightarrow$$

$$\left(\sum_{cyc}a^{2n+2}\cdot c^n\right)\cdot(a+b+c)\overset{Chebyshev}{\ge}$$

$$\ge\frac{1}{3}(a^3+b^3+c^3)\left(\sum_{cyc}(a^{2n-1}\cdot c^n)\right)\cdot(a+b+c)\overset{AM\ge GM}{\ge}$$

$$\ge\frac{1}{3}(a^3+b^3+c^3)\cdot 3\cdot\sqrt[3]{a^{3n-1}\cdot b^{3n-1}\cdot c^{3n-1}}\cdot 3\cdot\sqrt[3]{abc}=$$

$$=3\cdot(a^3+b^3+c^3)\cdot\sqrt[3]{a^{3n}\cdot b^{3n}\cdot c^{3n}}=3\cdot(a^3+b^3+c^3)\cdot a^n\cdot b^n\cdot c^n$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

We know, $(a^2+b^2+c^2)^2\ge 3(a^3b+b^3c+c^3a)$ ***(1)***

1)

$$\sum_{cyc}\frac{a^3}{b}=\sum_{cyc}\frac{a^6}{a^3b}\overset{BERGSTROM}{\ge}\frac{(a^3+b^3+c^3)^2}{\sum a^3b}\ge\frac{3(a^3+b^3+c^3)^2}{(a^2+b^2+c^2)^2}$$

[applying relation (1)]

we need to prove, $\frac{3(a^3+b^3+c^3)^2}{(a^2+b^2+c^2)^2}\ge\frac{3(a^3+b^3+c^3)}{a+b+c}$

$$\Leftrightarrow\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^3\right)\ge\left(\sum_{cyc}a^2\right)^2,$$

which is true by Cauchy – Schwarz

$\therefore\sum_{cyc}\frac{a^3}{b}\ge\frac{3(a^3+b^3+c^3)}{a+b+c}$ ***(proved)***

2) $\sum_{cyc}\frac{b^5}{a^3}=\sum_{cyc}\frac{b^6}{a^3b}\overset{BERGSTROM}{\geq}\frac{\left(\sum a^3\right)^2}{\sum a^3b}\geq\frac{3\left(\sum a^3\right)^2}{\left(\sum a^2\right)^2}$ **. We need to prove**

$$\frac{3(\sum a^3)^2}{(\sum a^2)^2}\geq\frac{3(\sum a^3)}{\sum a}\Leftrightarrow\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^3\right)\geq\left(\sum_{cyc}a^2\right)^2$$

which is true by Cauchy – Schwarz

$$\therefore\sum_{cyc}\frac{b^5}{a^3}\geq\frac{3(a^3+b^3+c^3)}{a+b+c}$$

(Proved)

114. **Prove that for any positive real numbers x,y,z**

$$\frac{x^2\sqrt{y^2+z^2}+y^2\sqrt{z^2+x^2}+z^2\sqrt{x^2+y^2}}{x^3+y^3+z^3}\leq\sqrt{2}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los numerous R^+ x,y,z la siguiente desigualdad

$$x^2\sqrt{y^2+z^2}+y^2\sqrt{z^2+x^2}+z^2\sqrt{x^2+y^2}\leq\sqrt{2}(x^3+y^3+z^3)$$

Recordar la siguiente desigualdad:

$$(a^4+b^4)\leq 2(a^2-ab+b^2)^2\Leftrightarrow(a-b)^4\geq 0$$

Por lo tanto

$$\sum x^2\sqrt{y^2+z^2}\leq\sqrt{2}x^2(y+z-\sqrt{yz})+\sqrt{2}y^2(z+x-\sqrt{zx})+\sqrt{2}z^2(x+y-\sqrt{xy})$$

Es necesario demostrar lo siguiente

$$\sqrt{2}(x^3+y^3+z^3)\geq\sqrt{2}x^2(y+z-\sqrt{yz})+\sqrt{2}y^2(z+x-\sqrt{zx})+\sqrt{2}z^2(x+y-\sqrt{xy})$$

$$\Leftrightarrow x^3+y^3+z^3+x^2\sqrt{yz}+y^2\sqrt{zx}+z^2\sqrt{xy}\geq xy(x+y)+yz(y+z)+zx(z+x)$$

Por MA $\geq$ MG

$$\Leftrightarrow x^3+y^3+z^3+x^2\sqrt{yz}+y^2\sqrt{zx}+z^2\sqrt{xy}\geq x^3+y^3+z^3+3xyz$$

Por ultimo

$$x^3+y^3+z^3+3xyz\geq xy(x+y)+yz(y+z)+zx(z+x)$$

(Válid por desigualdad de Schur)

Solution 2 Soumitra Mandal-Chandar Nagore-India

We know, $\sqrt{x}$ as a concave function for all $x>0$

$$\therefore \frac{x^2\sqrt{y^2+z^2}+y^2\sqrt{x^2+z^2}+z^2\sqrt{x^2+y^2}}{x^3+y^3+z^3}=\sum_{cyc}\frac{x^3}{x^3+y^3+z^3}\sqrt{\frac{y^2+z^2}{x^2}}$$

$$\overset{WEIGHTED\ JENSEN\ INEQUALITY}{\leq}\sqrt{\sum_{cyc}\frac{x^3}{x^3+y^3+z^3}\left(\frac{y^2+z^2}{x^2}\right)}$$

$$=\sqrt{\frac{xy(x+y)+yz(y+z)+zx(z+x)}{x^3+y^3+z^3}}\leq\sqrt{2}\ \textit{(proved)}$$

$$\begin{bmatrix}\because x^3+y^3\geq xy(x+y)\\ y^3+z^3\geq yz(y+z)\\ z^3+x^3\geq zx(z+x)\ and\ adding\end{bmatrix}$$

Solution 3 by Sanong Haueray-Nakonpathom-Thailand

Since

$$x^6+x^3y^3+x^3y^3\geq 3x^4y^2$$

$$x^6+x^3z^3+x^3z^3\geq 3x^4z^2$$

$$y^6+x^3y^2+x^2y^3\geq 3x^4y^2$$

$$y^6+y^3z^3+y^2z^3\geq 3y^4z^2$$

$$z^6+x^3z^3+x^3z^3\geq 3z^4x^2$$

$$z^6+y^3z^3+y^3z^3\geq 3z^4y^2$$

Hence,

$$2(x^3+y^3+z^3)^2 \geq 3(x^4y^2+x^4z^2+\cdots+z^4y^2)$$

Consider

$$\frac{x^2\sqrt{y^2+z^2}+y^2\sqrt{z^2+x^2}+z^2\sqrt{x^2+y^2}}{x^3+y^3+z^3}$$

$$\leq \sqrt{\frac{3(x^4y^2+x^4z^2+\cdots+z^4x^2+z^4y^4)}{(x^2+y^2+z^3)^2}}$$

$$= \sqrt{\frac{6(x^4y^2+\cdots+z^4y^2)}{2(x^3+y^3+z^3)^2}} \leq \sqrt{\frac{(x^4y^2+\cdots+z^4y^2)}{3(x^4y^2+\cdots+z^4y^2)}} = \sqrt{2}$$

Solution 4 by Erbolat Darin-Ulanbaatar-Mongolia

$$A = \frac{x^2\sqrt{y^2+z^2}+y^2\cdot\sqrt{z^2+x^2}+z^2\sqrt{y^2+x^2}}{x^3+y^3+z^3} \leq \sqrt{2}$$

$$\Rightarrow$$

$$B = x^3+y^3+z^3 = \frac{4(x^3+y^3+z^3)}{4} =$$

$$= \frac{2x^3+2y^3+2z^3+(x^3+y^3)+(y^3+z^3)+(z^3+x^3)}{4} \geq$$

$$\geq \frac{2x^3+2y^3+2z^3+(x+y)xy+yz(y+z)+xz(x+z)}{4} =$$

$$= \frac{2x^3+2y^3+2z^3+x\cdot(y^2+z^2)+y\cdot(z^2+x^2)+z\cdot(x^2+y^2)}{4} \geq$$

$$\geq \frac{\sum 2\cdot\sqrt{2x^3\cdot x(y^2+z^2)}}{4} = \frac{\sum 2\sqrt{2}x^2\sqrt{y^2+z^2}}{4} = \frac{\sum x^2\sqrt{y^2+z^2}}{\sqrt{2}}$$

$$A \leq \frac{\sum x^2\sqrt{y^2+z^2}}{\frac{\sum x^2\sqrt{y^2+z^2}}{\sqrt{2}}} = \sqrt{2}$$

115. **Let x, y, z be positive real numbers such that: $x + y + z = 3$.**

Prove that: $\frac{x^3y^3}{x^4+y^3-x+2} + \frac{y^3z^3}{y^4+z^3-y+2} + \frac{z^3x^3}{z^4+x^3-z+2} \le \frac{x^4+y^4+z^4+3xyz}{6}$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Hoang Le Nhat Tung – Hanoi – Vietnam

**** Let** $x, y, z > 0$ **We will prove that:***

$$x^4 + y^4 + z^4 + xyz(x + y + z) \ge xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \quad (1)$$

$$(1) \Leftrightarrow x^4 + y^4 + z^4 + xyz(x + y + z) - xy(x^2 + y^2) - yz(y^2 + z^2) - zx(z^2 + x^2) \ge 0$$

$$\Leftrightarrow x^2(x^2 - xy - xz + yz) + y^2(y^2 - yz - yx + zx) + z^2(z^2 - zx - zy + xy) \ge 0$$

$$\Leftrightarrow x^2(x - y)(x - z) + y^2(y - z)(y - x) + z^2(z - x)(z - y) \ge 0 \quad (2)$$

***Let** $x \ge y \ge z > 0$*

$$+ \text{We have: } \begin{cases} z \le x \\ z \le y \end{cases} \Leftrightarrow \begin{cases} z - x \le 0 \\ z - y \le 0 \end{cases} \Rightarrow (z - x)(z - y) \ge 0 \Rightarrow z^2(z - x)(z - y) \ge 0 \quad (3)$$

$$+ \text{Other: } x^2(x - y)(x - z) + y^2(y - z)(y - x)$$

$$= (x - y)[x^2(x - z) - y^2(y - z)] = (x - y)[(x^3 - y^3) - z(x^2 - y^2)]$$

$$= (x - y)[(x - y)(x^2 + xy + y^2) - z(x - y)(x + y)] = (x - y)^2(x^2 + xy + y^2 - zx - zy) \ge 0 \quad (4)$$

$(x \ge y \ge z > 0 \Rightarrow x^2 + xy + y^2 - zx - zy = x(x - z) + y(x - z) + y^2 \ge y^2 > 0$ **và** $(x - y)^2 \ge 0)$

– Since (3), (4): $\Rightarrow x^2(x - y)(x - z) + y^2(y - z)(y - x) + z^2(z - x)(z - y) \ge 0$

$\Rightarrow$ ***(2) True*** $\Rightarrow$ ***(1) True.***

- Since (1), AM-GM:

$$x^4 + y^4 + z^4 + xyz(x + y + z) \ge xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \ge xy \cdot 2xy + yz \cdot 2yz + zx \cdot 2zx$$

$$\Leftrightarrow x^2y^2 + y^2z^2 + z^2x^2 \le \frac{x^4 + y^4 + z^4 + 3xyz}{2} (x + y + z = 3)$$

**** We have:***

$$x^4 - x^3 - x + 1 = x^3(x - 1) - (x - 1) = (x - 1)(x^3 - 1) = (x - 1)^2(x^2 + x + 1) \ge 0$$

(Do** $(x - 1)^2 \ge 0$**)

$$\Rightarrow x^4 - x^3 - x + 1 \ge 0 \Rightarrow x^4 + y^3 - x + 2 \ge x^3 + y^3 + 1 \ge 3 \cdot \sqrt[3]{x^3 \cdot y^3 \cdot 1} = 3xy$$

$$\Leftrightarrow \frac{x^3y^3}{x^4+y^3-x+2} \le \frac{x^3y^3}{3xy} = \frac{x^2y^2}{3} \quad (6)$$

$+$ ***Similar:*** $\frac{y^3z^3}{y^4+z^3-y+2} \le \frac{y^2z^2}{3}; \frac{z^3x^3}{z^4+x^3-z+2} \le \frac{z^2x^2}{3}$ ***(7)***

$-$ ***Since (6), (7):*** $\Rightarrow \frac{x^3y^3}{x^4+y^3-x+2} + \frac{y^3z^3}{y^4+z^3-y+2} + \frac{z^3x^3}{z^4+x^3-z+2} \le \frac{x^2y^2+y^2z^2+z^2x^2}{3}$ ***(8)***

$-$ ***Since (5), (8):*** $\Rightarrow \frac{x^3y^3}{x^4+y^3-x+2} + \frac{y^3z^3}{y^4+z^3-y+2} + \frac{z^3x^3}{z^4+x^3-z+2} \le \frac{x^4+y^4+z^4+3xyz}{6}$

$\Rightarrow$ ***We get the result***

$+$ ***Iquality occurs if:*** $\begin{cases} x, y, z > 0; x + y + z = 3 \\ x = y = z \end{cases} \Leftrightarrow x = y = z = 1.$

116. Let a, b, c be non-negative real numbers such that

$$(a + b)(b + c)(c + a) = 8.$$

Prove that

$$abc(a^2 + bc)(b^2 + ca)(c^2 + ab) \le 8.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c números reales no negativos, de tal manera que

$$(a + b)(b + c)(c + a) = 8$$

Probar que

$$abc(a^2 + bc)(b^2 + ca)(c^2 + ab) \le 8$$

Utilizando la siguiente desigualdad

$$4xy \le (x + y)^2 \Leftrightarrow (x - y)^2 \ge 0$$

$$4(a^2 + bc)(ab + ac) \le \left((a^2 + bc) + (ab + ac)\right)^2 = (a + b)^2(a + c)^2$$

(A)

$$4(b^2 + ca)(bc + ba) \le \left((b^2 + ca) + (bc + ba)\right)^2 = (b + c)^2(b + a)^2$$

(B)

$$4(c^2+ab)(ca+cb)\le\left((c^2+ab)+(ca+cb)\right)^2=(c+a)^2(c+b)^2\quad \textbf{(C)}$$

Multiplicando (A) · (B) · (C)

$$64abc(a^2+bc)(b^2+ca)(c^2+ab)\le(a+b)^3(b+c)^3(c+a)^3=512$$

$$\Leftrightarrow abc(a^2+bc)(b^2+ca)(c^2+ab)\le 8\quad \textbf{(LQQD)}$$

La igualdad se alcanza cuando $a=b=c=1$.

Solution 2 by Sanong Hauerai-Nakonpathom-Thailand

From $(a+b)(b+c)(c+a)=8$***, we get***

$$8=(a+b)(b+c)(c+a)\ge 8\sqrt{a^2b^2c^2}$$

Hence, $abc\le 1$

and get $a^4b^4c+ab^4c^4+a^4bc^4+a^5b^2c^2+a^2b^5c^2+a^2b^2c^5\le$

$$\le a^2c+b^2a+c^2b+a^2b+c^2a+b^2c$$

Hence

$$a^3b^3+b^3c^3+c^3a^3+a^4bc+ab^4c+abc^4\le$$

$$\le\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{a}{c}+\frac{a}{b}+\frac{b}{a}\right)$$

Hence

$$(abc)^2+(ab)^3+(bc)^3+(ca)^3+a^4bc+ab^4c+abc^4\le$$

$$\le\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}+2$$

Hence

$$(a^2+bc)(b^2+ca)(c^2+ab)\le\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{a}{c}\right)$$

$$\le\frac{(a+b)(b+c)(c+a)}{abc}$$

That

$$(a^2+bc)(b^2+ca)(c^2+ab) \le (a+b)(b+c)(c+a) = 8$$

Then from it is to be true

117. Prove that for all non-negative real numbers a, b, c

$$\sqrt{\frac{a^2+2}{b+c+1}} + \sqrt{\frac{b^2+2}{c+a+1}} + \sqrt{\frac{c^2+2}{a+b+1}} \ge 3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los numerous R^+ a, b, c:

$$\sqrt{\frac{a^2+2}{b+c+1}} + \sqrt{\frac{b^2+2}{c+a+1}} + \sqrt{\frac{c^2+2}{a+b+1}} \ge 3$$

Por la desigualdad de Cauchy:

$$(a^2+1+1)(1+b^2+1) \ge (a+b+1)^2 \quad \textit{(A)}$$

De forma análoga:

$$(b^2+1+1)(1+c^2+1) \ge (b+c+1)^2 \quad \textit{(B)}$$

$$(c^2+1+1)(1+a^2+1) \ge (c+a+1)^2 \quad \textit{(C)}$$

Multiplicando (A) (B) (C):

$$(a^2+2)^2(b^2+2)^2(c^2+2)^2 \ge (b+c+1)^2(c+a+1)^2(a+b+1)^2$$

$$\Rightarrow (a^2+2)(b^2+2)(c^2+2) \ge (b+c+1)(c+a+1)(a+b+1)$$

De la desigualdad propuesta … Por: MA ≥ MG

$$\sqrt{\frac{a^2+2}{b+c+1}} + \sqrt{\frac{b^2+2}{c+a+1}} + \sqrt{\frac{c^2+2}{a+b+1}} \ge 3\sqrt[3]{\sqrt{\frac{(a^2+2)(b^2+2)(c^2+2)}{(b+c+1)(c+a+1)(a+b+1)}}} \ge 3$$

(LQQD)

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\sqrt{\frac{a^2+2}{b+c+1}} \overset{AM\geq GM}{\geq} 3\sqrt[3]{\prod_{cyc}\sqrt{\frac{a^2+2}{b+c+1}}}$$

We need to prove,

$$(a^2+2)(b^2+2)(c^2+2) \geq (a+b+1)(b+c+1)(c+a+1)$$

Now, $(a^2+1+1)(b^2+1+1) \overset{CAUCHY\ SCHWARZ}{\geq} (a+b+1)^2$

$(b^2+1+1)(c^2+1+1) \overset{CAUCHY\ SCHWARZ}{\geq} (b+c+1)^2$ ***and***

$(c^2+1+1)(a^2+1+1) \overset{CAUCHY\ SCHWARZ}{\geq} (c+a+1)^2$***. So,***

$$\therefore \prod_{cyc}(a^2+1) \geq \prod_{cyc}(a+b+1).$$

So,

$$\sum_{cyc}\sqrt{\frac{a^2+2}{b+c+1}} \geq 3$$

(Proved)

118. For $a,b,c \geq 0 \wedge a+b+c=1$. Prove:

$$a^4+b^3+c+2(a^2b^2+b^2c^2+c^2a^2)+\frac{2}{a^2+b^2+c^2} \geq 3$$

Proposed by Nho Nguyen Van - Nghe An - Vietnam

Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

We have: $\begin{cases} a,b,c \geq 0 \\ a+b+c=1 \end{cases} \Rightarrow a,b,c \in [0;1] \Rightarrow a^4+b^3+c \geq a^4+b^4+c^4 \Rightarrow$

$$\Rightarrow LHS \geq \sum a^4 + 2\sum a^2b^2 + \frac{2}{\sum a^2} = \left(\sum a^2\right)^2 + \frac{1}{\sum a^2} + \frac{1}{\sum a^2} \geq 3$$

The equality holds for $(a,b,c) = (1,0,0)$ ***and any cyclic permutations.***

119. Let a,b,c,d be positive real numbers such that $a+b+c+d=2$.

Prove that:

$$\frac{a}{\sqrt{b+\sqrt[3]{cda}}} + \frac{b}{\sqrt{c+\sqrt[3]{dab}}} + \frac{c}{\sqrt{d+\sqrt[3]{abc}}} + \frac{d}{\sqrt{a+\sqrt[3]{bcd}}} \geq 2.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c,d ***números*** R^+ ***de tal manera que*** $a+b+c+d=2$.

Probar que

$$\frac{a}{\sqrt{b+\sqrt[3]{cda}}} + \frac{b}{\sqrt{c+\sqrt[3]{dab}}} + \frac{c}{\sqrt{d+\sqrt[3]{abc}}} + \frac{d}{\sqrt{a+\sqrt[3]{bcd}}} \geq 2.$$

Por la desigualdad de Holder

$$\left(\sum \frac{a}{\sqrt{b+\sqrt[3]{cda}}}\right)^2 \left(a\left(b+\sqrt[3]{cda}\right) + b\left(c+\sqrt[3]{dab}\right) + c\left(d+\sqrt[3]{abc}\right) + d\left(a+\sqrt[3]{bcd}\right)\right) \geq$$

$$\geq (a+b+c+d)^3 = 8$$

Es suficiente demostrar que

$$a\left(b+\sqrt[3]{cda}\right) + b\left(c+\sqrt[3]{dab}\right) + c\left(c+\sqrt[3]{abc}\right) + d\left(a+\sqrt[3]{bcd}\right) \leq 2 \quad \textit{(A)}$$

Aplicando MA ≥ MG

$$a\left(b+\sqrt[3]{cda}\right) \leq a\left(b+\frac{c+d+a}{3}\right) = a\left(\frac{2b+(a+b+c+d)}{3}\right) =$$

$$= \frac{a(2b+2)}{3} = \frac{2ab}{3} + \frac{2a}{3}$$

$$\Leftrightarrow b\left(c+\sqrt[3]{dab}\right)\le\frac{b(2c+2)}{3}=\frac{2bc}{3}+\frac{2b}{3}$$

$$c\left(d+\sqrt[3]{abc}\right)\le\frac{c(2d+2)}{3}=\frac{2dc}{3}+\frac{2c}{3}$$

$$d\left(a+\sqrt[3]{bcd}\right)\le\frac{2ad}{3}+\frac{2d}{3}$$

$$\Leftrightarrow\sum a\left(b+\sqrt[3]{cda}\right)\le\frac{2(a+b+c+d)}{3}+\frac{2(ab+bc+cd+ad)}{3}=$$

$$=\frac{4}{3}+\frac{2(a+c)(b+d)}{3}\le\frac{4}{3}+\frac{2}{3}\cdot\frac{2(a+b+c+d)^2}{4}=2\ \textit{(LQQD)}$$

Por lo tanto $\rightarrow\left(\sum\frac{a}{\sqrt{b+\sqrt[3]{cda}}}\right)^2\ge\frac{8}{A}\ge4\Leftrightarrow\sum\frac{a}{\sqrt{b+\sqrt[3]{cda}}}\ge2$

Solution 2 by Sanong Haueray-Nakonpathom-Thailand

Leading fact when $a,b,c,d\in R^+,a+b+c+d=2$

$$+(a+b+c+d)^3\ge4^2(a^2b+b^2c+c^2d+d^2a)$$

$$\frac{8}{16}=\frac{1}{2}\ge a^4b+b^2c+c^2d+d^2a$$

$$2(a+b+c+d)^9\ge4^8(a^7cd+b^7da+c^7ab+b^7bc)$$

$$\frac{1}{27}\ge(a^7cd+b^7da+c^7db+d^7bc)$$

$$\frac{1}{2}\ge\sqrt[3]{a^7cd}+\sqrt[3]{b^7da}+\sqrt[3]{c^7ab}+\sqrt[3]{d^7bc}$$

Consider

$$\frac{a}{\sqrt{b+\sqrt[3]{cda}}}+\frac{b}{\sqrt{c+\sqrt[3]{dab}}}+\cdots+\frac{d}{\sqrt{a+\sqrt[3]{bcd}}}=$$

$$=\frac{a^2}{a\sqrt{b+\sqrt[3]{cda}}}+\cdots+\frac{d^2}{d\sqrt{a+\sqrt[3]{bcd}}}$$

$$\geq \frac{(a+b+c+d)^2}{\sqrt{a^2b + a^2\sqrt[3]{cda}} + \cdots + \sqrt{d^2a + d^2\sqrt[3]{bcd}}}$$

$$\geq \frac{4}{\sqrt{4\left(a^2b + b^2c + c^2d + d^2a + a^2\sqrt[3]{cda} + \cdots + d^2\sqrt[3]{bcd}\right)}}$$

$$\geq \frac{4}{2\sqrt{\frac{1}{2} + \frac{1}{2}}} = 2$$

120. If $a, b, c > 0, a + b + c = 3$ then:

$$\sum \left(2^a + \sqrt{2^{b+c+2}}\right) \geq 18$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdul Aziz-Semarang-Indonesia

Let $x = 2^a, y = 2^b, z = 2^e$

Then $a + b + c = 3 \Leftrightarrow \log_2(xyz) = 3 \Leftrightarrow xyz = 8$

Now,

$$2^a + 2^b + 2^c + 2 \cdot 2^{\frac{b+c}{2}} + 2 \cdot 2^{\frac{a+c}{2}} + 2 \cdot 2^{\frac{a+b}{2}}$$

$$= x + y + z + 2\sqrt{xy} + 2\sqrt{yz} + 2\sqrt{xz}$$

$$= \left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right)^2 \geq \left(3\sqrt[6]{xyz}\right)^2 = 9 \cdot \sqrt[3]{xyz} = 9 \cdot 2 = 18$$

Equality holds when $a = b = c = 1$

Solution 2 by Chris Kyriazis-Greece

Using only AM-GM, we have

$$\sum \left(2^a + \sqrt{2^{b+c+2}}\right) \geq 3\sqrt[3]{2^{a+b+c}} + 3\sqrt[3]{2^{a+b+c+3}} =$$

$$\overset{a+b+c=3}{=} 3\cdot 2+3\cdot 2^2=18$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\textit{Let } f(x)=2^x+2^{\sqrt{5-x}}, 0\le x\le 3$$

$$f'(x)=2^x\ln 2+\left(2^{\sqrt{5-x}}\ln 2\right)\left(\frac{-1}{2\sqrt{5-x}}\right)$$

$$=(\ln 2)\left[2^x-\frac{2^{\sqrt{5-x}}}{2\sqrt{5-x}}\right], 0<x<3$$

$$f'(x)=0 \textit{ if } x=1$$

$$\min f(x)=\min\{f(0),f(1),f(3)\}$$

$$=f(1)=6$$

Now,

$$\sum\left(2^a+2^{\sqrt{b+c+2}}\right)=\sum\left(2^a+2^{\sqrt{5-a}}\right)\ge 3(6)=18$$

Solution 4 by Ngo Minh Ngoc Bao-Vietnam

$$\sum\left(2^a+\sqrt{2^{b+c+2}}\right)\ge 18 \quad (*)$$

$$(*)\Leftrightarrow \sum\left(2^a+\sqrt{2^{5-a}}\right)\ge 18$$

$$\textit{Considering function } f(a)=2^a+\sqrt{2^{5-a}}, \forall\, a\in(0,3)$$

$$\Rightarrow f'(a)=2^a\ln 2-\frac{2^{4-a}\ln 2}{\sqrt{2^{5-a}}}$$

$$f'(a)=0\Leftrightarrow a=1$$

a	0	1	3
$f'(a)$	$-$	0	$+$
$f(a)$	↘	6	↗

Similarly $f(b), f(c) \geq 6 \Rightarrow \sum\left(2^a + \sqrt{2^{b+c+2}}\right) \geq 18$

Solution 5 by Soumitra Mandal-Chandar Nagore-India

By AM ≥ GM

$$\sum_{cyc} 2^a + \sum_{cyc} \sqrt{2^{b+c+2}} \geq 3 \cdot 2^{\frac{a+b+c}{3}} + 3 \cdot 2^{\frac{\sum(b+c+2)}{6}}$$

$$= 3 \cdot 2 + 3 \cdot 2^{\frac{a+b+c+3}{3}} = 6 + 12 = 18$$

(Proved)

121. **Given $a, b \,\&\, c > 0$ such that $a^3 + b^3 + c^3 = 3$**

Prove that

$$\frac{a^5+1}{b^2+c} + \frac{b^5+1}{c^2+a} + \frac{c^5+1}{a^2+b} \geq 3$$

Proposed by Imad Zak-Saida-Lebanon

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Dado que $a, b, c > 0$ de tal manera que $a^3 + b^3 + c^3 = 3$. Probar que

$$\frac{a^5+1}{b^2+c} + \frac{b^5+1}{c^2+a} + \frac{c^5+1}{a^2+b} \geq 3$$

Por la desigualdad de Holder

$$(a^3+b^3+c^3)(1+1+1)(1+1+1) \geq (a+b+c)^3 \Leftrightarrow 3 \geq a+b+c$$

$$(a^3+b^3+c^3)(a^3+b^3+c^3)(1+1+1) \geq (a^2+b^2+c^2)^3 \Leftrightarrow 3 \geq a^2+b^2+c^2$$

$$(a^3+b^3+c^3)(b^3+c^3+a^3)(1+1+1) \geq (ab+bc+ca)^3 \Leftrightarrow 3 \geq ab+bc+ca$$

Por $MA \geq MG$

$$b^3 + b^3 + a^3 \geq 3b^2a \quad (M)$$

$$c^3 + c^3 + b^3 \geq 3c^2b \quad (N)$$

$$a^3+a^3+c^3\ge 3a^2c \quad \textbf{(P)}$$

Sumando (M) + (N) + (P) $\rightarrow 3(a^3+b^3+c^3)\ge 3b^2a+3c^2b+3a^2c \Leftrightarrow$

$$\Leftrightarrow 3\ge b^2a+c^2b+a^2c$$

Aplicando la desigualdad de Cauchy

$$\frac{a^5}{b^2+c}+\frac{b^5}{c^2+a}+\frac{c^5}{a^2+b}\ge\frac{(a^3+b^3+c^3)^2}{b^2a+c^2b+a^2c+a+b+c}=$$

$$=\frac{9}{b^2a+c^2b+a^2c+a+b+c}\ge\frac{9}{3+3}=\frac{3}{2} \quad \textbf{(A)}$$

$$\frac{1}{b^2+c}+\frac{1}{c^2+a}+\frac{1}{a^2+b}\ge\frac{9}{a^2+b^2+c^2+a+b+c}\ge\frac{9}{3+3}=\frac{3}{2}\ \textbf{(B)}$$

Sumando (A) + (B) $\rightarrow \frac{a^5+1}{b^2+c}+\frac{b^5+1}{c^2+a}+\frac{c^5+1}{a^2+b}\ge 3$ ***(LQQD)***

Solution 2 by Sanong Hauerai-Nakonpathom-Thailand

Give $a,b,c>0$ ***and*** $a^3+b^3+c^3$

Prove that $\frac{a^5+1}{b^2+c}+\frac{b^5+1}{c^2+a}+\frac{c^5+1}{a^2+b}\ge 3$

Consider $(a+b+c)^3\le 9(a^3+b^3+c^3)^3=27$

Hence $a+b+c\le 3$

And $(a^3+b^3+c^3)(a+b+c)\ge(a^2+b^2+c^2)^2$

Hence $3(a+b+c)\ge(a^2+b^2+c^2)^2$

$$\sqrt{3\times 3}\ge\sqrt{3(a+b+c)}\ge a^2+b^2+c^2$$

Hence $a^2+b^2+c^2\le 3$

$$a^2+b+b^2+c+c^2+a\le b$$

$$\frac{1}{a^2+b}+\frac{1}{b^2+c}+\frac{1}{c^2+a}\ge\frac{3}{2}\ldots \textbf{(A)}$$

$$\frac{a^5}{b^2+c}+\frac{b^5}{c^2+a}+\frac{c^5}{a^2+b}=\frac{a^6}{ab^2+ac}+\frac{b^6}{bc^2+ab}+\frac{c^6}{ca^2+bc}$$

$$\geq \frac{(a^3+b^2+c^2)^2}{\left((ab^2+bc^2+ca^2)+(ab+bc+ca)\right)} \geq \frac{9}{6} = \frac{3}{2} \dots \textit{(B)}$$

Because $3 = a^3 + b^2 + c^2 \geq ab^2 + bc^2 + ca^2$

$$3 \geq a + b + c \geq ab + bc + ca$$

Therefore $\frac{a^5+1}{b^2+c} + \frac{b^5+1}{c^2+a} + \frac{c^5+1}{a^2+b} \geq 3 \dots (A + B)$

122. If $a, b, c, d > 0, a + b + c + d = 3$ then:

$$27 + 3(abc + abd + acd + bcd) \geq a^3 + b^3 + c^3 + d^3 + 54\sqrt{abcd}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Si $a, b, c, d > 0$***, de tal manera que*** $a + b + c + d = 3$***. Probar que***

$$27 + 3(abc + abd + acd + bcd) \geq a^3 + b^3 + c^3 + d^3 + 54\sqrt{abcd}$$

Sabemos la siguiente identidad

$(x + y)^3 = x^3 + y^3 + 3xy(x + y)$***, donde*** $x = a + b, y = c + d$

$$\Leftrightarrow (a + b + c + d)^3 = (a + b)^3 + (c + d)^3 + 3(a + b)(c + d)(a + b + c + d)$$

$$\Leftrightarrow 27 = a^3 + b^3 + c^3 + d^3 + 3ab(a + b) + 3cd(c + d) + 9(a + b)(c + d)$$

$$\Leftrightarrow 27 = a^3 + b^3 + c^3 + d^3 + 3ab(a + b) + 3cd(c + d) + 9(ac + ad + bc + bd)$$

$$\Leftrightarrow 27 + 3(abc + abd + acd + bcd) = \sum a^3 + 3ab(a + b + c + d) + 3cd(a + b + c + d) +$$

$$+9(ac + ad + bc + bd)$$

$$\Leftrightarrow 27 + 3(abc + abd + acd + bcd) = a^3 + b^3 + c^3 + d^3 + 9(ab + cd + ac + ad + bc + bd)$$

Es suficiente demostrar lo siguiente

$$9(ab + ac + ad + bc + bd + cd) \geq 54\sqrt{abcd} \Leftrightarrow$$

$\Leftrightarrow$ ***(Lo cual es válido por MA*** $\geq$ ***MG)***

123. If $a,b,c>0$ then:

$$\sum\left(\frac{a^2(1+b^2)}{1+a}\right)\cdot\left(\frac{b^2(1+a^2)}{1+b}\right)\ge 4(3-2\sqrt{2})abc(a+b+c)$$

Proposed by Daniel Sitaru – Romania

Solution by Redwane El Mellas-Morroco

$$\therefore \sum\frac{a^2(b^2+1)b^2(a^2+1)}{(a+1)(b+1)}=\sum\frac{(ab)^2}{\frac{(a+1)(b+1)}{(a^2+1)(b^2+1)}}\overset{Cauchy}{\ge}\frac{(\sum ab)^2}{\sum\frac{(a+1)(b+1)}{(a^2+1)(b^2+1)}}$$

$$\text{Let } f(x>0)=\frac{x+1}{x^2+1}$$

$$\text{Since } f'(x)=-\frac{x^2+2x-1}{(x^2+1)^2}=-\frac{\left(x-(\sqrt{2}-1)\right)\left(x+(\sqrt{2}+1)\right)}{(x^2+1)^2}$$

$$\Rightarrow 0<f(x>0)\le f\left(\sqrt{2}-1\right)=\frac{\sqrt{2}}{4-2\sqrt{2}}$$

So,

$$\frac{1}{\sum\frac{(a+1)(b+1)}{(a^2+1)(b^2+1)}}=\frac{1}{\sum f(a)f(b)}\ge\frac{1}{3\left(\frac{\sqrt{2}}{4-2\sqrt{2}}\right)^2}=\frac{4\left(3-2\sqrt{2}\right)}{3}$$

$$\text{Also, } [2,2,0]\ge[2,1,1]\Rightarrow\left(\sum ab\right)^2=\sum(ab)^2+2\sum a^2bc\ge 3\sum a^2bc$$

Finally,

$$\sum\frac{a^2(b^2+1)b^2(a^2+1)}{(a+1)(b+1)}\ge 4\left(3-2\sqrt{2}\right)\sum a^2bc=4\left(3-2\sqrt{2}\right)abc\sum a$$

124. Prove that for any positive real numbers a,b,c,x,y,z

$$(a^3+3x^3)(b^3+3y^3)(c^3+3z^3)\ge(ayz+bzx+cxy+xyz)^3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Nirapada Pal-Jhargram-India

$$[(a^3+3x^3)(b^3+3y^3)(c^3+3z^3)]^{\frac{1}{3}}$$

$$=(a^3+3x^3)^{\frac{1}{3}}(b^3+3y^3)^{\frac{1}{3}}(c^3+3z^3)^{\frac{1}{3}}$$

$$=xyz\left[\left(\frac{a}{x}\right)^3+1^3+1^3+1^3\right]^{\frac{1}{3}}\left[1^3+\left(\frac{b}{y}\right)^3+1^3+1^3\right]^{\frac{1}{3}}\left[1^3+1^3+\left(\frac{c}{z}\right)^3+1^3\right]^{\frac{1}{3}}$$

$$\overset{Holder}{\overbrace{\geq}}\ xyz\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}+1\right)=(ayz+bzx+cxy+xyz)$$

$$\therefore (a^3+3x^3)(b^3+3y^3)(c^3+3z^3)\geq(ayz+bzx+cxy+xyz)^3$$

Solution 2 by Fotini Kaldi-Greece

Holder

$$(ayz+bzx+cxy+xyz)^3\leq(a^3+x^3+x^3+x^3)(b^3+y^3+y^3+y^3(c^3+z^3+z^3+z^3)$$

$$\text{“=”}\Leftrightarrow\frac{a}{y}=\frac{x}{b}=\frac{x}{y}\wedge\frac{a}{z}=\frac{x}{z}=\frac{x}{c}\Leftrightarrow b=y\wedge a=x\wedge z=c$$

Solution 3 by Uche Eliezer Okeke-Anambra-Nigeria

$$LHS=\prod_{cyc}(a^3+x^3+x^3+x^3)\ \overset{Holder}{\overbrace{\geq}}\ \left\{\sum_{cyc}[a^3y^3z^3]^{\frac{1}{3}}\right\}^3$$

$$=(ayz+bzx+cxy+xyz)^3$$

125. Let a,b,c be positive real numbers such that

$$a^2+b^2+c^2+abc=4.$$

Prove that

$$a+b+c\geq a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c números R^+ de tal manera que $a^2+b^2+c^2+abc=4.$

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Probar que

$$a + b + c \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}$$

En un triángulo ABC se cumple la siguiente identidad

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$$

Realizando las siguentes cambios de variables

$$a = 2\cos x > 0, b = 2\cos y > 0, c = 2\cos z > 0 \Leftrightarrow$$

$\Leftrightarrow$ **(Válido en triángulo acutángulo)**

Las desigualdad pedida es equivalente

$$\cos x + \cos y + \cos z \geq 2\cos x\sqrt{\cos y\cos z} + 2\cos y\sqrt{\cos z\cos x} + 2\cos z\sqrt{\cos x\cos y}$$

Aplicando la desigualdad Woltehnshome

Siendo m, n, p números $R \wedge x + y + z = \pi$ se verifica lo siguiente

$$m^2 + n^2 + p^2 \geq 2np\cos x + 2mp\cos y + 2mn\cos z,\ \textbf{donde}$$

$$m = \sqrt{\cos x} > 0, n = \sqrt{\cos y} > 0, p = \sqrt{\cos z} > 0$$

$$\Leftrightarrow \cos x + \cos y + \cos z \geq 2\cos x\sqrt{\cos y\cos z} + 2\cos y\sqrt{\cos z\cos x} + 2\cos z\sqrt{\cos x\cos y}$$

(LQQD)

Solution 2 by Imad Zak-Saida-Lebanon

$$\left(\sum a^2\right) + abc = 4 \overset{AM-GM}{\Rightarrow} 4 \geq r + 3r^{\frac{2}{3}}$$

$$\Leftrightarrow \left(\sqrt[3]{r} + 2\right)^2\left(1 - \sqrt[3]{r}\right) \geq 0 \Rightarrow r \leq 1$$

Moreover $\sum a^2 + abc = 4 \Rightarrow r = 4 - \sum a^2$

$$= 4 - (p^2 - 29)$$

$$= 4 - p^2 + 29 \leq 1$$

but $29 \leq \frac{2p^2}{3} \Rightarrow 4 - p^2 + \frac{2p^2}{3} \leq 1 \Rightarrow 3 \leq \frac{p^2}{3} \Rightarrow p \geq 3$

Now $a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}=\sqrt{a}\cdot\sqrt{r}+\sqrt{b}\cdot\sqrt{r}+\sqrt{c}\cdot\sqrt{r}\overset{C-B-S}{\le}$

$$\sqrt{a+b+c}\cdot\sqrt{3r}\overset{r\le 1}{\le}\sqrt{p}\cdot\sqrt{3}=\sqrt{3p}$$

we need to prove $\sqrt{3p}\le p\Leftrightarrow 3\le p$ **true**

<< = >> at $(1;1;1)$

Solution 3 by Marian Dincă – Romania

Let $\frac{a}{2}=\cos\alpha,\frac{b}{2}=\cos\beta,\frac{c}{2}=\cos\gamma,\alpha,\beta,\gamma$ **the angles acute triangle**

Let: $\alpha=\frac{\pi-A}{2},\beta=\frac{\pi-B}{2},\gamma=\frac{\pi-C}{2},A,B,C$ **the angles of triangle**

Result: $\alpha=2\sin\frac{A}{2},b=2\sin\frac{B}{2},c=2\sin\frac{C}{2}$

The inequality is equivalent to:

$$2\sin\frac{A}{2}+2\sin\frac{B}{2}+2\sin\frac{C}{2}\ge\sum_{ciclic}2\sin\frac{A}{2}\sqrt{2\sin\frac{B}{2}\cdot 2\sin\frac{C}{2}}$$

or: $\frac{1}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)\ge\sum_{ciclic}\sin\frac{A}{2}\sqrt{\sin\frac{B}{2}\cdot\sin\frac{C}{2}}$

$\sin\frac{B}{2}\cdot\sin\frac{C}{2}\le\sin^2\left(\frac{B+C}{4}\right)$ **and similarly**

we obtain:

$$\sum_{cyclic}\sin\frac{A}{2}\sqrt{\sin\frac{B}{2}\cdot\sin\frac{C}{2}}\le\sum_{cyclic}\sin\frac{A}{2}\sin\left(\frac{B+C}{4}\right)$$

and use Cebyshev inequality, result:

$$\sum_{cyclic}\sin\frac{A}{2}\sin\left(\frac{B+C}{4}\right)\le\frac{1}{3}\left(\sum_{ciclic}\sin\frac{A}{2}\right)\left(\sum_{ciclic}\sin\left(\frac{B+C}{4}\right)\right)$$

and: $\frac{1}{3}\left(\sum_{ciclic}\sin\left(\frac{B+C}{4}\right)\right) \le \sin\left(\frac{\sum_{ciclic}\left(\frac{B+C}{4}\right)}{3}\right) = \sin\left(\frac{A+B+C}{6}\right) = \frac{1}{2}$

Jensen inequality

126. If $a, b, c > 0, a + b + c = 1$ then:

$$a^{2a} + b^{2b} + c^{2c} + \frac{4}{3}\left(a^b b^c c^a + a^c b^a c^b\right) \le 1$$

Proposed by Hung Nguyen Viet – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo $a, b, c > 0$ ***de tal manera que*** $a + b + c = 1$. ***Probar que***

$$a^{2a}b^{2b}c^{2c} + \frac{4}{3}\left(a^b b^c c^a + a^c b^a c^b\right) \le 1$$

Siendo $\rightarrow a_1, a_2, a_3 \dots a_n > 0, x_1, x_2, x_3 \dots x_n > 0 \wedge a_1 + a_2 + a_3 + \dots + a_n = 1$

Se cumple la siguiente desigualdad

$$x_1^{a_1} \cdot x_2^{a_2} \cdot x_3^{a_3} \dots x_n^{a_n} \le a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n$$

Para $n = 3$

1) $a^a b^b c^c \le a^2 + b^2 + c^2 \Leftrightarrow$

$$\Leftrightarrow a^{2a}b^{2b}c^{2c} \le (a^2 + b^2 + c^2)^2 = \left((a + b + c)^2 - 2(ab + bc + ca)\right)^2$$

$$\Leftrightarrow a^{2a}b^{2b}c^{2c} \le \left(1 - 2(ab + bc + ca)\right)^2 =$$

$$= 1 - 4(ab + bc + ca) + 4(ab + bc + ca)^2 \ \textbf{(A)}$$

2) $a^b b^c c^a \le ab + bc + ca$ ***(B),***

3) $a^c b^a c^b \le ca + ab + bc$ ***(C)***

De (A), (B), (C)

$$a^{2a}b^{2b}c^{2c} + \frac{4}{3}\left(a^b b^c c^a + a^c b^a c^b\right) \le 1 - 4(ab + bc + ca) + 4(ab + bc + ca)^2 + \frac{8}{3}(ab + bc + ca)$$

$$a^{2a}b^{2b}c^{2c}+\frac{4}{3}\left(a^b b^c c^a+a^c b^a c^b\right)\le 1+4(ab+bc+ca)\left((ab+bc+ca)-\frac{1}{3}\right)\le 1$$

Lo cual es cierto ya que $\rightarrow ab+bc+ca\le\frac{(a+b+c)^2}{3}=\frac{1}{3}\wedge ab+bc+ca>0$

127. Prove that if $x,y,z\in(1,\infty)$ then:

$$\sum\left(\frac{\ln x}{\ln y\ln z}+\frac{\ln y}{\ln x\ln z}\right)\ge\frac{18}{\ln(xyz)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar para todo $x,y,z\in<1,\infty>$ ***lo siguiente***

$$\sum\left(\frac{\ln x}{\ln y\ln z}+\frac{\ln y}{\ln x\ln z}\right)\ge\frac{18}{\ln(xyz)}$$

De las condiciones se puede deducir que

$$a=\ln x>0,b=\ln y>0,c=\ln z>0\Leftrightarrow a+b+c=\ln(xyz)$$

La desigualdad propuesta es equivalente

$$\left(\frac{a}{bc}+\frac{b}{ca}\right)+\left(\frac{b}{ca}+\frac{c}{ab}\right)+\left(\frac{c}{ab}+\frac{a}{bc}\right)\ge\frac{18}{a+b+c}$$

$$\Leftrightarrow\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)(a+b+c)\ge 9\quad\textbf{(A)}$$

Aplicando la desigualdad de Cauchy y MA $\ge$ MG en (A)

$$\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)(a+b+c)\ge\frac{(a+b+c)^2}{3abc}\cdot(a+b+c)=\frac{(a+b+c)^3}{3abc}\ge 9$$

(LQQD)

Solution 2 by Nirapada Pal-Jhargram-India

Let $\ln x=a,\ln y=b,\ln z=c$

Since $x,y,z>1$ ***so*** $a,b,c>0$

Now the inequality reduces to

$$\sum\left(\frac{a}{bc}+\frac{b}{ca}\right)\geq\frac{18}{a+b+c}$$

$$LHS=\sum\left(\frac{a}{bc}+\frac{b}{ca}\right)$$

$$=\sum\frac{a^2+b^2}{abc}$$

$$\geq\sum\frac{2ab}{abc}.\ Since(A^2+B^2\geq 2AB)=2\sum\frac{1}{a}$$

$$\overset{AM-HM}{\geq}2\times\frac{9}{a+b+c}=\frac{18}{a+b+c}$$

$\therefore$ ***we get***

$$\sum\left(\frac{\ln x}{\ln y\ln z}+\frac{\ln y}{\ln z\ln x}\right)\geq\frac{18}{\ln x+\ln y+\ln z}=\frac{18}{\ln(xyz)}$$

Solution 3 by Nirapada Pal-Jhargram-India

Let $\ln x=a,\ln y=b,\ln z=c$

Since $x,y,z>1$ **so** $a,b,c>0$

Now the inequality reduces to

$$\sum\left(\frac{a}{bc}+\frac{b}{ca}\right)\geq\frac{18}{a+b+c}$$

$$LHS=\sum\left(\frac{a}{bc}+\frac{b}{ca}\right)=2\sum\frac{a}{bc}=\frac{2}{abc}\sum a^2$$

$$\geq\frac{2}{abc}\sum ab\ Since\sum A^2\geq\sum AB=2\sum\frac{1}{a}$$

$$\overset{AM-HM}{\geq}2\times\frac{9}{a+b+c}=\frac{18}{a+b+c}$$

$\therefore$ ***we get***

$$\sum\left(\frac{\ln x}{\ln y\ln z}+\frac{\ln y}{\ln z\ln x}\right)\geq\frac{18}{\ln x+\ln y+\ln z}=\frac{18}{\ln(xyz)}$$

Solution 4 by Nikolaos Skoutaris-Greece

If $x, y, z \in (1, +\infty)$

(1) Prove $\sum\left(\frac{\ln x}{\ln y \ln z}+\frac{\ln y}{\ln x \cdot \ln z}\right) \geq \frac{18}{\ln(xyz)}$

Let $a=\ln x, b=\ln y, c=\ln z$

Then

(1) becomes:

$$\left(\frac{a}{bc}+\frac{b}{ca}\right)+\left(\frac{b}{ca}+\frac{c}{ab}\right)+\left(\frac{c}{ab}+\frac{a}{bc}\right) \geq \frac{18}{a+b+c}$$

$$\frac{a}{bc}+\frac{b}{ca}=\frac{a^2}{abc}+\frac{b^2}{abc} \geq \frac{(a+b)^2}{2abc} \geq \frac{4ab}{2abc}=\frac{2}{c}$$

$$\frac{b}{ca}+\frac{c}{ab}=\frac{b^2}{abc}+\frac{c^2}{abc} \geq \frac{(b+c)^2}{2abc} \geq \frac{4abc}{2abc}=\frac{2}{a}$$

$$\frac{c}{ab}+\frac{a}{bc}=\frac{c^2}{abc}+\frac{a^2}{abc} \geq \frac{(c+a)^2}{2abc} \geq \frac{4ac}{2abc}=\frac{2}{b}$$

$(+)$

$$LHS \geq 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=2 \cdot 3 \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$$

$$\Rightarrow LHS \geq 6 \cdot \frac{3}{\frac{1}{\frac{1}{a}}+\frac{1}{\frac{1}{b}}+\frac{1}{\frac{1}{c}}} \Rightarrow LHS \geq \frac{18}{a+b+c}$$

Solution 5 by Nguyen Thanh Nho-Tra Vinh-Vietnam

$x, y, z \in (1; \infty) \Rightarrow \ln x, \ln y, \ln z > 0$

$$\sum\left(\frac{\ln x}{\ln y \ln z}+\frac{\ln y}{\ln x \ln z}\right) \overset{AM-GM}{\geq} 2\left(\frac{1}{\ln z}+\frac{1}{\ln x}+\frac{1}{\ln y}\right)$$

$$\overset{C-S}{\geq} 2\cdot\frac{(1+1+1)^2}{\ln x+\ln y+\ln z}=\frac{18}{\ln(xyz)}$$

$$\text{“ = “}\Leftrightarrow \ln x=\ln y=\ln z\Leftrightarrow x=y=z$$

Solution 6 by Soumava Chakraborty-Kolkata-India

Let $a=\ln x,\ b=\ln y,\ c=\ln z$

$\therefore$ ***given inequality becomes*** $\sum\frac{a}{bc}+\sum\frac{b}{ca}\geq\frac{18}{a+b+c}$

$$\Leftrightarrow 2\sum\frac{a}{bc}\geq\frac{18}{\sum a}\Leftrightarrow\frac{\sum a^2}{abc}\geq\frac{9}{\sum a}$$

$$\Leftrightarrow \sum a^2\cdot\sum a\geq 9abc\quad \textbf{(1)}$$

But $\sum a^2 \underset{(i)}{\overset{A-G}{\geq}} 3\sqrt[3]{a^2b^2c^2}$ **, and** $\sum a \underset{(ii)}{\overset{A-G}{\geq}} 3\sqrt[3]{abc}$

$(i)\times(ii)\Rightarrow\sum a^2\cdot\sum a\geq 9abc\Rightarrow$ ***(1) is true (Proved)***

Solution 7 by Eliezer Okeke-Anambra-Nigerie

Let $a=\ln x;\ b=\ln y;\ c=\ln z\Rightarrow\sum a=\ln(xyz)$

$$LHS=\sum\left\{\frac{a}{bc}+\frac{b}{ac}\right\}$$

$$=\frac{2}{abc}\sum\frac{a^2}{1}\overset{BEG}{\geq}\frac{2}{abc}\cdot\frac{(\sum a)^2}{3}$$

$$\overset{\substack{REV\\(AM-GM)}}{\geq}\frac{2}{3}\cdot\frac{(\sum a)^2}{(\sum a)^3}\cdot\frac{27}{1}=\frac{18}{\sum a}=\frac{18}{\ln(xyz)}$$

Solution 8 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\ln x=a$$

$$\ln y=b$$

$$\ln z=c$$

$$\sum\left(\frac{a}{bc}+\frac{b}{ac}\right)\geq\frac{18}{a+b+c}$$

$$\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\geq\frac{9}{a+b+c}$$

$$\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\overset{CHEBYSHEV}{\geq}\frac{1}{3}(a+b+c)\left(\frac{1}{bc}+\frac{1}{ac}+\frac{1}{ab}\right)\overset{C-B-C}{\geq}\frac{3(a+b+c)}{ab+bc+ac}$$

$$\frac{3(a+b+c)}{ab+bc+ac}\geq\frac{9}{a+b+c}$$

$$(a+b+c)^2\overset{?}{\geq}3(ab+bc+ac)\rightarrow \textbf{true}\overset{because}{\rightarrow}a^2+b^2+c^2\geq ab+bc+ac$$

Solution 9 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\left.\begin{matrix}\ln x=t_1\\ \ln y=t_2\\ \ln z=t_3\end{matrix}\right\}\sum\left(\frac{t_1}{t_2\cdot t_3}+\frac{t_2}{t_1\cdot t_3}\right)=\sum\frac{1}{t_3}\cdot\left(\frac{t_1}{t_2}+\frac{t_2}{t_1}\right)\geq$$

$$\overset{AM\geq GM}{\geq}2\cdot\sum\frac{1}{t_3}\overset{Bergstrom}{\geq}2\cdot\frac{9}{\sum t_1}=\frac{18}{\ln(xyz)}$$

Solution 10 by Geanina Tudose-Romania

Denote

$$\ln x=a$$

$$\ln y=b$$

$$\ln z=c$$

$$a,b,c>0$$

We have $\sum\left(\frac{a}{bc}+\frac{b}{ac}\right)\geq\frac{18}{a+b+c}\Leftrightarrow\sum\left(\frac{a^2+b^2}{abc}\right)\geq\frac{18}{a+b+c}$

$$\Leftrightarrow\frac{2(a^2+b^2+c^2)}{abc}\geq\frac{18}{a+b+c}\Leftrightarrow(a^2+b^2+c^2)(a+b+c)\geq 9abc$$

But $a^2+b^2+c^2\overset{AM-GM}{\geq}3\sqrt[3]{a^2b^2c^2}$

$$a+b+c\geq 3\sqrt[3]{abc}$$

The conclusion follows.

128. **Prove that for all positive real numbers a, b, c the inequality holds**

$$\frac{b^3+c^3}{a}+\frac{c^3+a^3}{b}+\frac{a^3+b^3}{c}\geq 2(a^2+b^2+c^2)+3(a-b)^2+3(b-c)^2+3(c-a)^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar para todos las números R^+ a, b, c la siguiente desigualdad

$$\frac{b^3+c^3}{a}+\frac{c^3+a^3}{b}+\frac{a^3+b^3}{c}\geq 2(a^2+b^2+c^2)+3(a-b)^2+3(b-c)^2+3(c-a)^2$$

La desigualdad es equivalente

$$\left(\frac{a^3}{b}+\frac{b^3}{a}+6ab\right)+\left(\frac{b^3}{c}+\frac{c^3}{b}+6bc\right)+\left(\frac{c^3}{a}+\frac{a^3}{c}+6ca\right)\geq 8(a^2+b^2+c^2)$$

Como $a, b, c > 0$

Es suficiente demostrar lo siguiente

$$\frac{a^3}{b}+\frac{b^3}{a}+6ab\geq 4(a^2+b^2)\Leftrightarrow a^4+b^4+6a^2b^2\geq 4(a^2+b^2)ab$$

$$\Leftrightarrow a^4-4a^3b+6a^2b^2-4ab^3+b^4=(a-b)^4\geq 0$$

Por lo tanto

$$\sum\left(\frac{a^3}{b}+\frac{b^3}{a}+6ab\right)\geq 4\sum(a^2+b^2)=4\cdot 2(a^2+b^2+c^2)=8(a^2+b^2+c^2)$$

(LQQD)

129. **Prove that for all positive real numbers a, b, c the inequality holds**

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\leq\frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los números R^+ a, b, c la siguiente desigualdad

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca}$$

Como** $a,b,c>0\to$ **multiplicamos** $(ab+bc+ca)$**, sin alterar el sentido

$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)(ab+bc+ca)\le\frac{ab+bc+ca}{2}+a^2+b^2+c^2$$

$$\left(\frac{a}{b+c}\right)(a(b+c)+bc)+\left(\frac{b}{c+a}\right)(b(c+a)+ca)+\left(\frac{c}{a+b}\right)(c(a+b)+ab)\le$$

$$\le\frac{ab+bc+ca}{2}+a^2+b^2+c^2$$

$$a^2+b^2+c^2+abc\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$$

$$\le a^2+b^2+c^2+\frac{ab+bc+ca}{2}$$

$$\Leftrightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}$$

Aplicando la desigualdad de Cauchy

$\frac{1}{a}+\frac{1}{b}\ge\frac{4}{a+b}$ ***(A),***

$\frac{1}{b}+\frac{1}{c}\ge\frac{4}{b+c}$ ***(B),***

$\frac{1}{c}+\frac{1}{a}\ge\frac{4}{c+a}$ ***(C)***

Sumando (A) + (B) + (C)

$\to\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}$ ***(LQQD)***

Solution 2 by Nguyen Ngoc Tu-Ha Giang-Vietnam

We have

$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)(ab+bc+ca)=a^2+b^2+c^2+abc\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

$$\le a^2+b^2+c^2+\frac{abc}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\leq a^2+b^2+c^2+\frac{1}{2}(ab+bc+ca) \Rightarrow \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \leq \frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca}$$

130. Prove that for all positive real numbers a, b, c the inequality holds

$$\frac{a(b+c)^2}{b^2+bc+c^2}+\frac{b(c+a)^2}{c^2+ca+a^2}+\frac{c(a+b)^2}{a^2+ab+b^2} \geq \frac{4(ab+bc+ca)}{a+b+c}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los números $\mathbb{R}^+$ a, b, c la siguiente desigualdad

$$\frac{a(b+c)^2}{b^2+bc+c^2}+\frac{b(c+a)^2}{c^2+ca+a^2}+\frac{c(a+b)^2}{a^2+ab+b^2} \geq \frac{4(ab+bc+ca)}{a+b+c}$$

Como $a, b, c > 0$

Por la desigualdad de Cauchy

$$\frac{(ab+ac)^2}{ab^2+abc+ac^2}+\frac{(bc+ba)^2}{bc^2+bca+ba^2}+\frac{(ca+cb)^2}{ca^2+cab+cb^2} \geq$$

$$\geq \frac{4(ab+bc+ca)^2}{(a+b+c)(ab+bc+ca)}=\frac{4(ab+bc+ca)}{a+b+c}$$

(LQQD)

131. If $a, b, c > 0$ then:

$$6+\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab} \geq 3\sqrt[3]{6(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-27}$$

Proposed by Adil Abdulayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c números R^+, probar la siguiente desigualdad

$$6+\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\ge 3\sqrt[3]{6(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-27}$$

Por la desigualdad de Schur

$$a^3+b^3+c^3+3abc\ge ab(a+b)+bc(b+c)+ca(c+a)$$

$$\Leftrightarrow \frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}+6\ge\left(\frac{a+b}{c}+1\right)+\left(\frac{b+c}{a}+1\right)+\left(\frac{c+a}{b}+1\right)=$$

$$=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Es suficiente probar

$x\ge 3\sqrt[3]{6x-27}$, ***donde*** $x=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 9$

(Válido por MA $\ge$ MG)

$$x^3\ge 27(6x-27)\Leftrightarrow x^3-162x+729=(x-9)(x^2+9x-81)\ge 0$$

Lo cual es cierto ya que $x\ge 9\wedge x^2+9x-81\ge 81>0$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$6+\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\underbrace{\ge}^{(1)} 3\sqrt[3]{6(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-27}$$

(1) $\Leftrightarrow \frac{\sum a^3+6abc}{abc}\ge 3\sqrt[3]{\frac{6(\sum a)(\sum ab)-27abc}{abc}}$

$$\Leftrightarrow \frac{\sum a^3+6abc}{abc}\underbrace{\ge}^{(2)} 3\sqrt[3]{\frac{6(\sum a^2b+\sum ab^2)-9abc}{abc}}$$

LHS of (2) $\underbrace{\overbrace{\ge}^{Schur}}_{(3)} \frac{\sum a^2b+\sum ab^2+3abc}{abc}$

(2), (3) $\Rightarrow$ ***it suffices to prove:***

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$\frac{p+3q}{q} \geq 3\sqrt[3]{\frac{6p-9q}{q}}$ ***(where*** $p = \sum a^2b + \sum ab^2$ ***and*** $q = abc$***)***

$$\Leftrightarrow \frac{(p+3q)^3}{q^3} \geq \frac{27(6p-9q)}{q}$$

$$\Leftrightarrow p^3 + 9p^2q - 135pq^2 + 270q^3 \geq 0$$

$$\Leftrightarrow t^3 + 9t^2 - 135t + 270 \geq 0 \text{ (where } t = \tfrac{p}{q}\text{)}$$

$$\Leftrightarrow (t-6)\{(t-6)(t+21)+81\} \geq 0$$

$$\rightarrow \textbf{true} \because t = \frac{p}{q} = \frac{\sum a^2b + \sum ab^2}{abc} \geq 6 \text{ by } A-G$$

(Proved)

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Schur's Inequality

$$\sum_{cyc} a^3 + 3abc \geq \sum_{cyc} ab(a+b)$$

$$\textbf{Let } t = \left(\textstyle\sum_{cyc} a\right)\left(\textstyle\sum_{cyc} \frac{1}{a}\right)$$

$$\sum_{cyc} a^3 + 6abc \geq (a+b+c)(ab+bc+ca)$$

$$\therefore 6 + \sum_{cyc} \frac{a^2}{bc} \geq \left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right)$$

$$\textbf{we will show } t \geq 3\sqrt[3]{6t-27}$$

$$\Leftrightarrow t^3 - 162t + 729 \geq 0 \Leftrightarrow t^2(t-9) + 9t(t-9) - 81(t-9) \geq 0$$

$$\Leftrightarrow (t-9)\{(t-9)(t+9)+9t\} \geq 0, \textbf{ which is true since } t \geq 9$$

$$\therefore 6+\sum_{cyc}\frac{a^2}{bc}\ge 3\sqrt[3]{6\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{a}\right)-27}$$

(Proved)

132. Let $a,b>0$. Prove:

$$\sqrt{ab}-\frac{2(a^2-ab+b^2)}{ab(a+b)}\le\frac{\sqrt{3}}{9}(a^2+b^2)$$

Proposed by Le Minh Cuong-Ho Chi Minh-Vietnam

Solution by Do Quoc Chinh-Ho Chi Minh-Vietnam

Using the AM-GM inequality, we have:

$$\frac{a^2-ab+b^2}{ab(a+b)}=\frac{1}{8}\cdot\frac{2(a^2+b^2)+(6a^2-8ab+6b^2)}{ab(a+b)}$$

$$\ge\frac{1}{8}\cdot\frac{2(a^2+b^2)+(a^2+b^2+10ab-8ab)}{ab(a+b)}$$

$$=\frac{1}{8}\cdot\frac{2(a^2+b^2)+(a+b)^2}{ab(a+b)}\ge\frac{\sqrt{2(a^2+b^2)(a+b)^2}}{4ab(a+b)}=\frac{\sqrt{a^2+b^2}}{2\sqrt{2}ab}$$

Therefore, we have:

$$\frac{a^2+b^2}{3\sqrt{3}}+\frac{2(a^2-ab+b^2)}{ab(a+b)}\ge\frac{a^2+b^2}{3\sqrt{3}}+\frac{\sqrt{a^2+b^2}}{ab\sqrt{2}}\ge\frac{2ab}{3\sqrt{3}}+\frac{1}{\sqrt{ab}}=$$

$=\frac{ab}{3\sqrt{3}}+\frac{ab}{3\sqrt{3}}+\frac{1}{\sqrt{ab}}\ge\sqrt{ab}$. ***The equality holds for*** $a=b=\sqrt{3}$.

133. From the book: "Math Accent"

If $a,b,c\in(0,\infty)$, $abc=1$ then:

$$\sum\left(a+\sqrt[3]{a}+\sqrt[3]{a^2}\right)\ge 9\sum\frac{1}{1+\sqrt[3]{b^2}+\sqrt[3]{c}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nguyen Tien Lam-Vietnam

Let $\sqrt[3]{a} = x, \sqrt[3]{b} = y, \sqrt[3]{c} = z$ **then** $xyz = 1$ **and** $x, y, z > 0$

By Cauchy – Schwarz, we have

$$(1 + y^2 + z)(x^2 + 1 + z) \ge (x + y + z)^2 \ge 9$$

which implies $\frac{9}{1+z^2+x} \le y^2 + 1 + x, \frac{9}{1+x^2+y} \le z^2 + 1 + y$

Adding above inequalities, we obtain

$$9\sum \frac{9}{1 + y^2 + z} \le 3 + x + y + z + x^2 + y^2 + z^2$$

$$\le x^3 + y^3 + z^3 + x + y + z + x^2 + y^2 + z^2$$

Thus, we get the desired inequality.

Solution 2 by Rozeta Atanasova-Skopje

$$LHS = \left(a + \sqrt[3]{b} + \sqrt[3]{c^2}\right) + \left(b + \sqrt[3]{c} + \sqrt[3]{a^2}\right) + \left(c + \sqrt[3]{a} + \sqrt[3]{b^2}\right) \ge (AM - GM)$$

$$3\left(\sqrt[9]{a^3bc^2} + \sqrt[9]{b^3ca^2} + \sqrt[9]{c^3ab^2}\right) =$$

$$3\left(\sqrt[9]{\frac{a^3bc^2}{(abc)^3}} + \sqrt[9]{\frac{b^3ca^2}{(abc)^3}} + \sqrt[9]{\frac{c^3ab^2}{(abc)^3}}\right) =$$

$$3\left(\sqrt[3]{\frac{1}{b^{\frac{2}{3}}c^{\frac{1}{3}}}} + \sqrt[3]{\frac{1}{c^{\frac{2}{3}}c^{\frac{1}{3}}}} + \sqrt[3]{\frac{1}{a^{\frac{2}{3}}b^{\frac{1}{3}}}}\right) \ge (GM - HM)$$

$$9\left(\frac{1}{1 + \sqrt[3]{c} + \sqrt[3]{b^2}} + \frac{1}{1 + \sqrt[3]{a} + \sqrt[3]{c^2}} + \frac{1}{1 + \sqrt[3]{b} + \sqrt[3]{a^2}}\right) = RHS$$

134. Let a, b, c be non-negative real numbers, no two of which are zero.

Prove that

$$\sum_{cyc}\sqrt{\frac{ab+bc+ca}{b^2+bc+c^2}} \geq 2+\frac{2}{a+b+c}\left(\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\right)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Como $a, b, c \geq 0 \Leftrightarrow ab+bc+ca>0$, ya que 2 de ellos son diferentes de zero.

La desigualdad es equivalente

$$\sum\frac{ab+bc+ca}{\sqrt{(b^2+c^2+bc)(ab+bc+ca)}} \geq 2+\frac{2}{a+b+c}\left(\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\right)$$

Luego por MA $\geq$ MG

$$\sum\frac{ab+bc+ca}{\sqrt{(b^2+c^2+bc)(ab+bc+ca)}} \geq \sum\frac{2(ab+bc+ca)}{(b+c)^2+a(b+c)} =$$

$$=\sum\frac{2(ab+bc+ca)}{(a+b+c)(b+c)}$$

$$\Leftrightarrow \sum\frac{2(ab+bc+ca)}{(a+b+c)(b+c)} =$$

$$=\frac{2}{a+b+c}\left[\frac{a(b+c)+bc}{b+c}+\frac{b(c+a)+ca}{c+a}+\frac{c(a+b)+ab}{a+b}\right]$$

$$\Leftrightarrow \sum\frac{2(ab+bc+ca)}{(a+b+c)(b+c)}=\frac{2}{a+b+c}\left[(a+b+c)+\left(\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\right)\right]=$$

$$=2+\frac{2}{a+b+c}\left(\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\right)$$

La igualdad se alcanza cuando $a=b=c$.

135. **Let a, b, c be positive real numbers such that**

$$\frac{1}{\sqrt{1+a^3}}+\frac{1}{\sqrt{1+b^3}}+\frac{1}{\sqrt{1+c^3}}\le 1$$

Prove that

$$a^2+b^2+c^2\ge 12$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Abdul Aziz-Semarang-Indonesia

A fact

$$(a^2-2a)^2\ge 0 \Leftrightarrow a^4+4a^2\ge 4a^3$$

$$\Leftrightarrow a^4+4a^2+4\ge 4(a^3+1)\Leftrightarrow (a^2+2)\ge 2\sqrt{a^3+1}$$

Analogoue,

$$(b^2+2)\ge 2\sqrt{b^3+1}$$

$$(c^2+2)\ge 2\sqrt{c^3+1}$$

------------------------------ +

$$a^2+b^2+c^2+6\ge 2\left(\sqrt{a^3+1}+\sqrt{b^3+1}+\sqrt{c^3+1}\right)\quad \textit{(1)}$$

$$1\ge\frac{1}{\sqrt{a^3+1}}+\frac{1}{\sqrt{b^3+1}}+\frac{1}{\sqrt{c^3+1}}\ge\frac{(1+1+1)^3}{\sqrt{a^3+1}+\sqrt{b^3+1}+\sqrt{c^3+1}}$$

$$\sqrt{a^3+1}+\sqrt{b^3+1}+\sqrt{c^3+1}\ge 9$$

By (1) and (2),

$$a^2+b^2+c^2\ge 2\left(\sqrt{a^3+1}+\sqrt{b^3+1}+\sqrt{c^3+1}\right)-6\ge 18-6=12$$

Equality holds when $a=b=c=2$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c números R^+ de tal manera que

$$\frac{1}{\sqrt{1+a^3}}+\frac{1}{\sqrt{1+b^3}}+\frac{1}{\sqrt{1+c^3}}\le 1$$

Probar que $\rightarrow a^2+b^2+c^2\ge 12$

Por la desigualdad de Cauchy

$$9\le\sqrt{1+a^3}+\sqrt{1+b^3}+\sqrt{1+c^3}=$$

$$=\sqrt{(1+a)(a^2-a+1)}+\sqrt{(1+b)(b^2-b+1)}+\sqrt{(1+c)(c^2-c+1)}$$

Por MA $\ge$ ***MG***

$$9\le\sqrt{(1+a)(a^2-a+1)}+\sqrt{(1+b)(b^2-b+1)}+\sqrt{(1+c)(c^2-c+1)}\le$$

$$\le\frac{a^2+2}{2}+\frac{b^2+2}{2}+\frac{c^2+2}{2}$$

$$18\le a^2+b^2+c^2+6\Leftrightarrow a^2+b^2+c^2\ge 12$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$1\ge\sum_{sym}\frac{1}{\sqrt{1+a^3}}=\sum_{sym}\frac{1}{\sqrt{(1+a)\cdot(1-a+a^2)}}\overset{AM-GM}{\ge}$$

$$\ge\sum\frac{2}{2+a^2}=2\cdot\sum\frac{1}{2+a^2}\ge 2\cdot\frac{(1+1+1)^2}{6+\sum a^2}=\frac{2\cdot 9}{6+\sum a^2}$$

$$1\ge\frac{18}{6+\sum a^2}\Rightarrow\sum a^2\ge 12$$

136. If $a,b,c>0$ then:

$$9(a^2+b^2+c^2)\ge 3(a+b+c)^2+\sum|a-b|^2$$

Proposed by D.M. Bătinețu – Giurgiu and Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $a,b,c>0$. ***Probar que***

$$2(a^2+b^2+c^2)\geq 2(ab+bc+ca)\geq \frac{1}{3}\left(\sum|a-b|\right)^2$$

$$\Leftrightarrow 3((a-b)^2+(b-c)^2+(c-a)^2)\geq\left(\sum|a-b|\right)^2$$

$$\Leftrightarrow 3((|a-b|)^2+(|b-c|)^2+(|c-a|)^2)^2\geq(|a-b|+|b-c|+|c-a|)^2$$

(Lo cual es válido por Cauchy)

Solution 2 by Mihalcea Andrei Stefan-Romania

$$\frac{1}{3}\left(\sum|a-b|\right)^2\overset{C-B-S}{\leq}\sum|a-b|^2=2\sum a^2-2\sum ab\,|+2\sum ab$$

$$\Rightarrow 2\sum ab+\frac{1}{3}\left(\sum|a-b|\right)^2\leq 2\sum a^2$$

Solution 3 by Serban George Florin-Romania

$$9\cdot(a^2+b^2+c^2)\geq 3\cdot(a+b+c)^2+\sum|a-b|^2$$

$$9a^2+9b^2+9c^2\geq 3a^2+3b^2+3c^2+6ab+6bc+6ac+$$

$$+|a-b|^2+|b-c|^2+|a-c|^2$$

$$6a^2+6b^2+6c^2\geq 6ab+6bc+6ac+a^2-2ab+b^2+$$

$$+b^2-2bc+c^2+a^2-2ac+c^2$$

$$\Rightarrow 4a^2+4b^2+4c^2\geq 4ab+4bc+4ac|:2$$

$$\Rightarrow 2a^2+2b^2+2c^2\geq 2ab+2bc+2ac\geq 0$$

$$\Rightarrow a^2+a^2+b^2+b^2+c^2+c^2-2ab-2bc-2ac\geq 0$$

$$\Rightarrow (a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ac+a^2)\geq 0$$

$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2\geq 0\ \ \textit{(A)}$$

Solution 4 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$a\geq b\geq c$$

$$9a^2+9b^2+9c^2\geq 3a^2+3b^2+3c^2+6ab+6ac+6bc+2\sum a^2-2\sum ab$$

$$4a^2+4b^2+4c^2 \geq 4ab+4bc+4ac$$

$$a^2+b^2+c^2 \geq ab+bc+ac$$

(Proved)

Solution 5 by Soumava Chakraborty-Kolkata-India

$$2(a^2+b^2+c^2) \overset{(1)}{\gtreqless} 2(ab+bc+ca)+\frac{1}{3}\left(\sum|a-b|\right)^2 \quad \forall a,b,c \in \mathbb{R}$$

$$\textit{(1)} \Leftrightarrow 3\left(2\sum a^2-2\sum ab\right) \geq |a-b|^2+|b-c|^2+|c-a|^2$$

$$+2(|a-b|\,|b-c|+|b-c||c-a|+|c-a||a-b|)$$

$$\Leftrightarrow 3\{(a-b)^2+(b-c)^2+(c-a)^2\}-(|a-b|^2+|b-c|^2+|c-a|^2)$$

$$\geq 2(|a-b||b-c|+|b-c||c-a|+|c-a||a-b|)$$

$$\Leftrightarrow |a-b|^2+|b-c|^2+|c-a|^2 \geq |a-b||b-c|+|b-c||c-a|+|c-a||a-b|$$

$$\rightarrow \textit{true} \because x^2+y^2+z^2 \geq xy+yz+zx,$$

$$\textit{where } x=|a-b|, y=|b-c|, z=|c-a|$$

Solution 6 by Soumava Pal-Kolkata-India

$$2(a-b)^2+2(b-c)^2+2(c-a)^2 \geq 0$$

$$\Rightarrow 3\sum(a-b)^2 \geq \sum(a-b)^2 = \sum|a-b|^2$$

$$\Rightarrow 6\sum a^2-6\sum ab \geq \sum|a-b|^2 \Rightarrow 6\sum a^2+3\sum a^2 \geq$$

$$\geq 3\sum a^2+6\sum ab+\sum|a-b|^2 = 3\left(\sum a\right)^2+\sum|a-b|^2$$

137. Let x, y, z be positive real numbers. Prove that:

$$\frac{x^2}{\sqrt{2(y^4+z^4)}+yz}+\frac{y^2}{\sqrt{2(z^4+x^4)}+zx}+\frac{z^2}{\sqrt{2(x^4+y^4)}+xy} \geq 1 \qquad \textbf{(1)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Hoang Le Nhat Tung – Hanoi – Vietnam

***** Lemma: Let** x, y, z **be positive real numbers. We have:***

$$x^4 + y^4 + z^4 + xyz(x + y + z) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \quad \textbf{(2)}$$

- Since AM – GM for 2 positive real number:

$$xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \geq xy \cdot 2xy + yz \cdot 2yz + zx \cdot 2zx =$$

$$= 2(x^2y^2 + y^2z^2 + z^2x^2)$$

$$\Leftrightarrow xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \geq 2(x^2y^2 + y^2z^2 + z^2x^2) \quad \textbf{(3)}$$

- Since (2), (3): $\Rightarrow x^4 + y^4 + z^4 + xyz(x + y + z) \geq 2(x^2y^2 + y^2z^2 + z^2x^2)$

$$\Leftrightarrow x^4 + y^4 + z^4 + 2(x^2y^2 + y^2z^2 + z^2x^2) \geq 4(x^2y^2 + y^2z^2 + z^2x^2) - xyz(x + y + z)$$

$$\Leftrightarrow (x^2 + y^2 + z^2)^2 \geq 4(x^2y^2 + y^2z^2 + z^2x^2) - xyz(x + y + z)$$

$$\Leftrightarrow \frac{(x^2+y^2+z^2)^2}{4(x^2y^2+y^2z^2+z^2x^2) - xyz(x+y+z)} \quad \textbf{(4)}$$

***** Since inequality Bunhiacopxki:***

$$\left(1 \cdot \sqrt{2(x^4 + y^4)} + 1 \cdot xy\right)^2 \leq (1^2 + 1^2) \cdot \left[\left(\sqrt{2(x^4 + y^4)}\right)^2 + (2xy)^2\right] =$$

$$= 2[2(x^4 + y^4) + 4x^2y^2]$$

$$\Leftrightarrow \left(\sqrt{2(x^4 + y^4)} + 2xy\right)^2 \leq 4(x^4 + 4x^2y^2 + y^4) \Leftrightarrow$$

$$\Leftrightarrow \left(\sqrt{2(x^4 + y^4)} + 2xy\right)^2 \leq 4(x^2 + y^2)^2$$

$$\Leftrightarrow \sqrt{2(x^4 + y^4)} + 2xy \leq \sqrt{4(x^2 + y^2)^2} = 2(x^2 + y^2) \Leftrightarrow$$

$$\Leftrightarrow \sqrt{2(x^4 + y^4)} + xy \leq 2x^2 - xy + 2y^2$$

$$\Leftrightarrow \frac{z^2}{\sqrt{2(x^4+y^4)}+2xy} \geq \frac{z^2}{2x^2-xy+2y^2} \quad \textbf{(5)}$$

- Similar: $\frac{y^2}{\sqrt{2(z^2+x^4)}+zx} \geq \frac{y^2}{2z^2-zx+2x^2}; \frac{x^2}{\sqrt{2(y^4+z^4)}+yz} \geq \frac{x^2}{2y^2-yz+2z^2}$ ***(6)***

- XXXXX (5), (6):

$$\Rightarrow \frac{x^2}{\sqrt{2(y^4 + z^4)} + yz} + \frac{y^2}{\sqrt{2(z^4 + x^4)} + zx} + \frac{z^2}{\sqrt{2(x^4 + y^4)} + xy} \geq$$

$$\geq \frac{x^2}{2y^2-yz+2z^2}+\frac{y^2}{2z^2-zx+2x^2}+\frac{z^2}{2x^2-xy+2y^2} \quad \textbf{(7)}$$

- Since inequality Cauchy – Schwarz we have

$$\frac{x^2}{2y^2-yz+2z^2}+\frac{y^2}{2z^2-zx+2x^2}+\frac{z^2}{2x^2-xy+2y^2}$$

$$=\frac{x^4}{2x^2y^2-x^2yz+2x^2z^2}+\frac{y^4}{2y^2z^2-y^2zx+2y^2x^2}+\frac{z^4}{2z^2x^2-z^2xy+2z^2y^2}\geq$$

$$\geq\frac{(x^2+y^2+z^2)^2}{(2x^2y^2-x^2yz+2x^2z^2)+(2y^2z^2-y^2zx+2y^2x^2)+(2z^2x^2-z^2xy+2z^2y^2)}$$

$$=\frac{(x^2+y^2+z^2)^2}{4(x^2y^2+y^2z^2+z^2x^2)-xyz(x+y+z)}$$

$$\Rightarrow \frac{x^2}{2y^2-yz+2z^2}+\frac{y^2}{2z^2-zx+2x^2}+\frac{z^2}{2x^2-xy+2y^2}\geq\frac{(x^2+y^2+z^2)^2}{4(x^2y^2+y^2z^2+z^2x^2)-xyz(x+y+z)} \quad \textbf{(8)}$$

- Since (7), (8):

$$\Rightarrow \frac{x^2}{\sqrt{2(y^4+z^4)}+yz}+\frac{y^2}{\sqrt{2(z^4+x^4)}+zx}+\frac{z^2}{\sqrt{2(x^4+y^4)}+xy}\geq\frac{(x^2+y^2+z^2)^2}{4(x^2y^2+y^2z^2+z^2x^2)-xyz(x+y+z)} \quad \textbf{(9)}$$

- (4), (9):

$$\Rightarrow \frac{x^2}{\sqrt{2(y^4+z^4)}+yz}+\frac{y^2}{\sqrt{2(z^4+x^4)}+zx}+\frac{z^2}{\sqrt{2(x^4+y^4)}+xy}\geq 1$$

⇒ Inequality (1) True and we get the result.

+ The occurs if:

$$\Leftrightarrow \begin{cases} x=y=z>0 \\ \sqrt{2(x^4+y^4)}=2xy;\sqrt{2(y^4+z^4)}=2yz;\sqrt{2(z^4+x^4)}=2zx \Leftrightarrow x=y=z>0. \\ \dfrac{1}{2y^2-yz+2z^2}=\dfrac{1}{2z^2-zx+2x^2}=\dfrac{1}{2x^2-xy+2y^2} \end{cases}$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Siendo x, y, z números R^+. Probar que

$$\frac{x^2}{\sqrt{2(y^4+z^4)}+yz}+\frac{y^2}{\sqrt{2(z^4+x^4)}+zx}+\frac{z^2}{\sqrt{2(x^4+y^4)}+xy}\geq 1$$

Teniendo en cuenta las siguientes desigualdades conocidas

$$\sqrt{2(y^4+z^4)} \le 2(y^2+z^2-yz),$$
$$\sqrt{2(z^4+x^4)} \le 2(z^2+x^2-xy),$$
$$\sqrt{2(x^4+y^4)} \le 2(x^2+y^2-xy)$$

Por lo tanto

$$\sum \frac{x^2}{\sqrt{2(y^4+z^4)}+yz} \ge \frac{x^2}{2(y^2+z^2)-yz} + \frac{y^2}{2(z^2+x^2)-xy} + \frac{z^2}{2(x^2+y^2)-xy} \ge 1$$

Por la desigualdad de Cauchy

$$\sum \frac{x^4}{2x^2(y^2+z^2)-x^2yz} \ge \frac{(x^2+y^2+z^2)^2}{4x^2y^2+4y^2z^2+4z^2x^2-xyz(x+y+z)} \ge 1$$
$$\Leftrightarrow (x^2+y^2+z^2)^2 \ge 4x^2y^2+4y^2z^2+4z^2x^2-xyz(x+y+z)$$
$$\Leftrightarrow xyz(x+y+z) \ge 2x^2y^2+2y^2z^2+2z^2x^2-x^4-y^4-z^4$$
$$\Leftrightarrow xyz(x+y+z) \ge (x+y+z)(x+y-z)(y+z-x)(z+x-y)$$
$$\Leftrightarrow xyz \ge (x+y-z)(y+z-x)(z+x-y)$$

(Lo cual es equivalente a la desigualdad de Schur)

La desigualdad pedida es equivalente

$$xyz \ge (x+y-z)(y+z-x)(z+x-y)$$
$$xyz \ge (y^2-(x-z)^2)(z+x-y)$$
$$xyz \ge (y^2-x^2-z^2+2xz)(z+x-y)$$
$$xyz \ge y^2z+y^2x-y^3-x^2z-x^3+x^2y-z^3-z^2x+z^2y+2xz^2+2x^2z-2xyz$$
$$\Leftrightarrow x^3+y^3+z^3+3xyz \ge xy(x+y)+yz(y+z)+zx(z+x)$$
$$\Leftrightarrow x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y) \ge 0$$

(Válido por desigualdad Schur)

Solution 3 by Boris Colakovic – Belgrade – Serbia

$$\sqrt{2(y^4+z^4)} = \sqrt{\frac{4(y^4+z^4)}{2}} = 2\sqrt{\frac{y^4+z^4}{2}} \overset{AM-GM}{\geq} 2yz$$

Analogous $\sqrt{2(z^4+x^4)} \geq 2zx; \sqrt{2(x^4+y^4)} \geq 2xy$

Now is $\sqrt{2(y^4+z^4)} + yz \geq 3yz \Rightarrow \frac{1}{\sqrt{2(y^4+z^4)}+yz} \leq \frac{1}{3yz} \Leftrightarrow \frac{x^2}{\sqrt{2(y^4+z^4)}+yz} \leq \frac{x^2}{3yz}$ *(1)*

Analogous $\frac{y^2}{\sqrt{2(z^4+x^4)}+zx} \leq \frac{y^2}{3zx}$ *(2);* $\frac{z^2}{\sqrt{2(x^4+y^4)}+xy} \leq \frac{z^2}{3xy}$ *(3)*

Adding (1) + (2) + (3) $\Rightarrow$

$$\frac{x^2}{3yz}+\frac{y^2}{3zx}+\frac{z^2}{3xy} \geq \overbrace{\frac{x^2}{\sqrt{2(y^4+z^4)}+yz}+\frac{y^2}{\sqrt{2(z^4+x^4)}+zx}+\frac{z^2}{\sqrt{2(x^4+y^4)}+xy}}^{S}$$

Now is $\frac{x^2}{3yz}+\frac{y^2}{3zx}+\frac{z^2}{3xy} \geq \frac{(x+y+z)^2}{3(xy+yz+zx)} \geq 1$

Now is $\frac{x^2}{3yz}+\frac{y^2}{3zx}+\frac{z^2}{3xy} \geq S \geq 1$

Equality holds for $x = y = z = 1$

138. **If $a, b, c \geq 2$ then:**

$$4(a+b+c) \leq 2(ab+bc+ca) \leq 3abc$$

$$(a+b+c)^3 \leq (ab+c)(bc+a)(ca+b)$$

Proposed by Maria Elena Panaitopol – Romania

Solution by SK Rejuan-West Bengal-India

Given that $a, b, c \geq 2$*, we have to prove that,*

$$4\sum a \leq 2\sum ab \leq 3abc$$

1st

$$4\sum a \leq 2\sum ab$$

$$\Leftrightarrow 2\sum a \leq \sum ab$$

$$\Leftrightarrow 0 \leq a(b-2) + b(c-2) + c(a-2)$$

[which is true $\because (b-2),(c-2),(a-2) \geq 0$***]***

$\therefore 4\sum a \leq 2\sum ab$ ***(1)***

2nd Again,

$2 \leq a \Rightarrow 2bc \leq abc \quad [\because a,b,c \geq 2]$

$2 \leq b \Rightarrow 2ac \leq abc$

$2 \leq c \Rightarrow 2ab \leq abc$

(adding) $2\sum ab \leq 3abc$

$\therefore 2\sum ab \leq 3abc$ ***(2)***

combining (1) & (2) we get

$$4\sum a \leq 2\sum ab \leq 3abc$$

[Proved]

$a,b,c \geq 2$

Now, $2(a+b+c) = 2\cdot a + 2\cdot b + 2\cdot c$

$\leq 2a + cb + bc \quad [\because 2 \leq b, 2 \leq c]$

$\Rightarrow 2(a+b+c) \leq 2(a+bc)$

$\Rightarrow (a+b+c) \leq (a+bc)$ ***(1)***

Similarly we can prove that

$(a+b+c) \leq (b+ca)$ ***(2)***

and $(a+b+c) \leq (c+ab)$ ***(3)***

Multiplying (1), (2) & (3) we get,

$$(a+b+c)^3 \leq (ab+c)(bc+a)(ca+b)$$

[Proved]

139. JBMO TEAM SELECTION TEST

Let a, b, c be positive real numbers. Prove that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9+3\sqrt[3]{\frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}}$$

Solution by Ravi Prakash-New Delhi-India

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=$$

$$=3+\left(\frac{b}{a}+\frac{a}{b}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)$$

$$=9+\frac{(a-b)^2}{ab}+\frac{(b-c)^2}{bc}+\frac{(c-a)^2}{ab}$$

$$\geq 9+3\left[\frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}\right]^{\frac{1}{3}}$$

$$[\because AM \geq GM]$$

140. If $x, y, z > 0, x \neq y, y \neq z, z \neq x$ then:

$$(xy+yz+zx)\left(\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}+\frac{x+y+z}{xyz}\right) > \frac{81}{4}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $x, y, z > 0, x \neq y, y \neq z, z \neq x$. Probar

$$P=(xy+yz+zx)\left(\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}+\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right) > \frac{81}{4}$$

Iran Inequality 1994

Siendo $x, y, z > 0$. Se cumple la siguiente desigualdad

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\ge\frac{9}{4}$$

La igualdad se alcanza cuando $x=y=z=k>0$

Dado que $x\neq y\neq z\neq 0$

$$\Leftrightarrow (xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)>\frac{9}{4}$$

Por la desigualdad de Cauchy

$$\frac{1}{(x-y)^2}+\frac{1}{2xy}+\frac{1}{2xy}\ge\frac{9}{(x-y)^2+4xy}=\frac{9}{(x+y)^2}\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{1}{(x-y)^2}+\sum\frac{1}{xy}\ge\sum\frac{9}{(x+y)^2}$$

Por lo tanto

$$\Leftrightarrow\left(\sum xy\right)\left(\sum\frac{1}{(x-y)^2}+\sum\frac{1}{xy}\right)\ge\left(\sum xy\right)\left(\sum\frac{9}{(x+y)^2}\right)>\frac{9\cdot 9}{4}=\frac{81}{4}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Lemma: Let $x,y,z>0$ ***then*** $(xy+yz+zx)\left(\sum_{cyc}\frac{1}{(x+y)^2}\right)\ge\frac{9}{4}$

$$(xy+yz+zx)\left(\sum_{cyc}\frac{1}{(x+y)^2}+\frac{x+y+z}{xyz}\right)=(xy+yz+zx)\sum_{cyc}\left(\frac{1}{(x-y)^2}+\frac{4}{4xy}\right)$$

$$\overset{BERGSTROM}{>}(xy+yz+zx)\sum_{cyc}\frac{(1+2)^2}{(x-y)^2+4xy}=9(xy+yz+zx)\sum_{cyc}\frac{1}{(x+y)^2}$$

$$>\frac{81}{4}\text{ (Proved)}$$

141. **Let a,b,c be real positive numbers.**

Prove that

$$6+\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\ge 3\cdot\sqrt[3]{6(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-27}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c nùmeros R^+, probar la siguiente desigualdad

$$6+\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}\geq 3\sqrt[3]{6(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-27}$$

Por la desigualdad de Schur

$$a^3+b^3+c^3+3abc\geq ab(a+b)+bc(b+c)+ca(c+a)$$

$$\Leftrightarrow \frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}+6\geq\left(\frac{a+b}{c}+1\right)+\left(\frac{b+c}{a}+1\right)+\left(\frac{c+a}{b}+1\right)=$$

$$=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Es suficiente probar

$x\geq 3\sqrt[3]{6x-27}$ **, donde** $x=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$

(Vàlido por MA $\geq$ MG)

$$x^3\geq 27(6x-27)\Leftrightarrow x^3-162x+729=(x-9)(x^2+9x-81)\geq 0$$

Lo cual es cierto ya que $x\geq 9\ \wedge x^2+9x-81\geq 81>0$

142. **Let a, b, c be real positive numbers. Prove that**

$$6+\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\geq 3\cdot\sqrt[3]{6(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-27}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c nùmeros R^+, probar la siguiente desigualdad

$$6+\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}\geq 3\sqrt[3]{6(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-27}$$

Por la desigualdad de Schur

$$a^3+b^3+c^3+3abc \geq ab(a+b)+bc(b+c)+ca(c+a)$$

$$\Leftrightarrow \frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}+6 \geq \left(\frac{a+b}{c}+1\right)+\left(\frac{b+c}{a}+1\right)+\left(\frac{c+a}{b}+1\right)=$$

$$=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Es suficiente probar

$x \geq 3\sqrt[3]{6x-27}$ ***, donde*** $x=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$

(Vàlido por MA ≥ MG)

$$x^3 \geq 27(6x-27) \Leftrightarrow x^3-162x+729=(x-9)(x^2+9x-81) \geq 0$$

Lo cual es cierto ya que $x \geq 9 \wedge x^2+9x-81 \geq 81>0$

143. If $a,b,c>0, a+b+c=3$ then:

$$\frac{1}{abc} \geq \sqrt[4]{\frac{a^3+b^3+c^3}{3}}$$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $a,b,c>0$***, de tal manera que*** $a+b+c=3$***. Probar que***

$$\frac{1}{abc} \geq \sqrt[4]{\frac{a^3+b^3+c^3}{3}} \Leftrightarrow 3 \geq (a^3+b^3+c^3)a^4b^4c^4$$

Nosotros sabemos que

$$(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)=27$$

Como $a,b,c>0$

Aplicando $MA \geq MG$

$$\left(a^3+b^3+c^3+\frac{3}{8}\cdot 8(a+b)(b+c)(c+a)\right)^9 \geq 9^9(a^3+b^3+c^3)\left(\frac{3(a+b)(b+c)(c+a)}{8}\right)^8$$

Utilizando la siguiente desigualdad $\forall\, a,b,c>0$

$$9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca)$$

$$\Rightarrow 9^9(a^3+b^3+c^3)\left(\frac{3(a+b)(b+c)(c+a)}{8}\right)^8 \geq$$

$$\geq 9^9(a^3+b^3+c^3)\left(\frac{3(a+b+c)(ab+bc+ca)}{9}\right)^8 =$$

$$= 9^9(a^3+b^3+c^3)(ab+bc+ca)^8$$

Por transitividad

$$\Leftrightarrow (a+b+c)^{27} \geq 9^9(a^3+b^3+c^3)(ab+bc+ca)^2 \geq$$

$$\geq 9^9(a^3+b^3+c^3)\big(3abc(a+b+c)\big)^4 = 9^{13}a^4b^4c^4 =$$

$$= 3^{26}(a^3+b^3+c^3)a^4b^4c^4$$

$$\Leftrightarrow 3^{27} \geq 3^{26}(a^3+b^3+c^3)a^4b^4c^4 \Leftrightarrow 3 \geq (a^3+b^3+c^3)a^4b^4c^4$$

(LQQD)

Solution 2 by Nguyen Ngoc Tu – HaGiang – Vietnam

**** Lemma. Let*** a,b,c ***be positive such that*** $a+b+c=3$***. Then***

$$(a^2b+b^2c+c^2a)(a^2c+b^2a+c^2b) \geq 9abc$$

Solution lemma.

$$(a^2b+b^2c+c^2a)(a^2c+b^2a+c^2b) \geq 9abc$$

$$\Leftrightarrow a^3b^3+b^3c^3+c^3a^3+3a^2b^2c^2+abc(a^3+b^3+c^3) \geq 9abc$$

Use Schur inequality for $n=3$ ***we have***

$x^3+y^3+z^3+3xyz \geq x^2(y+z)+y^2(z+x)+z^2(x+y)$ ***with***

$x=ab, y=bc, z=ca$ ***we have***

$$a^3b^3+b^3c^3+c^3a^3+3a^2b^2c^2 \geq a^2b^2(bc+ca)+b^2c^2(ab+ca)+c^2a^2(ab+bc)$$

$$\Leftrightarrow a^3b^3 + b^3c^3 + c^3a^3 + 3a^2b^2c^2 \ge abc[a^2(b+c) + b^2(c+a) + c^2(a+b)]$$

Hence

$$a^3b^3 + b^3c^3 + c^3a^3 + 3a^2b^2c^2 + abc(a^3+b^3+c^3) \ge abc\left[\sum a^3 + \sum a^2(b+c)\right]$$

$$= abc(a+b+c)(a^2+b^2+c^2)$$

$$a^2+b^2+c^2 \ge \frac{1}{3}(a+b+c) = 1 \Rightarrow abc(a+b+c)(a^2+b^2+c^2) \ge 9abc$$

Solution problem

We have $\frac{1}{abc} \ge \sqrt[4]{\frac{a^3+b^3+c^3}{3}} \Leftrightarrow \frac{3}{(abc)^4} \ge a^3+b^3+c^3$

We have

$$(a^3+b^3+c^3)(a^2b+b^2c+c^2a)(a^2b+b^2c+c^2a)$$

$$\le \frac{(a^3+b^3+c^3+a^2b+b^2c+c^2a+a^2b+b^2c+c^2a)^3}{27}$$

$$= \frac{1}{27}(a+b+c)^3(a^2+b^2+c^2)^3 = (a^2+b^2+c^2)^3$$

$$\Rightarrow a^3+b^3+c^3 \le \frac{(a^2+b^2+c^2)^3}{(a^2b+b^2c+c^2a)(a^2b+b^2c+c^2a)} \le \frac{(a^2+b^2+c^2)^3}{9abc} \quad \textbf{(1)}$$

Use AM-GM inequality we have

$$(a^2+b^2+c^2)(ab+bc+ca)^2 \le \frac{(a+b+c)^6}{27} = 27,$$

$$(ab+bc+ca)^2 \ge 3abc(a+b+c) = 9abc$$

$$\Rightarrow a^2+b^2+c^2 \le \frac{27}{(ab+bc+ca)^2} \le \frac{27}{9abc} = \frac{3}{abc} \Rightarrow (a^2+b^2+c^2) \le \frac{27}{(abc)^3} \quad \textbf{(2)}$$

since (1) and (2) we have $a^3+b^3+c^3 \le \frac{3}{(abc)^4} \Rightarrow \frac{1}{abc} \ge \sqrt[4]{\frac{a^3+b^3+c^3}{3}}$

144. **If $a, b, c > 0$ then:**

$$\frac{a^9}{b^6c^2}+\frac{b^9}{c^6a^2}+\frac{c^9}{a^6b^2}\ge\sqrt[6]{\frac{a^{28}}{b^{17}c^5}}+\sqrt[6]{\frac{b^{28}}{c^{17}a^5}}+\sqrt[6]{\frac{c^{28}}{a^{17}b^5}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $a, b, c > 0$. Probar que

$$\frac{a^9}{b^6c^2}+\frac{b^9}{c^6a^2}+\frac{c^9}{a^6b^2}\ge\sqrt[6]{\frac{a^{28}}{b^{17}c^5}}+\sqrt[6]{\frac{b^{28}}{c^{17}a^5}}+\sqrt[6]{\frac{c^{28}}{a^{17}b^5}}$$

La desigualdad es equivalente

$$\frac{a^9}{b^6c^2}+\frac{b^9}{c^6a^2}+\frac{c^9}{a^6b^2}\ge\sqrt[6]{abc}\left(\sqrt{\frac{a^9}{b^6c^2}}+\sqrt{\frac{b^9}{c^6a^2}}+\sqrt{\frac{c^9}{a^6b^2}}\right)$$

Aplicando MA ≥ MG

$$\frac{a^9}{b^6c^2}+\frac{b^9}{c^6a^2}+\frac{c^9}{a^6b^2}\ge 3\sqrt[3]{abc}\quad \textit{(A)}$$

Aplicando desigualdad de Cauchy

$$3\left(\frac{a^9}{b^6c^2}+\frac{b^9}{c^6a^2}+\frac{c^9}{a^6b^2}\right)\ge\left(\sqrt{\frac{a^9}{b^6c^2}}+\sqrt{\frac{b^9}{c^6a^2}}+\sqrt{\frac{c^9}{a^6b^2}}\right)^2\quad \textit{(B)}$$

Multiplicando (A) × (B)

$$\left(\frac{a^9}{b^6c^2}+\frac{b^9}{c^6a^2}+\frac{c^9}{a^6b^2}\right)\ge\sqrt[3]{abc}\left(\sqrt{\frac{a^9}{b^6c^2}}+\sqrt{\frac{b^9}{c^6a^2}}+\sqrt{\frac{c^9}{a^6b^2}}\right)^2$$

$$\Rightarrow\frac{a^9}{b^6c^2}+\frac{b^9}{c^6a^2}+\frac{c^9}{a^6b^2}\ge\sqrt[6]{abc}\left(\sqrt{\frac{a^9}{b^6c^2}}+\sqrt{\frac{b^9}{c^6a^2}}+\sqrt{\frac{c^9}{a^6b^2}}\right)$$

Solution 2 by Ravi Prakash-New Delhi-India

$$4\frac{a^9}{b^6c^2}+\frac{b^9}{c^6a^2}+\frac{c^9}{a^6b^2}\geq 6\left(\frac{a^{36}}{b^2c^8}\cdot\frac{b^9}{c^6a^2}\cdot\frac{c^9}{a^6b^2}\right)^{\frac{1}{6}}=6\left(\frac{a^{28}}{b^{17}c^5}\right)^{\frac{1}{6}}\quad (1)$$

Similarly,

$$\frac{a^9}{b^6c^2}+4\frac{b^9}{c^6a^2}+\frac{c^9}{a^6b^2}\geq\left(\frac{b^{28}}{c^{17}a^5}\right)^{\frac{1}{6}}\quad (2)$$

$$\frac{a^9}{b^6c^2}+\frac{b^9}{c^6a^2}+4\frac{c^9}{a^6b^2}\geq 6\left(\frac{c^{28}}{a^{17}b^5}\right)^{\frac{1}{6}}\quad (3)$$

Adding (1), (2), (3) and dividing by 6 we get the desired inequality.

145. If $a,b,c>0$, $ab+bc+ca+2abc=1$ then:

$$\frac{1}{4a^3+4b^3+3c}+\frac{1}{4b^3+4c^3+3a}+\frac{1}{4c^3+4a^3+3b}\leq\frac{6}{5}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by Nguyen Ngoc Tu-Ha Giang-Vietnam

Since

$$ab+bc+ca+2abc=1\Rightarrow\exists x,y,z\geq 0:(a;b;c)=\left(\frac{x}{y+z};\frac{y}{z+x};\frac{z}{x+y}\right)\Rightarrow$$

$$\Rightarrow a+b+c\geq\frac{3}{2}$$

We have $4a^3\geq 3a-1\Leftrightarrow(2a-1)^2(a+1)\geq 0$***, similarly*** $4b^3\geq 3b-1$,

$$4c^3\geq 3c-1.$$

Hence $4a^3+4b^3+3c\geq 3(a+b+c)-2\geq\frac{5}{3}\Rightarrow\frac{1}{4a^3+4b^3+3c}\leq\frac{2}{5}$***, similarly***

$$\frac{1}{4b^3+4c^3+3a}\leq\frac{2}{5},\frac{1}{4c^3+4a^3+3b}\leq\frac{2}{5}.$$

$$\Rightarrow\frac{1}{4a^3+4b^3+3c}+\frac{1}{4a^3+4b^3+3c}+\frac{1}{4a^3+4b^3+3c}\leq\frac{6}{5}.$$

146. If $a,b,c>0$ then:

$$\sum\frac{a}{b+c}+\prod\frac{a}{b+c}\geq\frac{13}{8}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $a,b,c>0$. Probar la siguiente desigualdad

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{abc}{(a+b)(b+c)(c+a)}\geq\frac{13}{8}$$

Es suficiente demostrar la siguiente desigualdad $\forall\ a,b,c>0$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}\geq 2\ \ \textbf{(A)}$$

$$\Leftrightarrow a(a+b)(a+c)+b(b+a)(b+c)+c(c+a)(c+b)+4abc\geq$$

$$\geq 2(a+b)(b+c)(c+a)$$

$$\Leftrightarrow a^3+b^3+c^3+ab(a+b)+bc(b+c)+ca(c+a)+7abc\geq$$

$$\geq 2ab(a+b)+2bc(b+c)+2ca(c+a)+4abc$$

$$\Leftrightarrow a^3+b^3+c^3+3abc-ab(a+b)-bc(b+c)-ca(c+a)\geq 0$$

$$\Leftrightarrow a(a-b)(a-c)+b(b-a)(b-c)+c(c-a)(c-b)\geq 0$$

(Válido por desigualdad de Schur)

Además $\rightarrow\frac{-3abc}{(a+b)(b+c)(c+a)}\geq-\frac{3}{8}$ ***(B)***

Sumando (A) + (B)

$$\Rightarrow\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{abc}{(a+b)(b+c)(c+a)}\geq\frac{13}{8}$$

(LQQD)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{abc}{(a+b)(b+c)(c+a)}$$

$$= \frac{a(c+a)(a+b)+b(a+b)(b+c)+c(b+c)(c+a)+abc}{(a+b)(b+c)(c+a)}$$

$$= \frac{a(\sum ab+a^2)+b(\sum ab+b^2)+c(\sum ab+c^2)+abc}{2abc+\sum a^2b+\sum ab^2}$$

$$= \frac{(\sum ab)(\sum a)+\sum a^3+abc}{2abc+\sum a^2b+\sum ab^2} = \frac{\sum a^2b+\sum ab^2+3abc+\sum a^3+abc}{2abc+\sum a^2b+\sum ab^2}$$

$$= \frac{p+4abc+\sum a^3}{2abc+p} \text{ (where } p = \sum a^2b+\sum ab^2\text{)}$$

$$= \frac{p+2abc+\sum a^3+2abc}{p+2abc} = 1+\frac{\sum a^3+2abc}{p+2abc}$$

$$= 1+\frac{3\sum a^3+6abc}{3p+6abc} \overset{(1)}{=} 1+\frac{2(\sum a^3+3abc)+\sum a^3}{3p+6abc}$$

$$\text{Now, } \overbrace{\sum a^3+3abc \geq p}^{(a)} \text{ (Schur) and,}$$

$$\sum a^3 \geq 3abc \text{ (AM-GM)}$$

$$\text{Adding, } 2\sum a^3 \geq p \Rightarrow \sum a^3 \geq \frac{p}{2} \text{ (b)}$$

$$\text{(1), (a), (b)} \Rightarrow LHS \overset{(a),(b)}{\geq} 1+\frac{2p+\frac{p}{2}}{3p+6abc} = 1+\frac{5p}{6p+12abc}$$

$$\geq 1+\frac{5p}{6p+2p}\left(\because 12abc \overset{A-G}{\leq} 2p = 2\left(\sum a^2b+\sum ab^2\right)\right)$$

$$= 1+\frac{5}{8} = \frac{13}{8} = RHS$$

(Proved)

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{a}{b+c}+\prod_{cyc}\frac{a}{b+c} \geq \frac{13}{8}$$

$$\Leftrightarrow \frac{a(a+b)(a+c)+b(b+c)(b+a)+c(c+a)(c+a)}{(a+b)(b+c)(c+a)} \geq \frac{13}{8}$$

$$\Leftrightarrow \frac{a^3+b^3+c^3+(a+b+c)(ab+bc+ca)+abc}{(a+b)(b+c)(c+a)} \geq \frac{13}{8}$$

$\Leftrightarrow \frac{p^2-2pq+4r}{pq-r} \geq \frac{13}{8}$, ***where*** $a+b+c=p$, $ab+bc+ca=q$ ***and*** $abc=r$

$\Leftrightarrow 8p^3+45r \geq 29pq$. ***Now, from Schur*** $p^3+9r \geq 4^{pq} \Rightarrow$

$$\Rightarrow 45r \geq 20^{pq}-5p^3$$

$$8p^3+45r \geq 3p^3+20pq \geq 29pq \; [\because p^2 \geq 3q]$$

$$\therefore \sum_{cyc}\frac{a}{b+c}+\prod_{cyc}\frac{a}{b+c} \geq \frac{13}{8}$$

(Proved)

Solution 4 by Nguyen Ngoc Tu-Ha Giang-Vietnam

We have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{abc}{(a+b)(b+c)(c+a)} \geq \frac{13}{8}$$

$$\Leftrightarrow \sum a(a+b)(a+c)+abc \geq \frac{13}{8}(a+b)(b+c)(c+a)$$

$$\Leftrightarrow \sum a^3+\sum a\sum ab+abc \geq \frac{13}{8}\left[\sum a\sum ab-8abc\right]$$

$$\Leftrightarrow 8\sum a^3+6abc \geq 5\left[\sum a^2(b+c)\right]$$

$$\Leftrightarrow 2\left(\sum a^3+3abc\right)+6\sum a^3 \geq 5\left[\sum a^2(b+c)\right]$$

Use Schur inequality, we have $\sum a^3+3abc \geq \sum a^2(b+c)$ ***and***

$a^3+b^3 \geq ab(a+b) \Rightarrow 2\sum a^3 \geq \sum a^2(b+c) \Rightarrow 6\sum a^3 \geq 3\sum a^2(b+c)$

Done.

Solution 5 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum a^3 + 3abc \overset{Schur}{\geq} \sum (a^2b + ab^2) \,|\cdot 5$$

$$8\cdot\sum a^3 + 6abc \overset{AM\geq GM}{\geq} 5\sum a^3 + 15abc \geq 5\cdot\sum (a^2b + ab^2)$$

$$3\cdot(a^3 + b^3 + c^3) \geq 9abc$$

$$8\cdot\sum a^3 + 6abc \geq 5\cdot\sum(a^2b + ab^2) \quad (*)$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{abc}{(a+b)(b+c)(c+a)} - \frac{13}{8} \geq 0$$

$$\frac{8\cdot\sum a\cdot(a+b)(a+c) + 8abc}{8(a+b)(b+c)(c+a)} - \frac{13(a+b)(b+c)(c+a)}{8\cdot(a+b)(b+c)\cdot(ca)} =$$

$$= \frac{8\cdot(\sum a^3 + \sum(a^2b + ab^2) + 4abc) - 13\cdot(\sum(a^2b + ab^2) + 2abc)}{8(a+b)\cdot(b+c)\cdot(c+a)} =$$

$$= \frac{8\cdot\sum a^3 + 6abc - 5[\sum(a^2b + ab^2)]}{8\cdot\prod(a+b)} \overset{(*)}{\geq} 0$$

147. If $a, b, c > 0, a + b + c + abc = 4$ then:

$$\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} \leq \frac{3}{2}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo $a, b, c > 0$, de tal manera que $ab + bc + ca + abc = 4$. Probar que

$$\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} \leq \frac{3}{2}$$

La condición es equivalente

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}=1 \quad (A)$$

La desigualdad se puede expresar como

$$\Leftrightarrow \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\geq\frac{3}{2}$$

Por la desigualdad de Cauchy

$$\frac{1}{a+2}=\frac{1}{\frac{a+1}{2}+\frac{a+1}{2}+1}\leq\frac{1}{9\left(\frac{a+1}{2}\right)}+\frac{1}{9\left(\frac{a+1}{2}\right)}+\frac{1}{9}=\frac{4}{9(a+1)}+\frac{1}{9}$$

Análogamente para los siugientes términos

$$\frac{1}{b+1}\leq\frac{4}{9(b+1)}+\frac{1}{9},\frac{1}{c+2}\leq\frac{4}{9(c+1)}+\frac{1}{9}$$

Sumando dichas desigualdades

$$\Rightarrow 1=\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\leq\frac{4}{9(a+1)}+\frac{4}{9(b+1)}+\frac{4}{9(c+1)}+\frac{1}{3}$$

$$\Leftrightarrow \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\geq\frac{3}{2}$$

(LQQD)

148. Let a,b,c be positive numbers such that $(a+b)(c+a)(c+a)=8$.

Prove that

$$\left(\sqrt{\frac{1}{2}ab(a+b)}+1\right)\left(\sqrt{\frac{1}{2}bc(b+c)}+1\right)\left(\sqrt{\frac{1}{2}ca(c+a)}+1\right)\leq abc+7$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c números R^+ de tal manera que $(a+b)(b+c)(c+a)=8$.

Probar que

$$\left(\sqrt{\frac{1}{2}ab(a+b)}+1\right)\left(\sqrt{\frac{1}{2}bc(b+c)}+1\right)\left(\sqrt{\frac{1}{2}ca(c+a)}+1\right)\le 7+abc$$

La condición es equivalente

$$\frac{ab(a+b)}{2}+\frac{bc(b+c)}{2}+\frac{ca(c+a)}{2}+abc=4\Leftrightarrow x^2+y^2+z^2+xyz=4$$

Donde

$$x=\sqrt{\frac{ab(a+b)}{2}}>0, y=\sqrt{\frac{bc(b+c)}{2}}>0, z=\sqrt{\frac{ca(c+a)}{2}}>0\Leftrightarrow xyz=abc$$

Realizamos la siguiente sustitución trigonométrica en un Δ acutángulo ABC

$$x=\sqrt{\frac{ab(a+b)}{2}}=2\cos A>0, y=2\cos B=\sqrt{\frac{bc(b+c)}{2}}>0, z=2\cos C=\sqrt{\frac{ca(c+a)}{2}}>0$$

La desigualdad propuesta es equivalente

$$(1+x)(1+y)(1+z)\le 7+xyz\Leftrightarrow x+y+z+xy+yz+zx\le 6$$

(LQQD)

Lo cual es cierto ya que

$$\rightarrow \sum x=2\sum\cos A\le 3\wedge\sum xy=4\sum\cos A\cos B\le 3$$

149. **If $x,y,z>0, x+y+z=3$ then**

$$\frac{x^3}{\sqrt[3]{4(y^6+1)}}+\frac{y^3}{\sqrt[3]{4(z^6+1)}}+\frac{z^3}{\sqrt[3]{4(x^6+1)}}\ge\frac{3}{2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Hoang Le Nhat Tung – Hanoi – Vietnam

We have: $\sqrt[3]{4(y^6+1)}=\sqrt[3]{4(y^2+1)(y^2-y\sqrt{3}+1)(y^2+y\sqrt{3}+1)}$

$$= 2\cdot\sqrt[3]{\frac{(y^2+1)}{2}\cdot(2+\sqrt{3})(y^2-y\sqrt{3}+1)\cdot(2-\sqrt{3})(y^2+y\sqrt{3}+1)}$$

$$\le 2\cdot\frac{\frac{y^2+1}{2}+(2+\sqrt{3})(y^2-y\sqrt{3}+1)+(2-\sqrt{3})(y^2+y\sqrt{3}+1)}{3}=3y^2-4y+3$$

$$\Leftrightarrow \frac{1}{\sqrt[3]{4(y^6+1)}}\ge\frac{1}{3y^2-4y+3}\Leftrightarrow\frac{x^3}{\sqrt[3]{4(y^6+1)}}\ge\frac{x^3}{3y^2-4y+3}$$

Similar: $\frac{y^3}{\sqrt[3]{4(z^6+1)}}\ge\frac{y^3}{3z^2-4z+3};\frac{z^3}{\sqrt[3]{4(x^6+1)}}\ge\frac{z^3}{3x^2-4x+3}$

Therefore: $\Rightarrow P=\frac{x^3}{\sqrt[3]{4(y^6+1)}}+\frac{y^3}{\sqrt[3]{4(z^6+1)}}+\frac{z^3}{\sqrt[3]{4(x^6+1)}}\ge\frac{x^3}{3y^2-4y+3}+\frac{y^3}{3z^2-4z+3}+\frac{z^3}{3x^2-4x+3}$ ***(1)***

Other:

$$\frac{x^3}{3y^2-4y+3}+\frac{y^3}{3z^2-4z+3}+\frac{z^3}{3x^2-4x+3}=\frac{x^4}{3xy^2-4xy+3x}+\frac{y^4}{3yz^2-4yz+3y}+$$

$$+\frac{z^4}{3zx^2-4zx+3z}\ge\frac{(x^2+y^2+z^2)^2}{3xy^2-4xy+3x+3yz^2-4yz+3y+3zx^2-4zx+3z}$$

$$\Rightarrow\frac{x^3}{3y^2-4y+3}+\frac{y^3}{3z^2-4z+3}+\frac{z^3}{3x^2-4x+3}\ge\frac{(x^2+y^2+z^2)^2}{3(xy^2+yz^2+zx^2)-4(xy+yz+zx)+3(x+y+z)}\quad (2)$$

Since (1), (2) $\Rightarrow P\ge\frac{(x^2+y^2+z^2)^2}{3(xy^2+yz^2+zx^2)-4(xy+yz+zx)+3(x+y+z)}$ ***(3)***

We will prove that: $\frac{(x^2+y^2+z^2)^2}{3(xy^2+yz^2+zx^2)-4(xy+yz+zx)+3(x+y+z)}\ge\frac{3}{2}$ ***(4)***

$$\Leftrightarrow 2(x^2+y^2+z^2)^2\ge 3\left(3(xy^2+yz^2+zx^2)-4(xy+yz+zx)+3(x+y+z)\right)$$

$$\Leftrightarrow 2(x^2+y^2+z^2)^2+12(xy+yz+zx)\ge 9(xy^2+yz^2+zx^2)+9(x+y+z)$$

$$\Leftrightarrow 6(x^2+y^2+z^2)^2+36(xy+yz+zx)\ge 27(xy^2+yz^2+zx^2)+27(x+y+z)$$

$$\Leftrightarrow 6(x^2+y^2+z^2)^2+4(x+y+z)^2(xy+yz+zx)\ge 9(x+y+z)(xy^2+yz^2+zx^2)+(x+y+z)^4$$

$$\Leftrightarrow 5(x^4+y^4+z^4)+5(x^2y^2+y^2z^2+z^2x^2)\ge xyz(x+y+z)+9(xy^3+yz^3+zx^3)\quad (5)$$

Other:

$$5(x^4+y^4+z^4)+5(x^2y^2+y^2z^2+z^2x^2)=5x^2(x^2+z^2)+5y^2(y^2+x^2)+5z^2(z^2+y^2)\ge$$

$$\ge 5x^2\cdot 2xz+5y^2\cdot 2yx+5z^2\cdot 2zy=10(xy^3+yz^3+zx^3)\quad (6)$$

Cauchy – Schwarz: $\frac{y^2}{z}+\frac{z^2}{x}+\frac{x^2}{y}\ge\frac{(y+z+x)^2}{z+x+y}=x+y+z\Rightarrow xy^3+yz^3+zx^3\ge xyz(x+y+z)$

(7)

Since (6), (7) $:\Rightarrow$

$$\Rightarrow 5(x^4+y^4+z^4)+5(x^2y^2+y^2z^2+z^2x^2)\ge xyz(x+y+z)+9(xy^3+yz^3+zx^3)$$

$\Rightarrow$ *(5) True* $\Rightarrow$ *(4) True.*

Since (3), (4) $:\Rightarrow P=\frac{x^3}{\sqrt[3]{4(y^6+1)}}+\frac{y^3}{\sqrt[3]{4(z^6+1)}}+\frac{z^3}{\sqrt[3]{4(x^6+1)}}\ge\frac{3}{2}\Rightarrow$ *QED.*

150. If $a,b,c>,a+b+c=3$ then:

$$\sum\frac{a^5+a-1}{a^3+a^2-1}\ge ab+bc+ca$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$a^5+a-1=a^5+a^4-a^2-a^4-a^3+a+a^3+a^2-1$$

$$=a^2(a^3+a^2-1)-a(a^3+a^2-1)+1(a^3+a^2-1)$$

$$=(a^3+a^2-1)(a^2-a+1)$$

Similarly, $b^5+b-1=(b^3+b^2-1)(b^2-b+1)$

$$c^5+c-1=(c^3+c^2-1)(c^2-c+1)$$

$$\therefore LHS=\sum\frac{(a^3+a^2-1)(a^2-a+1)}{(a^3+a^2-1)}$$

$=\sum a^2-\sum a+3$ (*$\because a^3+a^2-1$ etc $\neq 0$ as, the LHS would then be undefined)*

$$=\sum a^2\left(\because\sum a=3\right)\ge\sum ab$$

$$\left(\because\sum a^2-\sum ab=\frac{1}{2}\left\{\sum(a-b)^2\right\}\ge 0\right)$$

151. Let a, b, c be real numbers such that

$$(a-2b+c)(b-2c+a)(c-2a+b)\neq 0 \text{ and}$$

$$a^2+b^2+c^2=ab+bc+ca+3. \text{ Prove that}$$

$$\frac{1}{(a-2b+c)^2}+\frac{1}{(b-2c+a)^2}+\frac{1}{(c-2a+b)^2}\geq\frac{3}{4}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Abdul Aziz-Semarang-Indonesia

Let $x=a-2b+c$

$$y=b-2c+a$$

$$z=c-2a+b$$

Clear that:

$$x+y+z=0$$

$$x^2+y^2+z^2=18\Rightarrow z^2=18-(x^2+y^2)$$

$$xy+yz+xz=-9\Rightarrow xy=-9-z(x+y)$$

$$xy=z^2-9$$

$$z^2=18-(x^2+y^2)\leq 18-2xy=18-2z^2+18$$

$$\Leftrightarrow 3z^2\leq 36$$

$$\Leftrightarrow z^2\leq 12\begin{cases} z^2-9\leq 3\Leftrightarrow \dfrac{1}{z^2-9}\geq\dfrac{1}{3} \\ \dfrac{1}{z^2}\geq\dfrac{1}{12}\end{cases}$$

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\geq\frac{2}{xy}+\frac{1}{z^2}=\frac{2}{z^2-9}+\frac{1}{z^2}\geq\frac{2}{3}+\frac{1}{12}=\frac{3}{4}$$

Equality holds when $x=y=\sqrt{3}$ *and* $z=-2\sqrt{3}$

Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam

$$\frac{1}{(a-2b+c)^2}+\frac{1}{(b-2c+a)^2}+\frac{1}{(c-2a+b)^2}\ge\frac{3}{4}\quad \textbf{(1)}$$

$$\textbf{Put}\begin{cases}a-2b+c=x\\ b-2c+a=y\\ c-2a+b=z\end{cases}\Rightarrow x+y+z=0\Rightarrow z=-(x+y)\quad \textbf{(2)}$$

We have:

$$x^2+y^2+z^2=(a-2b+c)^2+(b-2c+a)^2+(c-2a+b)^2=$$

$$=6(a^2+b^2+c^2)-6(ab+bc+ca)$$

$$=6(ab+bc+ca+18-6(ab+bc+ca)=18$$

$$\Rightarrow x^2+y^2+z^2=18\quad \textbf{(3), (2)}\Rightarrow x^2+y^2+(-x-y)^2=18$$

$$\Rightarrow x^2+xy+y^2=9\Leftrightarrow -xy=9-(x+y)^2\Leftrightarrow xy=(x+y)^2-9\quad \textbf{(4)}$$

$$\textbf{(1)}\Leftrightarrow\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\ge\frac{3}{4}\Leftrightarrow\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{(-x-y)^2}\ge\frac{3}{4}$$

$$\Leftrightarrow\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{(x+y)^2}\ge\frac{3}{4}\Leftrightarrow\frac{(x+y)^2-2xy}{x^2y^2}+\frac{1}{(x+y)^2}\ge\frac{3}{4}$$

$$\Leftrightarrow\frac{(x+y)^2-2[(x+y)^2-9]}{[(x+y)^2-9]^2}+\frac{1}{(x+y)^2}\ge\frac{3}{4}$$

$$\Leftrightarrow\frac{18-(x+y)^2}{[(x+y)^2-9]^2}+\frac{1}{(x+y)^2}\ge\frac{3}{4}\Leftrightarrow\frac{18-a}{(a-9)^2}+\frac{1}{a}\ge\frac{3}{4}$$

$$(a=(x+y)^2>0)$$

$$\Leftrightarrow\frac{a(18-a)+(a-9)^2}{a(9-a)^2}\ge\frac{3}{4}\leftrightarrow\frac{81}{a(9-a)^2}\ge\frac{3}{4}\leftrightarrow a(9-a)^2\le 108\quad \textbf{(5)}$$

Because by AM – GM $\forall a>0$; ***propose*** $xy<0\to(x+y)^2<9\to 9-a>0$

$$2a(a-9)^2=2a(9-a)(9-a)\le\frac{(2a+9-a+9-a)^3}{27}=\frac{18^3}{27}=216$$

$$\Rightarrow a(9-a)^2\le 108\Rightarrow \textbf{(5) true}\Rightarrow\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\ge\frac{3}{4}$$

$$\Rightarrow \frac{1}{(a-2b+c)^2} + \frac{1}{(b-2c+a)^2} + \frac{1}{(c-2a+b)^2} \geq \frac{3}{4}$$

$\Rightarrow$ ***Q.E.D.***

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\sum (a+c-2b)^2 = 12$$

$$\sum (a+c-2b)^2 = \sum (a^2+c^2+4b^2+2ac-4ab-4bc) =$$

$$= +\begin{cases} 3b^2+3ac-3ab-3bc+3 \\ 3c^2+3ab-3bc-3ac+3 \\ 3a^2+3bc-3ac-3ab+3 \end{cases}$$

Note: $a^2+b^2+c^2 = \sum ab + 3$

$$12$$

$$LHS \overset{Cauchy}{\geq} \frac{(1+1+1)^2}{\sum (a-2b+c)^2} \geq \frac{9}{12} = \frac{3}{4}$$

152. **Let $a, b, c > 0$ such that $ab+bc+ca = 3$ and $k \in \mathbb{N}$,**

$$k \geq 6, n \geq \mathbb{N}, n \geq 2.$$

Prove that

$$a+b+c+\sqrt[k]{3^{k-1}\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right)} \geq 6$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $a, b, c > 0$ ***de tal manera que*** $ab+bc+ca = 3$ ***y*** $k \in N, n \in N, n \geq 2.$

Probar que

$$a+b+c+\sqrt[k]{3^{k-1}\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right)}\geq 6$$

Siendo $a,b,c>0$ se cumple lo siguiente

$$a+b+c\geq\sqrt{3(ab+bc+ca)}=\sqrt{9}=3$$

Aplicando la desigualdad de Holder

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)(ab+bc+ca)\geq(1+1+1)^3=27$$

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\cdot 3\geq 27\Leftrightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3$$

$$\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right)\cdot 3^{n-1}\geq\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^n\geq 3^n\Leftrightarrow\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\geq 3$$

Luego

$$a+b+c+\sqrt[k]{3^{k-1}\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right)}=a+b+c+\sqrt[k]{\frac{3^k}{3}\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right)}\geq 3+3=6$$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$a^2+b^2+c^2\geq ab+bc+ca=3\Rightarrow 3\sqrt[3]{a^2b^2c^2}\leq 3\Rightarrow abc\leq 1\quad (*)$$

$$(a+b+c)^2\geq 3ab+3bc+3ac=9\Rightarrow a+b+c\geq 3$$

$$3^{k-1}\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right)\geq 3^k$$

$$\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\geq 3$$

$$\frac{3}{a^{\frac{n}{3}}b^{\frac{n}{3}}c^{\frac{n}{3}}}\geq 3$$

$$(abc)^{\frac{n}{3}}\leq 1\Rightarrow (*)\ abc\leq 1$$

153. Let $a, b, c > 0$. Prove that

$$\sum \frac{a^2}{\sqrt{(a^2+ab+b^2)(a^2+ac+c^2)}} \geq 1.$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by Le Khanh Sy-Long An-Vietnam

Using Cauchy – Schwarz, we have

$$\sqrt{ab}\cdot\sqrt{ac}+\sqrt{a^2+ab+b^2}\sqrt{a^2+ac+c^2} \leq \sqrt{(a+b)^2(a+c)^2}$$

or

$$\sqrt{(a^2+ab+b^2)(a^2+ac+c^2)} \leq a^2+bc+a\left(b+c-\sqrt{bc}\right)$$

$$\leq a^2+bc+a\cdot\frac{(b^2+c^2)}{b+c}$$

$$=\frac{a^2(b+c)+b^2(c+a)+c^2(a+b)}{b+c}$$

Thus, it suffices to show that

$$\sum \frac{a^2(b+c)}{\sqrt{(a^2+ab+b^2)(a^2+bc+c^2)}} \geq \frac{\sum a^2(b+c)}{a^2(b+c)+b^2(c+a)+c^2(a+b)} = 1$$

We are done.

154. If $a, b, c > 0$ then:

$$\frac{a^3}{b^3+c^3}+\frac{b^3}{c^3+a^3}+\frac{c^3}{a^3+b^3} \geq \frac{(a+b+c)^4}{6(a^2+b^2+c^2)(ab+bc+ca)}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

If $a, b, c > 0$. Prove that:

$$\frac{a^3}{b^3+c^3}+\frac{b^3}{c^3+a^3}+\frac{c^3}{a^3+b^3}\geq\frac{(a+b+c)^4}{6(a^2+b^2+c^2)(ab+bc+ca)}$$

We have a Lemma: $\frac{a^3}{b^3+c^3}+\frac{b^3}{c^3+a^3}+\frac{c^3}{a^3+b^3}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$ ***(1)***

Therefore, by Cauchy – Schwarz:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a^2}{ab+ac}+\frac{b^2}{bc+ba}+\frac{c^2}{ca+cb}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)} \quad (2)$$

Then (1), (2): $\Rightarrow\frac{a^3}{b^3+c^3}+\frac{b^3}{c^3+a^3}+\frac{c^3}{a^3+b^3}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}$

We will prove: $\frac{(a+b+c)^2}{2(ab+bc+ca)}\geq\frac{(a+b+c)^4}{6(a^2+b^2+c^2)(ab+bc+ca)}\Leftrightarrow 1\geq\frac{(a+b+c)^2}{3(a^2+b^2+c^2)}$

$$\Leftrightarrow 3(a^2+b^2+c^2)\geq(a+b+c)^2\Leftrightarrow(a-b)^2+(b-c)^2+(c-a)^2\geq 0$$

(True)

Therefore:

$$\frac{a^3}{b^3+c^3}+\frac{b^3}{c^3+a^3}+\frac{c^3}{a^3+b^3}\geq\frac{(a+b+c)^4}{6(a^2+b^2+c^2)(ab+bc+ca)}$$

$\Rightarrow$ ***Q.E.D.***

155. If $a,b,c>0$ then

$$\frac{a^3}{b^3+c^3}+\frac{b^3}{c^3+a^3}+\frac{c^3}{a^3+b^3}\geq\frac{1}{3}\left(\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\right)+\frac{5}{6}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

Let $a,b,c>0$; Prove that:

$$\sum\frac{a^3}{b^3+c^3}\geq\frac{1}{3}\left(\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\right)+\frac{5}{6}$$

Lemma: $\forall\, a,b,c>0; n\in\mathbb{N}^*$. ***We have***

$$\frac{a^n}{b^n+c^n}+\frac{b^n}{c^n+a^n}+\frac{c^n}{a^n+b^n}\ge\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

$$\text{If } n=3\Rightarrow\sum\frac{a^3}{b^3+c^3}\ge\sum\frac{a}{b+c}\quad (1)$$

By Cauchy – Schwarz:

$$\sum\frac{a}{b+c}=\sum\frac{a^2}{ab+bc}\ge\frac{(\sum a)^2}{2\sum ab}=\frac{1}{2}\cdot\frac{\sum a^2}{\sum ab}+1\quad (2)$$

$$(1),(2)\Rightarrow\sum\frac{a^3}{b^3+c^3}\ge\frac{\sum a^2}{2\sum ab}+1$$

We need to prove: $\frac{\sum a^2}{2\sum ab}+1\ge\frac{1}{3}\left(\frac{\sum ab}{\sum a^2}+\frac{\sum a^2}{\sum ab}\right)+\frac{5}{6}$

$$\Leftrightarrow\frac{t}{2}+1\ge\frac{1}{3}\left(\frac{1}{t}+t\right)+\frac{5}{6}\qquad\left(t=\frac{\sum a^2}{\sum ab}\ge 1\right)$$

$$\Leftrightarrow\frac{t+2}{2}\ge\frac{t^2+t}{3t}+\frac{5}{6}\Leftrightarrow 3t^2+6t\ge 2t^2+5t+2$$

$$\Leftrightarrow t^2+t\ge 2\quad (\text{true because } t\ge 1)$$

$$\Rightarrow\sum\frac{a^3}{b^3+c^3}\ge\frac{1}{3}\left(\frac{\sum ab}{\sum a^2}+\frac{\sum a^2}{\sum ab}\right)+\frac{5}{6}\Rightarrow \text{Q.E.D.}$$

156. Let a,b,c be positive real numbers. Find the minimum of expression:

$$P=\frac{1}{\sqrt{2(a^4+b^4)}}+\frac{1}{\sqrt{2(b^4+c^4)}}+\frac{1}{\sqrt{2(c^4+a^4)}}+\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

By CBS we have:

$$\left(\sqrt{2(a^4+b^4)}+2ab\right)^2\le(1^2+1^2)\cdot(2(a^4+b^4)+4a^2b^2)=$$

$$=4(a^4+2a^2b^2+b^4)=4(a^2+b^2)^2$$

$$\Leftrightarrow \sqrt{2(a^4+b^4)}+2ab \le 2(a^2+b^2) \Leftrightarrow \sqrt{2(a^4+b^4)} \le 2(a^2-ab+b^2)$$

$\Leftrightarrow$

$$\Leftrightarrow \frac{1}{\sqrt{2(a^4+b^4)}} \ge \frac{1}{2(a^2-ab+b^2)}$$

Similar: $\frac{1}{\sqrt{2(b^4+c^4)}} \ge \frac{1}{2(b^2-bc+c^2)}; \frac{1}{\sqrt{2(c^4+a^4)}} \ge \frac{1}{2(c^2-ca+a^2)}$

Therefore:

$$\Rightarrow \frac{1}{\sqrt{2(a^4+b^4)}}+\frac{1}{\sqrt{2(b^4+c^4)}}+\frac{1}{\sqrt{2(c^4+a^4)}} \ge \frac{1}{2(a^2-ab+b^2)}+\frac{1}{2(b^2-bc+c^2)}+\frac{1}{2(c^2-ca+a^2)} \quad (1)$$

By $(m+n+p)^2 \le 3(m^2+n^2+p^2)$

$$\left(\frac{1}{\sqrt{a^2-ab+b^2}}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{\sqrt{c^2-ca+a^2}}\right)^2 \le \left(\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}\right)$$

$$\Leftrightarrow \frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2} \ge \frac{\left(\frac{1}{\sqrt{a^2-ab+b^2}}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{\sqrt{c^2-ca+a^2}}\right)^2}{3} \quad (2)$$

Then (1), (2):

$$\Rightarrow \frac{1}{\sqrt{2(a^4+b^4)}}+\frac{1}{\sqrt{2(b^4+c^4)}}+\frac{1}{\sqrt{2(c^4+a^4)}} \ge \frac{\left(\frac{1}{\sqrt{a^2-ab+b^2}}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{\sqrt{c^2-ca+a^2}}\right)^2}{2\cdot 3} \quad (3)$$

By inequality: $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \ge \frac{9}{x+y+z}$

$$\frac{1}{\sqrt{a^2-ab+b^2}}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{\sqrt{c^2-ca+a^2}} \ge \frac{9}{\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}} \quad (4)$$

Then (3), (4):

$$\Rightarrow \frac{1}{\sqrt{2(a^4+b^4)}}+\frac{1}{\sqrt{2(b^4+c^4)}}+\frac{1}{\sqrt{2(c^4+a^4)}} \ge \frac{\left(\frac{9}{\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}}\right)^2}{6}$$

$$\Leftrightarrow \frac{1}{\sqrt{2(a^4+b^4)}}+\frac{1}{\sqrt{2(b^4+c^4)}}+\frac{1}{\sqrt{2(c^4+a^4)}} \ge \frac{27}{2\left(\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}\right)^2} \quad (5)$$

By CBS we have:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} = \left(\frac{a^2}{b}-a+b\right)+\left(\frac{b^2}{c}-b+c\right)+\left(\frac{c^2}{a}-c+a\right) = \frac{a^2-ab+b^2}{b}+\frac{b^2-bc+c^2}{c}+\frac{c^2-ca+a^2}{a}$$

$$\Rightarrow \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{\left(\sqrt{a^2-ab+b^2}\right)^2}{b}+\frac{\left(\sqrt{b^2-bc+c^2}\right)^2}{c}+\frac{\left(\sqrt{c^2-ca+a^2}\right)^2}{a}\geq\frac{\left(\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}\right)^2}{b+c+a} \quad (6)$$

Other:

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}=$$

$$=\sqrt{\frac{3(a-b)^2}{4}+\frac{(a+b)^2}{4}}+\sqrt{\frac{3(b-c)^2}{4}+\frac{(b+c)^2}{4}}+\sqrt{\frac{3(c-a)^2}{4}+\frac{(c+a)^2}{4}}\geq$$

$$\geq\sqrt{\frac{(a+b)^2}{4}}+\sqrt{\frac{(b+c)^2}{4}}+\sqrt{\frac{(c+a)^2}{4}}=\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}=a+b+c$$

$$\Rightarrow a+b+c\leq\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2} \quad (7)$$

Then (6), (7):⇒

$$\Rightarrow \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{\left(\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}\right)^2}{\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}}$$

$$\Leftrightarrow \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2} \quad (8)$$

Then (5); (8):

$$\Rightarrow P=\frac{1}{\sqrt{2(a^4+b^4)}}+\frac{1}{\sqrt{2(b^4+c^4)}}+\frac{1}{\sqrt{2(c^4+a^4)}}+\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq$$

$$\geq\frac{27}{2\left(\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}\right)^2}+\left(\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}\right)$$

Put: $\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}=t>0$

By AM – GM:

$$\Rightarrow P\geq\frac{27}{2t^2}+t=\frac{27}{2t^2}+\frac{t}{2}+\frac{t}{2}\geq 3\cdot\sqrt[3]{\frac{27}{2t^2}\cdot\frac{t}{2}\cdot\frac{t}{2}}=3\sqrt[3]{\frac{27t^2}{8t^2}}=3\sqrt[3]{\frac{27}{8}}=3\cdot\frac{3}{2}=\frac{9}{2}$$

$\Rightarrow P\geq\frac{9}{2}\Rightarrow P_{Min}=\frac{9}{2}$. **Equality occurs if**

$$\Leftrightarrow\begin{cases}\sqrt{2(a^4+b^4)}=2ab;\sqrt{2(b^4+c^4)}=2bc;\sqrt{2(c^4+a^4)}=2ca\\ \dfrac{1}{\sqrt{a^2-ab+b^2}}=\dfrac{1}{\sqrt{b^2-bc+c^2}}=\dfrac{1}{\sqrt{c^2-ca+a^2}}\\ a-b=b-c=c-a=0\\ \dfrac{27}{2t^2}=\dfrac{t}{2}\end{cases}$$

$$\Leftrightarrow \begin{cases} a=b=c>0 \\ t=3 \end{cases} \Leftrightarrow \begin{cases} a=b=c>0 \\ \sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}=r \end{cases} \Leftrightarrow a=b=c=1$$

Minimum of P ***is:*** $\frac{9}{2}$ ***that*** $a=b=c=1$.

157. Let a,b,c be positive real numbers, prove that

$$\frac{(b+c)(b^2+ca)}{b^2+bc+c^2}+\frac{(c+a)(c^2+ab)}{c^2+ca+a^2}+\frac{(a+b)(a^2+bc)}{a^2+ab+b^2}\ge\frac{4(ab+bc+ca)}{a+b+c}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{(a+b)(a^2+bc)}{a^2+ab+b^2}\ge\frac{4(ab+bc+ca)}{a+b+c}$$

$$\Leftrightarrow\sum_{cyc}\frac{(a+b)(a^2+bc)}{4(ab+bc+ca)(a^2+ab+b^2)}\ge\frac{1}{a+b+c}$$

$$\sum_{cyc}\frac{(a+b)(a^2+bc)}{4(a^2+ab+b^2)(ab+bc+ca)}\overset{AM\ge GM}{\ge}\sum_{cyc}\frac{(a+b)(a^2+bc)}{\{(a+b)^2+c(a+b)\}^2}$$

$$=\sum_{cyc}\frac{a^2+bc}{(a+b)(a+b+c)^2}.$$

We need to prove,

$$\sum_{cyc}\frac{a^2+bc}{a+b}\ge a+b+c\Leftrightarrow\sum_{cyc}(b+c)(c+a)(a^2+bc)\ge(a+b+c)\prod_{cyc}(a+b)$$

$$\sum_{cyc}a^2b^2+\sum_{cyc}ab^3+\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)+\left(\sum_{cyc}ab\right)^2\ge$$

$$\ge(a+b+c)\prod_{cyc}(a+b)$$

$$\Leftrightarrow\left(\sum_{cyc}ab\right)^2-2abc(a+b+c)+\sum_{cyc}ab^3+\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}ab\right)-2\left(\sum_{cyc}ab\right)^2+\left(\sum_{cyc}ab\right)^2$$

$$\geq(a+b+c)\prod_{cyc}(a+b)$$

$$\Leftrightarrow\sum_{cyc}ab^3-abc(a+b+c)+\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}ab\right)-abc(a+b+c)\geq$$

$$\geq(a+b+c)\prod_{cyc}(a+b)$$

$$\Leftrightarrow\sum_{cyc}ab^3\geq abc(a+b+c)\Leftrightarrow\sum_{cyc}\frac{a^2}{b}\geq a+b+c,$$

which is true by BERGSTORM

$$\therefore\sum_{cyc}\frac{(a+b)(a^2+bc)}{a^2+ab+b^2}\geq\frac{4(ab+bc+ca)}{a+b+c}$$

(Proved)

158. Let a,b,c positive numbers such that: $a^2+b^2+c^2=3$. Prove that

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\geq\frac{a+b+c}{\sqrt{2}}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Abdul Aziz-Semarang-Indonesia

A fact

$$(a+b+c)^2\geq 3(ab+bc+ca)$$

$$a+b+c\geq\sqrt{3(ab+bc+ca)}$$

$$\frac{(a+b+c)^2}{\sqrt{6(ab+bc+ca)}} \geq \frac{a+b+c}{\sqrt{2}}$$

$$\frac{(a+b+c)^2}{\sqrt{(a^2+b^2+c^2)(ab+ac+bc+ba+ca+cb)}} \geq \frac{a+b+c}{\sqrt{2}} \quad \textbf{(1)}$$

by cs

$$\sqrt{(a^2+b^2+c^2)\big((ab+ac)+(bc+ba)+(ca+cb)\big)} \geq$$

$$\geq a\sqrt{ab+ac}+b\sqrt{bc+ba}+c\sqrt{ca+cb}$$

$$\Leftrightarrow \frac{(a+b+c)^2}{a\sqrt{ab+ac}+b\sqrt{bc+ba}+c\sqrt{ca+cb}} \geq$$

$$\geq \frac{(a+b+c)^2}{\sqrt{(a^2+b^2+c^2)(ab+ac+bc+ba+ca+cb)}} \quad \textbf{(2)}$$

by (1) and (2), we have

$$\frac{a+b+c}{\sqrt{2}} \leq \frac{(a+b+c)^2}{a\sqrt{ab+ac}+b\sqrt{bc+ba}+c\sqrt{ca+cb}}$$

by cs again

$$\frac{a+b+c}{\sqrt{2}} \leq \frac{(a+b+c)^2}{a\sqrt{ab+ac}+b\sqrt{bc+ba}+c\sqrt{ca+cb}} \leq$$

$$\leq \frac{a^2}{a\sqrt{ab+ac}}+\frac{b^2}{b\sqrt{bc+ba}}+\frac{c^2}{c\sqrt{ca+cb}}$$

$$\Leftrightarrow \frac{a+b+c}{\sqrt{2}} \leq \sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}$$

Equality holds when $a=b=c=1$

Solution 2 by Uche Eliezer Okeke-Anambra Nigeria

Lemma: $(a+b+c)^2 \geq 3(ab+bc+ca) \Rightarrow \frac{abc}{\sqrt{(a^2+b^2+c^2)(ab+bc+ca)}} \geq 1 \ldots$ ***(1)***

$$LHS=\sum_{cyc}\sqrt{\frac{a}{b+c}}=\sum_{cyc}\left(\frac{a^2}{a\sqrt{ab+ac}}\right)\overset{Titus}{\geq}\frac{(a+b+c)^2}{\sum_{cyc}\left(a\sqrt{ab+ac}\right)}$$

$$\Leftrightarrow\frac{(a+b+c)^2}{\sum_{cyc}\left(a\sqrt{ab+ac}\right)}\overset{C-B-S}{\geq}\frac{(a+b+c)^2}{\sqrt{(a^2+b^2+c^2)(2)(ab+bc+a)}}\overset{(1)}{\geq}\frac{a+b+c}{\sqrt{2}}=RHS$$

(Proved)

Solution 3 by Sanong Hauerai-Nakonpathom-Thailand

Because $a^2+b^2+c^2=3$, **we obtain that**

$$a+b+c\leq 3$$

and $ab+bc+ca\leq 3$

$$6(ab+bc+ca)\leq 18$$

$$\sqrt{6(ab+bc+ca)}\leq 3\sqrt{2}$$

$$\sqrt{ab+ac}+\sqrt{bc+ba}+\sqrt{ca+cb}\leq 3\sqrt{2}$$

$$\frac{9\sqrt{2}}{\sqrt{ab+ac}+\sqrt{bc+ba}+\sqrt{ca+cb}}\geq 3$$

$$\sqrt{\frac{2}{ab+ac}}+\sqrt{\frac{2}{bc+ba}}+\sqrt{\frac{2}{ca+cb}}\geq 3$$

give $x=\sqrt{\frac{2}{ab+ac}}, y=\sqrt{\frac{2}{ba+bc}}, z=\sqrt{\frac{2}{ca+cb}}$

we get $(ax+by+cz)+(ay+bz+cx)+(az+bx+cy)\geq 3(a+b+c)$

$$ax+by+cz\geq a+b+c$$

is $a\sqrt{\frac{2}{ab+ac}}+b\sqrt{\frac{b}{ba+bc}}+c\sqrt{\frac{2}{ca+cb}}\geq a+b+c$

$$\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}\geq a+b+c$$

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\ge\frac{a+b+c}{\sqrt{2}}$$

Solution 4 by Nguyen Thanh Nho-Vietnam

$$LHS=\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}$$

$$=\frac{a^2}{a\sqrt{ab+ac}}+\frac{b^2}{b\sqrt{bc+ba}}+\frac{c^2}{c\sqrt{ca+cb}}$$

$$\overset{C-S}{\ge}\frac{(a+b+c)^2}{a\sqrt{ab+ac}+b\sqrt{bc+ba}+c\sqrt{ca+cb}}\quad (*)$$

$$a\sqrt{ab+ac}+b\sqrt{bc+ba}+c\sqrt{ca+cb}\overset{BCS}{\ge}\sqrt{(a^2+b^2+c^2)\cdot 2(ab+bc+ca)}$$

$$\le\sqrt{3\cdot 2\cdot\frac{1}{3}(a+b+c)^2}=\sqrt{2}(a+b+c)\quad (**)$$

$$(*)\ \&\ (**)\Rightarrow LHS\ge\frac{(a+b+c)^2}{\sqrt{2}(a+b+c)}=\frac{a+b+c}{\sqrt{2}}$$

159. If $x,y,z>0$ then:

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{8xyz}{(x+y)(y+z)(z+x)}\ge 4$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

***Siendo** $x,y,z>0$. **Probar que** $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{8xyz}{(x+y)(y+z)(z+x)}\ge 4$*

***Como** $x,y,z>0$*

Aplicando MA ≥ MG

$$\frac{x}{y}+\frac{x}{y}+\frac{y}{z}\ge 3\sqrt[3]{\frac{x^2}{yz}}=3\sqrt[3]{\frac{x^3}{xyz}}=\frac{3x}{\sqrt[3]{xyz}}\quad (A),$$

$$\frac{y}{z}+\frac{y}{z}+\frac{z}{x}\geq\frac{3y}{\sqrt[3]{xyz}}\quad \textbf{(B)},$$

$$\frac{z}{x}+\frac{z}{x}+\frac{x}{y}\geq\frac{3z}{\sqrt[3]{xyz}}\quad \textbf{(C)}$$

$$(x+y+y+z+z+x)^3\geq 27(x+y)(y+z)(z+x)\Leftrightarrow$$

$$\Leftrightarrow 8(x+y+z)^3\geq 27(x+y)(y+z)(z+x)\quad \textbf{(D)}$$

Sumando (A) + (B) + (C)

$$\frac{3x}{y}+\frac{3y}{z}+\frac{3z}{x}\geq\frac{3x}{\sqrt[3]{xyz}}+\frac{3y}{\sqrt[3]{xyz}}+\frac{3z}{\sqrt[3]{xyz}}\Leftrightarrow\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq\frac{x+y+z}{\sqrt[3]{xyz}}$$

Es suficiente probar

$$\frac{x+y+z}{\sqrt[3]{xyz}}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 4$$

Nuevamente por MA ≥ MG y usando (D)

$$\frac{x+y+z}{3\sqrt[3]{xyz}}+\frac{x+y+z}{3\sqrt[3]{xyz}}+\frac{x+y+z}{3\sqrt[3]{xyz}}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq$$

$$\geq 4\sqrt[4]{\frac{(x+y+z)^3}{27xyz}\cdot\frac{8xyz}{(x+y)(y+z)(z+x)}}\geq 4$$

(LQQD)

Siendo $x,y,z>0$. Probar que

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 4\quad \textbf{(A)}$$

Siendo $x,y,z>0$, se cumple la siguiente desigualdad

$$\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 2$$

Aplicando la desigualdad de Cauchy en (A)

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq\frac{(x+y+z)^2}{xy+yz+zx}+\frac{8xyz}{(x+y)(y+z)(z+x)}=$$

$$=\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{8xyz}{(x+y)(y+z)(z+x)}+2\geq 4$$

Solution 2 by Nguyen Ngoc Tu-Ha Giang-Vietnam

Let $(a,b,c)=\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)\Rightarrow a,b,c>0, a+b+c\geq 3$ ***the inequality becomes***

$$a+b+c+\frac{8}{(a+1)(b+1)(c+1)}\geq 4\Leftrightarrow \sum(a+1)+\frac{8}{(a+1)(b+1)(c+1)}\geq 7$$

We have

$$\frac{a+1}{2}+\frac{b+1}{2}+\frac{c+1}{2}+\frac{8}{(a+1)(b+1)(c+1)}\geq$$

$$\geq 4\sqrt[4]{\frac{a+1}{2}\cdot\frac{b+1}{2}\cdot\frac{c+1}{2}\cdot\frac{8}{(a+1)(b+1)(c+1)}}=4$$

$$\sum\frac{a+1}{2}=\frac{a+b+c}{2}+\frac{3}{2}\geq 3\Rightarrow\sum(a+1)+\frac{8}{(a+1)(b+1)(c+1)}\geq 7$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Applying AM ≥ GM

$$\sum_{cyc}\frac{x+y}{y}+\frac{16xyz}{(x+y)(y+z)(z+x)}\geq 4\sqrt[4]{\left(\prod_{cyc}\frac{x+y}{y}\right)\frac{16xyz}{(x+y)(y+z)(z+x)}}$$

$$\Rightarrow\sum_{cyc}\frac{x}{y}+\frac{16xyz}{(x+y)(y+z)(z+x)}+3\geq 8$$

$$\Rightarrow\sum_{cyc}\frac{x}{y}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 5-\frac{8xyz}{(x+y)(y+z)(z+x)}=4$$

(proved)

$$[\because (x+y)(y+z)(z+x)\geq 8xyz]$$

160. In $\Delta\, ABC$:

$$1+\sqrt{1+\sqrt{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}\le\sqrt{3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$1+\sqrt{1+\sqrt{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}\le\sqrt{3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$$

WALKER INEQUALITY

Siendo a,b,c los lados de un $\Delta\, ABC$ se cumple la siguiente desigualdad

$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\ge(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ **(anteriormente demostrado)**

Pro ultimo demostraremos

$$1+\sqrt{1+\sqrt{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}\le\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

Aplicando la desigualdad de Cauchy

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\sqrt{(a+b+c)^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}=$$

$$=\sqrt{\left(\sum a^2+2\sum bc\right)\left(\sum\frac{1}{a^2}+2\sum\frac{1}{bc}\right)}\ge$$

$$\geq\sqrt{\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)}+2\sqrt{\left(\sum bc\right)\left(\sum\frac{1}{bc}\right)}$$

$$=\sqrt{\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)}+2\sqrt{\left(\sum a\right)\left(\sum\frac{1}{a}\right)}\Leftrightarrow$$

$$\Leftrightarrow\left(\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}-1\right)^2\geq 1+\sqrt{\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)}$$

$$\Rightarrow\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\geq 1+\sqrt{1+\sqrt{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}$$

(LQQD)

161. Let a,b,c be positive real numbers. Prove that

$$\sqrt{3(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+9}\geq\frac{b+c}{a}+\frac{c+a}{c}+\frac{a+b}{c}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c números R^+. Probar que

$$\sqrt{3(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+9}\geq\frac{b+c}{a}+\frac{c+a}{c}+\frac{a+b}{c}$$

La desigualdad propuesta es equivalente

$$\Leftrightarrow\sqrt{18+3\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+3\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+3\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)}\geq\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)$$

$$\Leftrightarrow \sqrt{3\left(\frac{a}{b}+\frac{b}{a}\right)^2+3\left(\frac{b}{c}+\frac{c}{b}\right)^2+3\left(\frac{c}{a}+\frac{a}{c}\right)^2} \geq \left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)$$

Aplicando la desigualdad de Cauchy

$$\Leftrightarrow \sqrt{(1+1+1)\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2\right)} \geq \left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)$$

(LQQD)

Solution 2 by Nguyen Ngoc Tu-Ha Giang-Vietnam

Let $x=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}, y=\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \Rightarrow x^2=\sum\frac{a^2}{b^2}+2y, y^2=\sum\frac{b^2}{a^2}+2x$

$$\Rightarrow \sum\frac{a^2}{b^2}=x^2-2y, \sum\frac{b^2}{a^2}=y^2-2x$$

Hence

$$(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=3+\sum\frac{a^2}{b^2}+\sum\frac{b^2}{a^2}=x^2+y^2-2x-2y+3$$

$$\Rightarrow (a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+3=x^2+y^2-2x-2y+6=$$

$$=(x-1)^2+(y-1)^2+4\geq 3(x+y)^2$$

$$\Rightarrow \sqrt{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+3} \geq \sqrt{3}(x+y)$$

$$\Rightarrow \sqrt{3(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+9} \geq \frac{b+c}{a}+\frac{c+a}{c}+\frac{a+b}{c}$$

Solution 3 by Ravi Prakash-New Delhi-India

For $x, y, z > 0$,

$$3(x^2+y^2+z^2)-(x+y+z)^2=2[x^2+y^2+z^2-xy-yz-zx]$$

$$=(x-y)^2+(y-z)^2+(z-x)^2\geq 0$$

$$\Rightarrow 3(x^2+y^2+z^2) \geq (x+y+z)^2$$

Put $x = \frac{b}{a} + \frac{a}{b}, y = \frac{b}{c} + \frac{c}{b}, z = \frac{a}{c} + \frac{c}{a}$

to obtain

$$3\left[\left(\frac{b}{a}+\frac{a}{b}\right)^2 + \left(\frac{b}{c}+\frac{c}{b}\right)^2 + \left(\frac{a}{c}+\frac{c}{a}\right)^2\right]$$

$$\geq \left(\frac{b}{a}+\frac{a}{b}+\frac{b}{c}+\frac{c}{b}+\frac{a}{c}+\frac{c}{a}\right)^2$$

$$\Rightarrow 3\left[\frac{b^2}{a^2}+\frac{a^2}{b^2}+2+\frac{b^2}{c^2}+\frac{c^2}{b^2}+2+\frac{a^2}{c^2}+\frac{c^2}{a^2}+2\right]$$

$$\geq \left[\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right]^2 \quad (1)$$

LHS of (1)

$$= 3\left[\frac{a^2}{a^2}+\frac{b^2}{a^2}+\frac{c^2}{a^2}+\frac{a^2}{b^2}+\frac{b^2}{b^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}+\frac{b^2}{c^2}+\frac{c^2}{c^2}+3\right]$$

$$= 3\left[(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+3\right]$$

Thus

$$\sqrt{3(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+9} \geq \frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$$

The equality holds when $a = b = c$

162. If $a_k, b_k, c_k > 0, k \in \overline{1,n}, n \in \mathbb{N}, n \geq 1$ then:

$$\sum_{k=1}^{n} \frac{1}{a_k b_k c_k} \sum_{k=1}^{n} (a_k+b_k+c_k)^3 \geq 27n^2$$

Proposed by Daniel Sitaru – Romania

Solution by Togrul Ehmedov-Baku-Azerbaidian

$$\left(\frac{1}{a_1b_1c_1}+\frac{1}{a_2b_2c_2}+\cdots+\frac{1}{a_nb_nc_n}\right)\left((a_1+b_1+c_1)^3+(a_2+b_2+c_2)^3+\cdots+(a_n+b_n+c_n)^3\right)$$

$$\overset{CBS}{\geq}\left(\sqrt{\frac{(a_1+b_1+c_1)^3}{a_1b_1c_1}}+\sqrt{\frac{(a_2+b_2+c_2)^3}{a_2b_2c_2}}+\cdots+\sqrt{\frac{(a_n+b_n+c_n)^3}{a_nb_nc_n}}\right)^2$$

We know that

$$a_n+b_n+c_n\geq 3\sqrt[3]{a_nb_nc_n}$$

$$\sqrt{(a_n+b_n+c_n)^3}\geq\sqrt{27}\sqrt{a_nb_nc_n}$$

Then

$$\sum_{k=1}^{n}\frac{1}{a_kb_kc_k}\sum_{k=1}^{n}(a_k+b_k+c_k)^3\geq\left(\sqrt{27}+\sqrt{27}+\cdots+\sqrt{27}\right)^2=27n^2$$

163. If $a,b,c>0$ then:

$$\sqrt[6]{ab^2c^3}+\sqrt[6]{a^3bc^2}+\sqrt[6]{a^2b^3c}\geq\sqrt[30]{a^9b^{10}c^{11}}+\sqrt[30]{a^{11}b^9c^{10}}+\sqrt[30]{a^{10}b^{11}c^9}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $a,b,c>0$. Probar que

$$\sqrt[6]{ab^2c^3}+\sqrt[6]{a^3bc^2}+\sqrt[6]{a^2b^3c}\geq\sqrt[30]{a^9b^{10}c^{11}}+\sqrt[30]{a^{11}b^9c^{10}}+\sqrt[30]{a^{10}b^{11}c^9}$$

Realizamos los siguientes cambios de variables

$$x^{90}=ab^2c^3>0,\ y^{90}=a^3bc^2>0,\ z^{90}=a^2b^3c\Leftrightarrow x,y,z>0$$

$$(xyz)^{90}=(abc)^6\Leftrightarrow(xyz)^{15}=abc\Leftrightarrow(xyz)^{120}=(abc)^8$$

La desigualdad es equivalente

$$x^{15}+y^{15}+z^{15}\geq(x^3+y^3+z^3)(xyz)^4$$

Aplicando MA ≥ MG

$$7x^{15}+4y^{15}+4z^5\geq 15\sqrt[15]{(x^{15})^7(y^{15})^4(z^{15})^4}=15x^7y^4z^4\quad (A)$$

$$7y^{15}+4z^{15}+4x^{15}\geq 15\sqrt[15]{(y^{15})^7(z^{15})^4(x^{15})^4}=15y^7z^4x^4\ \textbf{(B)}$$

$$7z^{15}+4x^{15}+4y^{15}\geq 15\sqrt[15]{(y^{15})^7(z^{15})^4(x^{15})^4}=15y^7z^4x^4\ \textbf{(C)}$$

Sumando (A) + (B) + (C)

$$\Rightarrow 15\left(x^{15}+y^{15}+z^{15}\right)\geq 15(x^3+y^3+z^3)(xyz)^4\Leftrightarrow x^{15}+y^{15}+z^{15}$$
$$\geq (x^3+y^3+z^3)(xyz)^4$$

(LQQD)

Solution 2 by Mohammad Jamal-Oujda-Morocco

Inequality is homogenous, let $abc=1$ inequality is equivalent to:

$$\sum\sqrt[6]{\frac{a}{b}}\geq\sum\sqrt[30]{\frac{a}{b}}$$

by AM-GM: $2\sqrt[6]{\frac{a}{b}}+\sqrt[6]{\frac{c}{a}}+\sqrt[6]{\frac{b}{c}}+1\geq 5\sqrt[30]{\frac{a}{b}}$ ***and similarly***

thus $4\sum\sqrt[6]{\frac{a}{b}}+3\geq 5\sum\sqrt[30]{\frac{a}{b}}$ ***so we need to show***

$\frac{1}{4}\left(5\sum\sqrt[30]{\frac{a}{b}}-30\right)\geq\sum\sqrt[30]{\frac{a}{b}}$ ***ie*** $\sum\sqrt[30]{\frac{a}{b}}\geq 3$ ***which is obvious***

Solution 3 by Nguyen Ngoc Tu-Ha Giang-Vietnam

Take $(x;y;z)=\left(\sqrt[30]{a};\sqrt[30]{b};\sqrt[30]{c}\right)\Rightarrow x,y,z>0$***, we have to prove***

$$(xyz)^5\left(x^5y^{10}+y^5x^{10}+z^5x^{10}\right)\geq(x^9y^{10}z^{11}+y^9z^{10}x^{11}+z^9x^{10}y^{11})$$

$$\Leftrightarrow x^5y^{10}+y^5x^{10}+z^5x^{10}\geq(xyz)^4(xy^2+yx^2+zx^2)$$

Assume that $xyz=1$, we have prove $\sum(xy^2)^5\geq\sum xy^2\Leftrightarrow$

$\Leftrightarrow X^5+Y^5+Z^5\geq X+Y+Z$ ***with***

$$(X;Y;Z)=(xy^2;yz^2;zx^2)\Rightarrow X,Y,Z>0,XYZ=1$$

$$\left(X^5+Y^5+Z^5\right)(X+Y+Z)\geq(X^3+Y^3+Z^3)^2\Rightarrow X^5+Y^5+Z^5\geq\frac{(X^3+Y^3+Z^3)^2}{X+Y+Z}$$

$$(X^3+Y^3+Z^3)(X+Y+Z)\geq(X^2+Y^2+Z^2)^2\geq\frac{(X+Y+Z)^4}{9}\Rightarrow X^3+Y^3+Z^3\geq\frac{(X+Y+Z)^3}{9}$$

$$\Rightarrow X^5+Y^5+Z^5 \geq \frac{(X^3+Y^3+Z^3)^2}{X+Y+Z} \geq \frac{(X+Y+Z)^5}{81} \geq X+Y+Z.$$

Solution 4 by Ravi Prakash-New Dehi-India

$$7(ab^2c^3)^{\frac{1}{6}}+4(a^3bc^2)^{\frac{1}{6}}+4(a^2b^3c)^{\frac{1}{6}} \geq$$

$$\geq 15(a^7b^{14}c^{21}a^{12}b^4c^8a^8b^{12}c^4)^{\frac{1}{6\times 15}}$$

$$=15(a^{27}b^{30}c^{33})^{\frac{1}{90}}=15(a^9b^{10}c^{11})^{\frac{1}{30}} \quad (1)$$

Similarly

$$4(ab^2c^3)^{\frac{1}{6}}+7(a^3bc^2)^{\frac{1}{6}}+4(a^2b^3c)^{\frac{1}{16}} \geq 15(a^{11}b^9c^{10})^{\frac{1}{30}} \quad (2)$$

and

$$4(ab^2c^3)^{\frac{1}{6}}+4(a^3bc^2)^{\frac{1}{6}}+7(a^2b^3c)^{\frac{1}{6}} \geq 15(a^{10}b^{11}c^{a})^{\frac{1}{30}} \quad (3)$$

Adding (1), (2), (3) and dividing by 15 ***we get the desired inequality.***

Solution 5 by Sanong Hauerai-Nakonpathom-Thailand

$$\sqrt[6]{ab^2c^3}+\sqrt[6]{a^3bc^2}+\sqrt[6]{a^2b^3c}=\sqrt[30]{a^5b^{10}c^{15}}+\sqrt[30]{a^{15}b^5c^{10}}+\sqrt[30]{a^{10}b^{15}c^5}$$

give $a^5b^{10}c^{15}=x,\ a^{15}b^5c^{10}=y,\ a^{10}b^{15}c^5=z$

consider $\sqrt[30]{x}+\sqrt[30]{x}+\sqrt[30]{y}+\sqrt[30]{y}+\sqrt[30]{z} \geq 5\sqrt[5]{\sqrt[30]{xxyyz}}=5\sqrt[30]{\sqrt[5]{xxyyz}}=5\sqrt[30]{a^{10}b^9c^{11}}$

Similarly $\sqrt[30]{y}+\sqrt[30]{y}+\sqrt[30]{z}+\sqrt[30]{z}+\sqrt[30]{x} \geq 5\sqrt[5]{\sqrt[30]{yyzzx}}=5\sqrt[30]{\sqrt[5]{yyzzx}}=5\sqrt[30]{a^{11}b^{10}c^9}$

$$\sqrt[30]{z}+\sqrt[30]{z}+\sqrt[30]{x}+\sqrt[30]{x}+\sqrt[30]{y} \geq 5\sqrt[3]{\sqrt[30]{zzxxy}}=5\sqrt[30]{\sqrt[5]{zzxxy}}=5\sqrt[30]{a^9b^{11}c^{10}}$$

Therefore

$$\sqrt[6]{ab^2c^3}+\sqrt[6]{a^3bc^2}+\sqrt[6]{a^2b^3c} \geq \sqrt[30]{a^9b^{10}c^{11}}+\sqrt[30]{a^{11}b^9c^{10}}+\sqrt[30]{a^{10}b^{11}c^9}$$

$$\sqrt[6]{ab^2c^3}+\sqrt[6]{a^3bc^2}+\sqrt[6]{a^2b^3c}=\sqrt[30]{a^5b^{10}c^{15}}+\sqrt[30]{a^{15}b^5c^{10}}+\sqrt[30]{a^{10}b^{15}c^5}$$

consider

$$a^5b^{10}c^{15}+a^5b^{10}c^{15}+a^{15}b^5c^{10}+a^{15}b^5c^{10}+a^{10}b^{15}c^5 \geq 5a^{10}b^9c^{11}$$

$$a^{15}b^5c^{10}+a^{15}b^5c^{10}+a^{10}b^{15}c^5+a^{10}b^{15}c^5+a^5b^{10}c^{15} \geq 5a^{11}b^{10}c^9$$

$$a^{10}b^{15}c^5+a^{10}b^{15}c^5+a^5b^{10}c^{15}+a^5b^{10}c^{15}+a^{15}b^5c^{10}\geq 5a^9b^{11}c^{10}$$

Hence

$$a^5b^{10}c^{15}+a^{15}b^5c^{10}+a^{10}b^{15}c^5\geq a^{10}b^9c^{11}+a^{11}b^{10}c^9+a^9b^{11}c^{10}$$

That is

$$\sqrt[30]{a^5b^{10}c^{15}}+\sqrt[30]{d^5b^5c^{10}}+\sqrt[30]{a^{10}b^{15}c^5}\geq\sqrt[30]{a^{10}b^9c^{11}}+\sqrt[30]{a^{11}b^{10}c^9}+\sqrt[30]{a^9b^{11}c^{10}}$$

Therefore

$$\sqrt[6]{ab^2c^3}+\sqrt[6]{a^3bc^2}+\sqrt[6]{a^2b^3c}\geq\sqrt[30]{a^9b^{10}c^{11}}+\sqrt[30]{a^{11}b^9c^{10}}+\sqrt[30]{a^{10}b^{11}c^9}$$

164. Let a,b,c positive numbers such that $a^4+b^4+c^4=3$. Prove that

$$\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\geq 9$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c números R^+ de tal manera que $a^4+b^4+c^4=3$. Probare que

$$\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\geq 9$$

Como $a,b,c>0$

Aplicando MA ≥ MG

$$3=a^4+b^4+c^4\geq 3\sqrt[3]{a^4b^4c^4}\Leftrightarrow 1\geq abc$$

$$\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\geq 3\sqrt[3]{\frac{1}{(bca)^2}}\cdot 3\sqrt[3]{\frac{1}{(abc)^2}}=9\sqrt[3]{\frac{1}{(abc)^4}}\geq 9$$

(LQQD)

Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam

Since AM-GM for 3 positive real numbers we have:

$$\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\ge 3\sqrt[3]{\frac{a^3}{b^5}\cdot\frac{b^3}{c^5}\cdot\frac{c^3}{a^5}}\cdot 3\sqrt[3]{\frac{b^3}{a^5}\cdot\frac{c^3}{b^5}\cdot\frac{a^3}{c^5}}$$

$$=9\cdot\sqrt[3]{\frac{1}{a^4b^4c^4}}\quad (1)$$

Other: $3=a^4+b^4+c^4\ge 3\sqrt[3]{a^4b^4c^4}$ ***(2)***

(1), (2) $\Rightarrow\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\ge 9\Rightarrow QED$

Solution 3 by Uche Eliezer Okeke-Anambra-Nigeria

From the condition

$$\sum_{cycl}a^4=3$$

$$\Leftrightarrow 3(3)=(3)\sum_{cycl}a^4\overset{C-B-S}{\ge}\left(\sum_{cycl}a^2\right)^2\Leftrightarrow\left[\sum_{cycl}a^2\le 3\ldots(1)\right]$$

We proceed thus with the inequality:

$$LHS=\sum_{cycl}\left(\frac{a^3}{b^5}\right)\sum_{cycl}\left(\frac{b^3}{a^5}\right)\overset{Holder}{\ge}\left(\sum_{cycl}\sqrt{\frac{a^3b^3}{b^5a^5}}\right)^2=\left(\sum_{cycl}\left(\frac{1}{ab}\right)\right)^2\ge\left(\frac{3^2}{\sum_{cycl}(ab)}\right)^2$$

$$\Leftrightarrow\left(\frac{3^2}{\sum_{cycl}(ab)}\right)^2\overset{C-B-S}{\ge}\left(\frac{3^2}{\sum a^2}\right)^2\overset{(1)}{\ge}\left(\frac{3^2}{3}\right)^2=9=RHS$$

Solution 4 by Boris Colakovic-Belgrade-Serbia

$$2\frac{a^3}{b^5}+\frac{3}{a^2}=\frac{a^3}{b^5}+\frac{a^3}{b^5}+\frac{1}{a^2}+\frac{1}{a^2}+\frac{1}{a^2}\overset{AM-GM}{\ge}5\frac{1}{\sqrt[5]{b^{10}}}=\frac{5}{b^2}$$

Similarly $2\frac{b^3}{c^5}+\frac{3}{b^2}\ge\frac{5}{c^2},2\frac{c^3}{a^5}+\frac{3}{c^2}\ge\frac{5}{a^2}$

$$2\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)+3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\ge 5\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\Leftrightarrow$$

$$\Leftrightarrow\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\ge\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Similarly $\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\ge\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$

Now is $\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\ge\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2$

Let's show $\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2\ge 9\Leftrightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge 3$

How is

$$\sqrt{\frac{a^4+b^4+c^4}{3}}\overset{QM-HM}{\ge}\frac{3}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\Rightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge 3$$

Equality holds for $a=b=c=1$

Solution 5 by Mohammed Jamal-Oujda-Morocco

By Cauchy Schwarz

$$\left(\sum\frac{a^3}{b^5}\right)\left(\sum\frac{b^3}{a^5}\right)\ge\left(\sum\frac{1}{ab}\right)^2\ge\frac{81}{(\sum ab)^2}$$

we have

$\sum a^4\ge\sum(ab)^2\ge\frac{(\sum ab)^2}{3}$ ***thus*** $\sum ab\le 3$ ***thus the conclusion***

Solution 6 by Nguyen Thanh Nho-Tra Vinh-Vietnam

AM-GM

$$3=a^4+b^4+c^4\ge 3\sqrt[3]{a^4b^4c^4}\Rightarrow abc\le 1$$

$$\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\ge 3\cdot\frac{1}{\sqrt[3]{(abc)^2}}\ge 3$$

$$\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\ge 3\cdot\frac{1}{\sqrt[3]{(abc)^2}}\ge 3$$

$$\Rightarrow\left(\frac{a^3}{b^5}+\frac{b^3}{c^5}+\frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5}+\frac{c^3}{b^5}+\frac{a^3}{c^5}\right)\ge 9$$

165. Prove that if $x,y,z\in(0,1)$ or $x,y,z\in(1,\infty)$ then:

$$\sum\frac{\log_y^3 x+\log_z^3 y}{\log_y^2 x+\log_z x+\log_z^2 y}\ge 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

De las condiciones dadas

$$a=\log_y x>0,\ b=\log_z y,\ c=\log_x z\Leftrightarrow abc=\log_z y\cdot\log_y x\cdot\log_x z=1$$

$$\Leftrightarrow ab=\log_z y\cdot\log_y x=\log_z x,\ bc=\log_x z\cdot\log_z y=\log_x y,$$

$$ca=\log_y x\cdot\log_x z=\log_y z$$

La desigualdad propuesta es equivalente

$$\sum\frac{a^3+b^3}{a^2+ab+b^2}\ge 2$$

Tener en cuenta la siguiente desigualdad

$$3(a^2-ab+b^2)\ge(a^2+ab+b^2)\Leftrightarrow 2(a-b)^2\ge 0$$

Como $a,b,c>0$

Por MA $\ge$ ***MG*** $\rightarrow a+b+c\ge 3\sqrt[3]{abc}=3$

Luego

$$\sum\frac{a^3+b^3}{a^2+ab+b^2}=\sum\frac{(a+b)(a^2-ab+b^2)}{a^2+ab+b^2}\ge\sum\frac{a+b}{3}=$$

$$=\frac{2(a+b+c)}{3}\ge\frac{2\cdot 3}{3}=2$$

(LQQD)

Solution 2 by Ravi Prakash-New Delhi-India

For $t\ge 1$,

$$3(t^3+1)-(t+1)(t^2+t+1)$$
$$=3(t+1)(t^2-t+1)-(t+1)(t^2+t+1)$$
$$=(t+1)(3t^2-3t+3-t^2-t-1)$$
$$=(t+1)(2t^2-4t+2)$$
$$=2(t+1)(t-1)^2\ge 0 \quad \textbf{(1)}$$

Let $a=\log_y x$

$$b=\log_z y$$
$$c=\log_x z$$

Note $a,b,c>0,\forall x,y,z\in(0,1)$ ***or*** $x,y,z\in(1,\infty)$

If $a\ge b$, ***let*** $t=\frac{a}{b}$ ***and if*** $a<b$, ***let*** $t=\frac{b}{a}$, ***then from (1),***

$$3(a^3+b^3)\ge(a+b)(a^2+ab+b^2)$$

$$\Rightarrow\frac{(\log_y x)^3+(\log_z y)^3}{(\log_y x)^2+(\log_y x)(\log_z y)+(\log_z y)^2}\ge\frac{1}{3}(\log_y x+\log_z y)$$

Since $(\log_y x)(\log_z y)=\frac{\log x}{\log y}\cdot\frac{\log y}{\log z}=\log_z x$,

We get

$$\frac{(\log_y x)^3+(\log_z y)^3}{(\log_y x)^3+\log_z x+(\log_z y)^3}\ge\frac{1}{3}\left(\frac{\log x}{\log y}+\frac{\log y}{\log z}\right)$$

$$\ge\frac{2}{3}\sqrt{\frac{\log x}{\log z}} \quad \textbf{(2)}$$

Similarly,

$$2^{nd}\ term \geq \frac{2}{3}\sqrt{\frac{\log y}{\log x}} \quad (3)$$

$$\text{and } 3^{rd}\ term \geq \frac{2}{3}\sqrt{\frac{\log z}{\log y}} \quad (4)$$

From (2), (3), (4) we get

$$\sum \frac{(\log_y x)^3 + (\log_z y)^3}{(\log_y x)^3 + \log_z x + (\log_z y)^3} \geq \frac{2}{3}\left[\sqrt{\frac{\log x}{\log z}} + \sqrt{\frac{\log y}{\log x}} + \sqrt{\frac{\log z}{\log y}}\right]$$

$$\geq \frac{2}{3}\left[3\left(\frac{\log x}{\log z}\right)\left(\frac{\log y}{\log x}\right)\left(\frac{\log z}{\log x}\right)\right]^{\frac{1}{6}} = 2$$

166. If $a, b, c > 0, a^3 + b^3 + c^3 = 3$ then:

$$\frac{1}{a^2(b^2-bc+c^2)} + \frac{1}{b^2(c^2-ca+a^2)} + \frac{1}{c^2(a^2-ab+b^2)} \geq \frac{1}{3}(ab + bc + ca)^2$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $a, b, c > 0$ de tal manera que $a^3 + b^3 + c^3 = 3$. Probar que

$$\frac{1}{a^2(b^2-bc+c^2)} + \frac{1}{b^2(c^2-ca+a^2)} + \frac{1}{c^2(a^2-ab+b^2)} \geq \frac{1}{3}(ab + bc + ca)^2$$

$$\Leftrightarrow \frac{1}{2}\left((b^3 + c^3) + (c^3 + a^3) + (a^3 + b^3)\right)\left(\frac{1}{a^2(b^2-bc+c^2)} + \frac{1}{b^2(c^2-ca+a^2)} + \frac{1}{c^2(a^2-ab+b^2)}\right) \geq$$

$$\geq (ab + bc + ca)^2$$

Tener en cuenta lo siguiente

$$b^3 + c^3 = (b + c)(b^2 - bc + c^2),$$

$$c^3 + a^3 = (c + a)(c^2 - ca + a^2),$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Por MA ≥ MG

$$3 = a^3 + b^3 + c^3 \geq 3abc \Leftrightarrow 1 \geq abc$$

Por la desigualdad de Holder

$$27 = (a^3 + b^3 + c^3)(b^3 + c^3 + a^3)(1 + 1 + 1) \geq (ab + bc + ca)^3 \Leftrightarrow$$

$$\Leftrightarrow 3 \geq ab + bc + ca \Leftrightarrow 9 \geq (ab + bc + ca)^2$$

Aplicando la desigualdad de Cauchy en la desigualdad propuesta

$$\frac{1}{2}\left((b^3 + c^3) + (c^3 + a^3) + (a^3 + b^3)\right)\left(\frac{1}{a^2(b^2-bc+c^2)} + \frac{1}{b^2(c^2-ca+a^2)} + \frac{1}{c^2(a^2-ab+b^2)}\right) \geq$$

$$\geq \frac{1}{2}\left(\sum \frac{\sqrt{b+c}}{a}\right)^2$$

Nuevamente por MA ≥ MG

$$\Rightarrow \frac{1}{2}\left(\sum \frac{\sqrt{b+c}}{a}\right)^2 \geq \frac{1}{2}\left(3\sqrt[3]{\frac{\sqrt{(b+c)(c+a)(a+b)}}{abc}}\right)^2 \geq \frac{9}{2}\left(\sqrt[3]{\frac{\sqrt{8abc}}{abc}}\right)^2$$

$$= \frac{9}{2}\left(\frac{\sqrt{2}}{\sqrt[6]{abc}}\right)^2 \geq 9 \geq (ab + bc + ca)^2$$

(LQQD)

Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam

If $a, b, c > 0; a^3 + b^3 + c^3 = 3$ ***then:***

$$\sum \frac{1}{a^2(b^2 - bc + c^2)} \geq \frac{1}{3}\left(\sum ab\right)^2$$

We have AM-GM for three positive real numbers

$$\sum \frac{1}{a^2(b^2-bc+c^2)} \geq 3\sqrt[3]{\prod \frac{1}{a^2(b^2-bc+c^2)}} = \frac{3}{\sqrt[3]{\prod a^2 \cdot \prod(b^2-bc+c^2)}} \quad (4)$$

Other:

$$\prod bc(b^2+c^2-bc)\le\prod\frac{(b^2-bc+c^2+bc)^2}{4}=\frac{\prod(b^2+c^2)^2}{64}$$

$$\Rightarrow\prod a^2\cdot\prod(b^2-bc+c^2)\le\frac{\prod(b^2+c^2)}{64}\le\frac{[(\sum(b^2+c^2))^3]^2}{64\cdot 27^2}=$$

$$=\frac{\left[8\left(\sum b^2\right)^3\right]^2}{64\cdot 27^2}=\frac{\left(\sum a^2\right)^6}{27^2}\quad \textbf{(2)}$$

We have:

$$\frac{2}{3}\cdot 3=\frac{2}{3}(a^3+b^3+c^3)=\frac{a^3+a^3+1}{3}+\frac{b^3+b^3+1}{3}+\frac{c^3+c^3+1}{3}-1$$

$$\ge\frac{3a^2}{3}+\frac{3b^2}{3}+\frac{3c^2}{3}-1=a^2+b^2+c^2-1$$

$$\Leftrightarrow 2\ge a^2+b^2+c^2-1\Leftrightarrow a^2+b^2+c^2\le 3\ \textbf{(2), (1)}$$

$$\textbf{(1), (2)}\Rightarrow\prod a^2\cdot\prod(b^2-bc+c^2)\le\frac{3^6}{27^2}=1\quad\textbf{(3)}$$

$$\textbf{(3), (4)}\Rightarrow\sum\frac{1}{a^2(b^2-bc+c^2)}\ge\frac{3}{1}=3\quad\textbf{(5)}$$

Because: $\sum ab\le\sum a^2\leftrightarrow\frac{1}{2}\sum(a-b)^2\ge 0$ ***(true)***

$$\Rightarrow\sum ab\le\sum a^2\le 3\to\frac{1}{3}\left(\sum ab\right)^2\le\frac{1}{3}\cdot 3^2=3\quad\textbf{(6)}$$

$$\textbf{(5), (6)}\Rightarrow\sum\frac{1}{a^2(b^2-bc+c^2)}\ge\frac{1}{3}\left(\sum ab\right)^2\Rightarrow\ \textbf{\textit{QED}}$$

Solution 3 by Aziz Abdul-Semarang-Indonesia

$$\frac{1}{a^2(b^2-bc+c^2)}+\frac{1}{b^2(c^2-ca+a^2)}+\frac{1}{c^2(a^2-ab+b^2)}$$

$$=\frac{\frac{1}{a^2}}{b^2-bc+c^2}+\frac{\frac{1}{b^2}}{c^2-ca+a^2}+\frac{1}{\frac{a}{a^2-ab+b^2}}\ge$$

$$\ge\frac{a^2}{b^2-bc+c^2}+\frac{b^2}{c^2-ca+a^2}+\frac{c^2}{a^2-ab+b^2}$$

$$=\frac{a^3}{ab^2-abc+ac^2}+\frac{b^3}{bc^2-abc+ba^2}+\frac{c^3}{ca^2-abc+cb^2}$$

$$\geq\frac{(a+b+c)^3}{3(ab^2+ac^2+ba^2+bc^2+ca^2+cb^2+3abc)}$$

$$\geq\frac{(a+b+c)^3}{\frac{3(a+b+c)^3}{9}}=\frac{9}{3}$$

$$\geq\frac{(ab+bc+ca)^2}{3}$$

Because $a^3+b^3+c^3=3\rightarrow a+b+c\leq 3$

$$\rightarrow(ab+bc+ca)\leq a+b+c\leq 3$$

$$\rightarrow(ab+bc+ca)^2\leq(a+b+c)^2\leq 9$$

and $a^2+b^2+c^2\leq 3$

$$\rightarrow l_{a^2}+l_{b^2}+l_{c^2}\geq 3\geq a^2+b^2+c^2$$

Solution 4 by Uche Eliezer Okeke-Anambra-Nigeria

Condition: $a^3+b^3+c^3=3$ ***(1)***

$$\Leftrightarrow 3=a^3+b^3+c^3\Leftrightarrow 3\overset{AM-GM}{\geq}3abc\Leftrightarrow[abc\leq 1\ldots(2)]$$

$$\Leftrightarrow 27=\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}b^3\right)\left(\sum_{cyc}1\right)\overset{Holder}{\geq}(ab+bc+ca)^3\Leftrightarrow$$

$$\Leftrightarrow[ab+bc+ca\leq 3\ldots(3)]$$

We proceed thus:

$$LHS=\sum_{cyc}\frac{1(b+c)}{a^2(b^2-bc+b^2)(b+c)}=\sum_{cyc}\frac{b+c}{a^2(b^3+c^3)}$$

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$$\Leftrightarrow LHS \overset{AM-GM}{\geq} \frac{(3)3\cdot\sqrt[3]{\prod_{cycl}(a+b)}}{\sqrt[3]{a^2b^2c^2}(3)\cdot\sqrt[3]{\prod_{cyc}(a^3+b^3)}} \overset{AM-GM}{\geq} \frac{3\cdot 3\cdot 2\sqrt[3]{abc}}{\left(\sqrt[3]{abc}\right)^2 2\sum_{cyc}a^3}$$

$$LHS = \frac{9}{\sqrt[3]{abc}\sum a^3} \overset{(1)(2)}{\geq} \frac{9}{3} = \frac{3^2}{3} \overset{(3)}{\geq} \frac{(ab+bc+ca)^2}{3} = RHS \quad \textit{(Proved)}$$

Solution 5 by Nguyen Thanh Nho-Tra Vinh-Vietnam

$$a,b,c>0, a^3+b^3+c^3=3$$

$$LHS = \frac{1}{a^2(b^2-bc+c^2)} + \frac{1}{b^2(c^2-ca+a^2)} + \frac{1}{c^2(a^2-ab+b^2)}$$

$$= \frac{b+c}{a^2(b^3+c^3)} + \frac{c+a}{b^2(c^3+a^3)} + \frac{a+b}{c^2(a^3+b^3)}$$

$$\Rightarrow LHS \overset{AM-GM}{\geq} 3\cdot\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{(abc)^2\cdot(a^3+b^3)(b^3+c^3)(c^3+a^3)}}$$

$$(a+b)(b+c)(c+a) \overset{AM-GM}{\geq} 8abc$$

$$(a^3+b^3)(b^3+c^3)(c^3+a^3) \overset{GM-AM}{\leq} \frac{8(a^3+b^3+c^3)^3}{27} = \frac{8\cdot 3^3}{27} = 8$$

$$3 = a^3+b^3+c^3 \geq 3abc \Rightarrow abc \leq 1$$

$$\Rightarrow LHS \geq 3\cdot\sqrt[3]{\frac{8abc}{(abc)^2\cdot 8}} = \frac{3}{\sqrt[3]{abc}} \geq 3 \quad (*)$$

$$a^3+b^3+1 \overset{AM-GM}{\geq} 3ab$$

$$b^3+c^3+1 \geq 3bc$$

$$c^3+a^3+1 \geq 3ca$$

$$\Rightarrow 2(a^3+b^3+c^3)+3 \geq 3(ab+bc+ca)$$

$$\Rightarrow 2\cdot 3+3 \geq 3(ab+bc+ca)$$

$$\Rightarrow ab+bc+ca \leq 3 \Rightarrow (ab+bc+ca)^2 \leq 9$$

$$\Rightarrow RHS = \frac{1}{3}(ab+bc+ca)^2 \le 3 \ \ (**)$$

$$(*)\ \&\ (**) \Rightarrow LHS \ge 3 \ge RHS$$

$$\Rightarrow LHS \ge RHS$$

$$\text{“} = \text{“} \Leftrightarrow a = b = c = 1$$

Solution 6 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc} a^3 = 3, \frac{a^3+b^3+c^3}{3} \ge \left(\frac{a+b+c}{3}\right)^3 \Rightarrow a+b+c$$

$$\Rightarrow 9 \ge (a+b+c)^2 \ge 3(ab+bc+ca) \Rightarrow 3 \ge ab+bc+ca$$

$$\Rightarrow 3 \ge \frac{1}{3}(ab+bc+ca)^2$$

$$(a+b)(b+c)(c+a) \ge 8abc$$

$$\sum_{cyc} \frac{1}{c^2(a^2-ab+b^2)} \overset{AM\ge GM}{\ge} 3\sqrt[3]{\frac{1}{(abc)^2(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)}}$$

$$= 3\sqrt[3]{8^2\left(\prod_{cyc}\frac{1}{(a+b)^2}\right)\left(\prod_{cyc}\frac{1}{(a^2-ab+b^2)}\right)}$$

$$= 12\sqrt[3]{\left(\prod_{cyc}\frac{1}{(a+b)}\right)\left(\prod_{cyc}\frac{1}{(a+b)(a^2-ab+b^2)}\right)}$$

$$\overset{REVERSE\ AM\ge GM}{\ge} \frac{12}{\frac{2}{3}(a+b+c)\cdot\frac{2}{3}(a^3+b^3+c^3)} = \frac{9}{a+b+c} \ge 3 \ge$$

$$\ge \frac{(ab+bc+ca)^2}{3}$$

(proved)

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167. Let $a, b, c > 0$ such that $a^2+b^2+c^2=3$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\frac{3}{2}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

Let $a, b, c > 0; a^2+b^2+c^2=3$***. Prove that***

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\frac{3}{2}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)}$$

$$\Leftrightarrow\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\geq\frac{3}{2}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)$$

$$\Leftrightarrow 9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\geq\frac{27}{2}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)\quad (1)$$

We have: $3=a^2+b^2+c^2\geq\frac{(a+b+c)^2}{3}$

$$\Rightarrow 9\geq(a+b+c)^2$$

$$\Rightarrow 9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\geq(a+b+c)^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

$$=\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}+3\right)^2\quad (2)$$

We will prove that:

$$\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}+3\right)^2\geq\frac{27}{2}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)\quad (3)$$

$$\Leftrightarrow (t+3)^2\geq\frac{27}{2}t\quad \left(\text{Put: }\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}=t>0\right)$$

$$\Leftrightarrow 2(t+3)^2\geq 27t\leftrightarrow(t-6)(2t-3)\geq 0$$ ***true because:***

$$t=\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}=\Bigg(\underbrace{\frac{a}{b}+\frac{b}{a}}_{\geq 2}\Bigg)+\Bigg(\underbrace{\frac{b}{c}+\frac{c}{b}}_{\geq 2}\Bigg)+\Bigg(\underbrace{\frac{c}{a}+\frac{a}{c}}_{\geq 2}\Bigg)\geq 6$$

$$\Rightarrow t\geq 6\Rightarrow (t-6)(2t-3)\geq 0$$

$$(2),(3)\Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\frac{3}{2}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)}$$

$\Rightarrow$ **Q.E.D.**

Solution 2 by Nguyen Minh Tri-Ho Chi Minh-Vietnam

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{2}\right)$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{9}{a+b+c}\geq 3$$

$a,b,c>0$ **such that** $a^2+b^2+c^2=3$**. Prove that**

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\frac{3}{2}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{a+c}{b}\right)}\Rightarrow a+b+c\leq 3$$

$$\Leftrightarrow\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\geq\frac{3}{2}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{a+c}{b}\right)$$

$$\Leftrightarrow 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2+9\geq 3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

We have:

$3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\leq 9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ **(because** $a+b+c\leq 3$**)**

We need to prove $2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2+9\geq 9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

$$\Rightarrow\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{2}\right)\geq 0$$

true because $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{9}{a+b+c}\geq\frac{9}{3}=3\Rightarrow$ **Q.E.D.**

Solution 3 by Mohammad Jamal-Oujada-Morocco

Squaring we get that inequality is equivalent

$$\sum \frac{1}{a^2} + 2\sum \frac{1}{ab} \ge \frac{3}{2}\sum \frac{a^2+b^2}{ab} = \frac{3}{2}\sum \frac{3-c^2}{ab}$$

$$\textit{ie} \sum \frac{1}{a^2} + \frac{3}{2}\sum \frac{c^2}{ab} \ge \frac{5}{2}\sum \frac{1}{ab}$$

by CS $(a+b+c)(a^3+b^3+c^3) \ge (a^2+b^2+c^2)^2 = 9$ ***or***

$(a+b+c)^2 \le 9$ ***thus*** $a^3+b^3+c^3 \ge \frac{9}{a+b+c} \ge a+b+c$

we conlude that $\frac{3}{2}\sum \frac{c^2}{ab} \ge \frac{3}{2}\sum \frac{1}{ab}$ ***or*** $\sum \frac{1}{a^2} \ge \sum \frac{1}{ab}$ ***sum*** $\min g$ ***up we get the desired inequality***

Solution 4 by Uche Eliezer Okeke-Anambra-Nigeria

$$\left.\begin{array}{l} 3(3) = 3(a^2+b^2+c^2) \Leftrightarrow 3(3) \overset{Cauchy}{\ge} (a+b+c)^2 \Leftrightarrow [p = a+b+c \le 3 \ldots (1)] \\ 3 = a^2+b^2+c^2 \Leftrightarrow 3 \overset{AM-GM}{\ge} 3\sqrt[3]{a^2b^2c^2} \Leftrightarrow [abc \le 1 \ldots (2)] \\ Let\ u = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \overset{Bergstrom}{\ge} \frac{3^2}{a+b+c} \overset{(1)}{\ge} \frac{9}{3} = 3 \Leftrightarrow [u \ge 3 \ldots (3)] \end{array}\right\}$$

We proceed thus:

consider $f(a,b,c) = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 - \frac{3}{2}\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)$ ***... (4)***

We need to show $f(a,b,c) \ge 0$

Variable transformation of (4) gives

$$f(a,b,c) = f(u) = u^2 - \frac{3}{2}[(a+b+c)u - 3] \overset{(1)}{\ge} u^2 - \frac{3}{2}(3u-3) \overset{AM-GM}{\ge} u^2 - \frac{3}{2}\left[\frac{(3+u)^2}{4} - 3\right]$$

$\Leftrightarrow f(u) \ge \frac{1}{8}(5u^2 - 18 + 9) = \frac{1}{8}(u-3)(5u-3) \overset{(3)}{\ge} 0$ ***(proof complete)***

Solution 5 by Soumitra Mandal-Chandar Nagore-India

We know, $\frac{a^2+b^2+c^2}{3} \ge \left(\frac{a+b+c}{3}\right)^2 \Rightarrow a+b+c$

$$\therefore (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9 \Rightarrow \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9 \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3$$

We need to prove, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\frac{3}{2}\sum_{cyc}\frac{a+b}{c}}$

$$\Leftrightarrow \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\geq\frac{3}{2}\sum_{cyc}\frac{a+b}{c}=\frac{3}{2}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{9}{2}$$

again, $3\geq a+b+c$. *So, we are left to prove,*

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\geq\frac{9}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{9}{2}\Rightarrow 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2-9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+9\geq 0$$

$$\Rightarrow\left(\frac{2}{a}+\frac{2}{b}+\frac{2}{c}-3\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)\geq 0,$$ *which is true*

hence proved.

168. Prove that if $a,b,c>0$ then:

$$\sum\left(\frac{a}{b}\right)^2\cdot\sum\left(\frac{a}{b}\right)^4\cdot\sum\left(\frac{a}{b}\right)^8\geq\sum\left(\frac{a}{c}\right)\cdot\sum\left(\frac{b}{a}\right)\cdot\sum\left(\frac{b}{c}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nirapada Pal-Jhargram-India

We have

$$\sum A^2\geq\sum AB$$

$$\left[\frac{a}{b},\frac{b}{c},\frac{c}{a}\to\frac{a}{c},\frac{b}{a},\frac{a}{c}\to\frac{b}{c},\frac{c}{a},\frac{a}{b}\to\frac{b}{a},\frac{c}{a},\frac{a}{c}\right]$$

$$\sum\left(\frac{a}{b}\right)^2\geq\sum\frac{a}{c}$$

$$\sum\left(\frac{a}{b}\right)^4\geq\sum\left(\frac{a}{c}\right)^2\geq\sum\frac{b}{c}$$

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$$\sum\left(\frac{a}{b}\right)^8 \geq \sum\left(\frac{a}{c}\right)^4 \geq \sum\left(\frac{b}{c}\right)^2 \geq \sum\frac{b}{a}$$

$$\therefore \sum\left(\frac{a}{b}\right)^2\sum\left(\frac{a}{b}\right)^4\sum\left(\frac{a}{b}\right)^8 \geq \sum\frac{a}{c}\sum\frac{b}{a}\sum\frac{b}{c}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\left(\frac{a}{b}\right)^2 \geq \sum_{cyc}\left(\frac{a}{b}\right)\left(\frac{b}{c}\right)\left[\because \sum_{cyc}x^2 \geq \sum_{cyc}xy\right] = \sum_{cyc}\frac{a}{c}$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^4 \geq \sum_{cyc}\left(\frac{a}{b}\right)^2\left(\frac{b}{c}\right)^2 = \sum_{cyc}\left(\frac{a}{c}\right)^2 \geq \frac{1}{3}\left(\sum_{cyc}\frac{b}{a}\right)^2 = \frac{1}{3}\left(\sum_{cyc}\frac{b}{a}\right)\left(\sum_{cyc}\frac{b}{a}\right)$$

$$\overset{AM\geq GM}{\geq} \sum_{cyc}\frac{b}{a}$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^8 \geq \sum_{cyc}\left(\frac{a}{b}\right)^4\left(\frac{b}{c}\right)^4 = \sum_{cyc}\left(\frac{a}{c}\right)^4 \geq \sum_{cyc}\left(\frac{a}{c}\right)^2\left(\frac{c}{b}\right)^2 = \sum_{cyc}\left(\frac{a}{b}\right)^2$$

$$\overset{AM\geq GM}{\geq} \frac{1}{3}\left(\sum_{cyc}\frac{a}{b}\right)^2 \overset{AM\geq GM}{\geq} \sum_{cyc}\frac{a}{b}$$

$$\therefore \left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^4\right)\left(\sum_{cyc}\left(\frac{a}{b}\right)^8\right) \geq \left(\sum_{cyc}\frac{a}{c}\right)\left(\sum_{cyc}\frac{b}{a}\right)\left(\sum_{cyc}\frac{b}{c}\right)$$

(proved)

169. **If $a,b,c\ge 1$ then:**

$$\frac{(1+a)(1+b)(1+c)\left(abc+\sqrt{abc}\right)}{\left(a+\sqrt{a}\right)\left(b+\sqrt{b}\right)\left(c+\sqrt{c}\right)}\ge 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{As } a\ge 1, a\ge\sqrt{a}\Rightarrow 1+a\ge 1+\sqrt{a}\Rightarrow\frac{1+a}{1+\sqrt{a}}\ge 1$$

$$\Rightarrow\frac{(1+a)(1+b)(1+c)}{\left(1+\sqrt{a}\right)\left(1+\sqrt{b}\right)\left(1+\sqrt{c}\right)}\ge 1$$

$$\text{Also } \sqrt{abc}+1\ge 2$$

Thus,

$$\frac{(1+a)(1+b)(1+c)\left(\sqrt{abc}+1\right)}{\left(1+\sqrt{a}\right)\left(1+\sqrt{b}\right)\left(1+\sqrt{c}\right)}\ge 2$$

Multiply the numerator and denominator by $\sqrt{a},\sqrt{b},\sqrt{c}$ we get

$$\frac{(1+a)(1+b)(1+c)\left(abc+\sqrt{abc}\right)}{\left(a+\sqrt{a}\right)\left(b+\sqrt{b}\right)\left(c+\sqrt{c}\right)}\ge 2$$

Solution 2 by Ngo Minh Ngoc Bao-Vietnam

We have:

$$\frac{(1+a)(1+b)(1+c)\left(abc+\sqrt{abc}\right)}{\left(a+\sqrt{a}\right)\left(b+\sqrt{b}\right)\left(c+\sqrt{c}\right)}\ge 2\Leftrightarrow$$

$$\Leftrightarrow\left(\sum\ln(a+1)+\frac{1}{2}\sum\ln a-\sum\ln\left(a+\sqrt{a}\right)\right)+\ln\left(\sqrt{abc}+1\right)\ge\ln 2$$

Considering the function:

$$f(t)=\ln(t+1)+\frac{1}{2}\ln t-\ln\left(t+\sqrt{t}\right),\forall\, t\in[1;+\infty)$$

$$\Rightarrow f'(t)=\frac{1}{2t}+\frac{1}{1+t}-\frac{2\sqrt{t}+1}{2t(\sqrt{t}+1)}=\frac{(t+1)(\sqrt{t}+1)+2t(\sqrt{t}+1)-(2\sqrt{t}+1)(t+1)}{2t(t+1)(\sqrt{t}+1)}=$$

$$=\frac{t+2\sqrt{t}-1}{2\sqrt{t}(t+1)(\sqrt{t}+1)}>0$$

$$\Rightarrow\left(\sum\ln(a+1)+\frac{1}{2}\sum\ln a-\sum\ln(a+\sqrt{a})\right)+\ln(\sqrt{abc}+1)\geq$$

$$\geq 3\ln 2-3\ln 2+\ln 2=\ln 2\quad(\sqrt{abc}+1\geq 2)$$

Solution 3 by Ngo Minh Ngoc Bao-Vietnam

We have: $f(a,b,c)=\frac{(1+a)(1+b)(1+c)(abc+\sqrt{abc})}{(a+\sqrt{a})(b+\sqrt{b})(c+\sqrt{c})}$***, we need to prove***

$$f(a,b,c)\geq 2.$$

Use: $(1+x^3)(1+y^3)(1+z^3)\geq(1+xyz)^3,(x,y,z>0)$ ***we have:***

$$(1+a)(1+b)(1+c)\geq\left(1+\sqrt[3]{abc}\right)^3$$

and $(a+\sqrt{a})(b+\sqrt{b})(c+\sqrt{c})\leq(a+a)(b+b)(c+c)=8abc$

$$\Rightarrow f(a,b,c)\geq\frac{\left(1+\sqrt[3]{abc}\right)^3(abc+\sqrt{abc})}{8abc}=\frac{1}{8}\left(1+\sqrt[3]{abc}\right)^3\left(1+\frac{1}{\sqrt{abc}}\right)$$

$$=\frac{1}{8}\left[\left(1+\sqrt[3]{abc}\right)^3+\left(\frac{1+\sqrt[3]{abc}}{\sqrt[6]{abc}}\right)^3\right]=\frac{1}{8}\left[\left(1+\sqrt[3]{abc}\right)^3+\left(\frac{1}{\sqrt[6]{abc}}+\sqrt[6]{abc}\right)^3\right]\geq$$

$$\geq\frac{1}{8}[(1+1)^3+(2)^3]=2$$

Solution 4 by Nguyen Ngoc Tu-Ha Giang-Vietnam

We have

$$\frac{(1+a)(1+b)(1+c)(abc+\sqrt{abc})}{(a+\sqrt{a})(b+\sqrt{b})(b+\sqrt{c})}\geq 2$$

$$\Leftrightarrow(1+a)(1+b)(1+c)(abc+\sqrt{abc})\geq 2(a+\sqrt{b})(b+\sqrt{c})$$

$$\Leftrightarrow \sqrt{abc}(1+a)(1+b)(1+c)(1+\sqrt{abc}) \geq 2\sqrt{abc}(1+\sqrt{a})(1+\sqrt{b})(1+\sqrt{c})$$

$$\Leftrightarrow (1+a)(1+b)(1+c)(1+\sqrt{abc}) \geq 2(1+\sqrt{a})(1+\sqrt{b})(1+\sqrt{c})$$

Use Cauchy – Schwarz inequality, we have

$$(1+\sqrt{a})^2 \leq 2(1+a) \Rightarrow 1+a \geq \frac{1}{2}(1+\sqrt{a})^2 \geq \frac{1}{2}(1+\sqrt{a})\cdot(1+1) = 1+\sqrt{a}$$

and $1+\sqrt{abc} \geq 2$ *by* $a \geq 1$*, similar we have*

$$(1+a)(1+b)(1+c)(1+\sqrt{abc}) \geq 2(1+\sqrt{a})(1+\sqrt{b})(1+\sqrt{c})$$

170. If $a, b, c > 0$ then:

$$\sum c\left(\frac{4a}{b^2}+\frac{3b}{a^2}\right) \geq 12 + 3\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum c\left(\frac{4a}{b^2}+\frac{3b}{a^2}\right) = c\left(\frac{4a}{b^2}+\frac{3b}{a^2}\right) + a\left(\frac{4b}{c^2}+\frac{3c}{b^2}\right) + b\left(\frac{4c}{a^2}+\frac{3a}{c^2}\right)$$

$$= \left(\frac{4ac}{b^2}+\frac{4bc}{a^2}+\frac{4ab}{c^2}\right) + \left(\frac{3ab}{c^2}+\frac{3bc}{a^2}+\frac{3ca}{b^2}\right) = 7\sum\left(\frac{ab}{c^2}\right)$$

$$AM \geq GM \Rightarrow \frac{4ac}{b^2}+\frac{4bc}{a^2}+\frac{4ab}{c^2} \geq 3\sqrt[3]{4^3} = 12 \quad (1)$$

$$\text{Again, } AM \geq GM \Rightarrow \frac{ab}{c^2}+\frac{ab}{c^2}+\frac{ca}{b^2} \geq 3\sqrt[3]{\frac{a^3}{c^3}} = 3\left(\frac{a}{c}\right) \quad (2)$$

$$AM \geq GM \Rightarrow \frac{bc}{a^2}+\frac{bc}{a^2}+\frac{ab}{c^2} \geq 3\sqrt[3]{\frac{b^3}{a^3}} = 3\left(\frac{b}{a}\right) \quad (3)$$

$$AM \geq GM \Rightarrow \frac{ca}{b^2}+\frac{ca}{b^2}+\frac{bc}{a^2} \geq 3\sqrt[3]{\frac{c^3}{b^3}} = 3\left(\frac{c}{b}\right) \quad (4)$$

$$(1)+(2)+(3)+(4) \Rightarrow 7\left(\sum\left(\frac{ab}{c^2}\right)\right) \geq 12 + 3\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)$$

(Proved)

171. Let a,b,c be positive real numbers such that: $a^2+b^2+c^2+2abc=1$

Prove that:

$$a^4+b^4+c^4+4a^2b^2c^2+\frac{1}{8}\ge ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\ (1)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution Hoang Le Nhat Tung – Hanoi – Vietnam

We will prove that: $4abc+1\ge 2(ab+bc+ca)$

We have: $2a-1;2b-1;2c-1.$

Then Dirichle, propose: $(2a-1)(2b-1)\ge 0\Leftrightarrow c(2a-1)(2b-1)\ge 0\ (c>0)$

$$\Leftrightarrow c(4ab-2a-2b+1)\ge 0\Leftrightarrow 4abc+1\ge 2ac+2bc-c+1\quad (2)$$

We need to prove: $2ac+2bc-c+1\ge 2(ab+bc+ca)\Leftrightarrow 1\ge c+2ab\quad (3)$

$$a^2+b^2+c^2+2abc=1\Leftrightarrow c^2+2abc+(a^2+b^2-1)=0\quad (4)$$

Other, because $a,b,c>0;a^2+b^2+c^2+2abc=1$ *therefore* $0<a,b,c<1$

$$\Delta'=(ab)^2-(a^2+b^2-1)=(1-a^2)(1-b^2)>0\ \ (0<a,b<1)$$

$$\Rightarrow \left[\begin{array}{l} c=-ab+\sqrt{(1-a^2)(1-b^2)} \\ c=-ab-\sqrt{(1-a^2)(1-b^2)}\end{array}\right.\ \textit{(absurd: } c=-ab-\sqrt{(1-a^2)(1-b^2)}<0\textit{)}$$

$$\Rightarrow c=-ab+\sqrt{(1-a^2)(1-b^2)}\quad (5)$$

Then (3), (5) $\Leftrightarrow 1\ge -ab+\sqrt{(1-a^2)(1-b^2)}+2ab\Leftrightarrow 1-ab\ge\sqrt{(1-a^2)(1-b^2)}$

$$\Leftrightarrow (1-ab)^2\ge(1-a^2)(1-b^2)\Leftrightarrow (ab)^2-2ab+1\ge 1-(a^2+b^2)+(ab)^2$$

$$\Leftrightarrow a^2-2ab+b^2\ge 0\Leftrightarrow (a-b)^2\ge 0\ \ \textit{(True } \forall\, a,b\textit{)}$$

$\Rightarrow$ *Inequality (3). Therefore:* $4abc+1\ge 2(ab+bc+ca)\qquad (6)$

By Bunhiacopxki we have:

$$\left(2abc+\frac{1}{2}\right)^2=\left(2abc+\frac{1}{4}+\frac{1}{4}\right)^2\le(1^2+1^2+1^2)\left[(2abc)^2+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2\right]$$

$$=3\left(4a^2b^2c^2+\frac{1}{8}\right)$$

$$\Leftrightarrow\frac{(4abc+1)^2}{4}\le 3\left(4a^2b^2c^2+\frac{1}{8}\right)\Leftrightarrow 4a^2b^2c^2+\frac{1}{8}\ge\frac{(4abc+1)^2}{12}\qquad (7)$$

Then (6), (7) $\Rightarrow 4a^2b^2c^2+\frac{1}{8}\ge\frac{(2(ab+bc+ca))^2}{12}=\frac{4(ab+bc+ca)^2}{12}=\frac{(ab+bc+ca)^2}{3}\quad (8)$

By inequality: $(ab + bc + ca)^2 \geq 3abc(a + b + c)$

Let (8) $\Rightarrow 4a^2b^2c^2 + \frac{1}{8} \geq \frac{3abc(a+b+c)}{3} = abc(a + b + c)$

$$\Rightarrow a^4 + b^4 + c^4 + 4a^2b^2c^2 + \frac{1}{8} \geq a^4 + b^4 + c^4 + abc(a + b + c) \quad (9)$$

Then (1), (9). We need to prove:

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \quad (10)$$

$$\Leftrightarrow a^2(a^2 - ab - ac + bc) + b^2(b^2 - bc - ca + ca) + c^2(c^2 - ca - cb + ab) \geq 0$$

$$\Leftrightarrow a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b) \geq 0$$

(True because this is Schur inequality)

Then (9), (10) $\Rightarrow a^4 + b^4 + c^4 + 4a^2b^2c^2 + \frac{1}{8} \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2)$

$\Rightarrow$ ***Inequality (1) True and we get the result***

172. Let a, b, c be positive such that $a + b + c = 3$. Prove that

$$\frac{\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c}}{ab + bc + ca} \geq \sqrt{\frac{ab + bc + ca}{3}}$$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

Solution by Nguyen Ngoc Tu – Ha Giang – Vietnam

Lemma. ***Let*** $x, y, z > 0$ ***such that*** $x^4 + y^4 + z^4 = 3$ ***then***

$$x^5y^5 + y^5z^5 + z^5x^5 \leq 3.$$

Solution Lemma.

Using AM-GM inequality, we have:

$$x \cdot y \cdot 1 \cdot 1 \leq \frac{x^4 + y^4 + 2}{4} = \frac{5 - z^4}{4} \Rightarrow x^5y^5 \leq \frac{5x^4y^4 - x^4y^4z^4}{4}$$

Same, we have $y^5z^5 \leq \frac{5y^5z^5 - x^5y^5z^5}{4}$, $z^5x^5 \leq \frac{5z^4x^4 - x^4y^4z^4}{4}$

$$\Rightarrow x^5y^5 + y^5z^5 + z^5x^5 \leq \frac{5}{4}(x^4y^4 + y^4z^4 + z^4x^4) - \frac{3}{4}x^4y^4z^4$$

Using Schur's inequality: $r \geq \frac{4pq-p^3}{9}$

With $r = x^4y^4z^4, p = x^4 + y^4 + z^4 = 3, q = x^4y^4 + y^4z^4 + z^4x^4 \Rightarrow$

$$\Rightarrow r \geq \frac{4}{3}q - 3$$

$$\Rightarrow x^4y^4 + y^4z^4 + z^4x^4 \leq \frac{5}{4}q - \frac{3}{4}\left(\frac{4}{3}q - 3\right) = \frac{1}{4}q + \frac{9}{4} \leq \frac{1}{4} \cdot \frac{p^2}{3} + \frac{9}{4} = 3$$

Solution problem.

The inequality given is equivalent to

$$\left(\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c}\right)^4 \geq \frac{1}{9}(ab + bc + ca)^6$$

$$\Leftrightarrow \left(\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c}\right)^4 (a + b + c)^{11} \geq \frac{1}{9}(ab + bc + ca)^6 \cdot 3^{11} = (ab + bc + ca)^6 \cdot 3^9$$

Using Holder's inequality, we have

$\left(\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c}\right)^4 (a + b + c)^{11} \geq \left(a^{\frac{4}{5}} + b^{\frac{4}{5}} + c^{\frac{4}{5}}\right)^{15}$ ***hence we need to prove***

$$\left(a^{\frac{4}{5}} + b^{\frac{4}{5}} + c^{\frac{4}{5}}\right)^{15} \geq (ab + bc + ca)^6 \cdot 3^9$$

Inequalities of the same rank hence we assume that $a^{\frac{4}{5}} + b^{\frac{4}{5}} + c^{\frac{4}{5}} = 3,$

then we need to prove $3^{15} \geq (ab + bc + ca)^6 \cdot 3^9 \Leftrightarrow ab + bc + ca \leq 3$

Let $(x, y, z) = \left(a^{\frac{1}{5}}, b^{\frac{1}{5}}, c^{\frac{1}{5}}\right) \Rightarrow x^4 + y^4 + z^4 = 3$ ***hence***

$x^5y^5 + y^5z^5 + z^5x^5 \leq 3$ ***or*** $ab + bc + ca \leq 3.$

Done!

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173. Let $a,b,c>0$ such that: $a+b+c=3$. Prove that:

$$\frac{a^4}{b^4(2ab-\sqrt{c}+2)}+\frac{b^4}{c^4(2bc-\sqrt{a}+2)}+\frac{c^4}{a^4(2ca-\sqrt{b}+2)}\ge\frac{a^2+b^2+c^2}{3}\quad (1)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 Hoang Le Nhat Tung – Hanoi – Vietnam

By Inequality Cauchy – Schwarz. We have:

$$\sum\frac{a^4}{b^4(2ab-\sqrt{c}+2)}=\sum\frac{\left(\frac{a^2}{b^2}\right)^2}{(2ab-\sqrt{c}+2)}\ge\frac{\left(\sum\frac{a^2}{b^2}\right)^2}{\sum(2ab-\sqrt{c}+2)}=\frac{\left(\sum\frac{a^2}{b^2}\right)^2}{2\sum ab-\sum\sqrt{a}+6}\quad (2)$$

Other, by AM-GM:

$$\sum\frac{a^2}{b^2}=\sum\frac{\frac{a^2}{b^2}+\frac{a^2}{b^2}+\frac{b^2}{c^2}}{3}\ge\sum\frac{3\cdot\sqrt[3]{\frac{a^2}{b^2}\cdot\frac{a^2}{b^2}\cdot\frac{b^2}{c^2}}}{3}=\sum\sqrt[3]{\frac{a^4}{b^2c^2}}=\frac{\sum a^2}{\sqrt[3]{a^2b^2c^2}}\quad (3)$$

$$3=a+b+c\ge 3\sqrt[3]{abc}\Rightarrow\sqrt[3]{abc}\le 1\Rightarrow\sqrt[3]{a^2b^2c^2}\le 1.$$

Let (3): $\Rightarrow\sum\frac{a^2}{b^2}\ge\sum a^2$

Let (2): $\Rightarrow\sum\frac{a^4}{b^4(2ab-\sqrt{c}+2)}\ge\frac{\left(\sum a^2\right)^2}{2\sum ab-\sum\sqrt{a}+6}\quad (4)$

By AM-GM and $a+b+c=3$. We have:

$$2\sum\sqrt{a}+\sum a^2=\sum\left(\sqrt{a}+\sqrt{a}+a^2\right)\ge\sum 3\cdot\sqrt[3]{\sqrt{a}\cdot\sqrt{a}\cdot a^2}=$$

$$=3\sum a=9=\left(\sum a\right)^2\Rightarrow\sum\sqrt{a}\ge\sum ab$$

Let (4):

$$\Rightarrow\sum\frac{a^4}{b^4(2ab-\sqrt{c}+2)}\ge\frac{\left(\sum a^2\right)^2}{2\sum ab-\sum ab+6}=\frac{\left(\sum a^2\right)^2}{\sum ab+6}\ge\frac{\left(\sum a^2\right)^2}{\sum a^2+6}\ \text{(because } \sum ab\le\sum a^2\text{)}$$

(5)

We will prove that:

$$\frac{(\sum a^2)^2}{\sum a^2+6}\ge\frac{\sum a^2}{3}\Leftrightarrow\frac{\sum a^2}{\sum a^2+6}\ge\frac{1}{3}\Leftrightarrow 3\sum a^2\ge\sum a^2+6\Leftrightarrow\sum a^2\ge 3$$

(True because by AM-GM: $\sum a^2\ge\frac{(\sum a)^2}{3}=\frac{3^2}{3}=3$***)***

Therefore, let (5): $\Rightarrow\sum\frac{a^4}{b^4(2ab-\sqrt{c}+2)}\ge\frac{\sum a^2}{3}$

$$\Leftrightarrow\frac{a^4}{b^4(2ab-\sqrt{c}+2)}+\frac{b^4}{c^4(2bc-\sqrt{a}+2)}+\frac{c^4}{a^4(2ca-\sqrt{b}+2)}\ge\frac{a^2+b^2+c^2}{3}\Rightarrow \textit{QED}$$

Solution 2 by Anh Tai Tran-Hanoi-Vietnam

$$LHS=\sum\frac{a^4}{b^4(2ab-\sqrt{c}+2)}\ge\frac{\left[\sum\left(\frac{a^2}{b^2}\right)\right]^2}{\sum(2\sum bc-\sum a+2)}\ge\frac{\left(\sum\frac{a^2}{b^2}\right)^2}{\sum bc+6}$$

$$LHS=\frac{a^2+b^2+c^2}{3}\le\frac{(a+b+c)^6}{81(ab+bc+ac)^2}=\frac{9}{(ab+bc+ac)^2}$$

So we are done if:

$$\left(\sum\frac{a^2}{b^2}\right)^2(ab+bc+ac)^2\ge 9\left(\sum bc+6\right)\quad (*)$$

By Cauchy Schwarz

$$\textit{LHS }(*)\ge\left(\sqrt{\frac{a^3}{b}}+\sqrt{\frac{b^3}{c}}+\sqrt{\frac{c^3}{a}}\right)^4$$

$$=\left(\frac{a^2}{\sqrt{ab}}+\frac{b^2}{\sqrt{bc}}+\frac{c^2}{\sqrt{ac}}\right)^4\ge\frac{(a^2+b^2+c^2)^8}{\left(\sum\sqrt{bc}\right)^4}\ge\frac{(a+b+c)^8}{(a+b+c)^4}=81$$

On the other hand,

$$\textit{RHS }(*)\le 9\left(\frac{(\sum a)^2}{3}+6\right)=81$$

So (*) is true

We are done!

174. Let $a,b,c>0$ sucht that: $a+b+c=3$. Prove that:

$$\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\geq\frac{7}{12}\quad(1)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Hoang Le Nhat Tung – Hanoi – Vietnam

We have:

$$b^6+c^6=(b^2+c^2)(b^4-b^2c^2+c^4)=(b^2+c^2)\left[(b^2+c^2)^2-\left(bc\sqrt{3}\right)^2\right]$$

$$=(b^2+c^2)\left(b^2-bc\sqrt{3}+c^2\right)\left(b^2+bc\sqrt{3}+c^2\right)$$

By inequality AM-GM for three positive real numbers:

$$\sqrt[3]{4(b^6+c^6)}=\sqrt[3]{(b^2+c^2)\cdot 2\left(2+\sqrt{3}\right)\left(b^2-bc\sqrt{3}+c^2\right)\cdot 2\left(2-\sqrt{3}\right)\left(b^2+bc\sqrt{3}+c^2\right)}\leq$$

$$\leq\frac{(b^2+c^2)+2\left(2+2\sqrt{3}\right)\left(b^2-bc\sqrt{3}+c^2\right)+2\left(2-\sqrt{3}\right)\left(b^2+bc\sqrt{3}+c^2\right)}{3}$$

$$=\frac{9b^2-12bc+9c^2}{3}$$

$$\Leftrightarrow\sqrt[3]{4(b^6+c^6)}\leq 3b^2-4bc+3c^2\Leftrightarrow\sqrt[3]{4(b^6+c^6)}+7bc\leq 3b^2+3bc+3c^2$$

$$\Leftrightarrow\frac{1}{\sqrt[3]{4(b^6+c^6)}+7bc}\geq\frac{1}{3(b^2+bc+c^2)}\Leftrightarrow\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}\geq\frac{a}{3(b^2+bc+c^2)}\quad(2)$$

Similar:

$$\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}\geq\frac{b}{3(c^2+ca+a^2)}\quad(3)$$

$$\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}\geq\frac{c}{3(a^2+ab+b^2)}\quad(4)$$

Then (2), (3), (4):

$$\Rightarrow\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}\geq$$

$$\geq\frac{a}{3(b^2+bc+c^2)}+\frac{b}{3(c^2+ca+a^2)}+\frac{c}{3(a^2+ab+b^2)}\quad(5)$$

Other, by Cauchy – Schwarz we have

$$\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}=\frac{a^2}{ab^2+abc+ac^2}+\frac{b^2}{bc^2+bca+ba^2}+\frac{c^2}{ca^2+cab+cb^2}\geq$$

$$\geq \frac{(a+b+c)^2}{(ab^2+abc+ac^2)+(bc^2+bca+ba^2)+(ca^2+cab+cb)^2} \quad (6)$$

That $\frac{(a+b+c)^2}{(ab^2+abc+ac^2)+(bc^2+bca+ba^2)+(ca^2+cab+cb^2)}$

$$=\frac{(a+b+c)^2}{ab(a+b)+bc(b+c)+ca(c+a)+3abc}=\frac{(a+b+c)^2}{(a+b+c)(ab+bc+ca)}=\frac{a+b+c}{ab+bc+ca} \quad (7)$$

Then (6), (7): $\Rightarrow \frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}\geq \frac{a+b+c}{ab+bc+ca}$ ***(8)***

And $a+b+c=3$. ***Then (8):***

$$\Rightarrow \frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}\geq \frac{3}{ab+bc+ca} \quad (9)$$

Then (5), (9): $\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}\geq \frac{1}{ab+bc+ca}$ ***(10)***

By AM-GM for five positive real numbers:

$$\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a}+a^2+a^2\geq 5\sqrt[5]{\sqrt[3]{a}\cdot\sqrt[3]{a}\cdot\sqrt[3]{a}\cdot a^2\cdot a^2}=5\sqrt[5]{a^5}=5a$$

$$\Leftrightarrow 3\sqrt[3]{a}+2a^2\geq 5a \Leftrightarrow 3\sqrt[3]{a}\geq 5a-2a^2 \quad (11)$$

Similar: $3\sqrt[3]{b}\geq 5b-2b^2; 3\sqrt[3]{c}\geq 5c-2c^2$ ***(12)***

Then (11), (12): $\Rightarrow 3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c})\geq 5(a+b+c)-2(a^2+b^2+c^2)$

$\Leftrightarrow 3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c})\geq 15-2(a^2+b^2+c^2)$ ***(Because*** $a+b+c=3$***)***

$$\Leftrightarrow 3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1)\geq 18-2(a^2+b^2+c^2)=2(a+b+c)^2-2(a^2+b^2+c^2)$$

(Because $a+b+c=3\Rightarrow 2(a+b+c)^2=18$***)***

$$\Leftrightarrow 3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1)\geq 2(a^2+b^2+c^2+2ab+2bc+2ca)-2(a^2+b^2+c^2)$$

$$\Leftrightarrow 3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1)\geq 4(ab+bc+ca)\Leftrightarrow 3(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c})\geq 4(ab+bc+ca)-3$$

$$\Leftrightarrow \frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\geq \frac{4(ab+bc+ca)-3}{36}\Leftrightarrow \frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\geq \frac{ab+bc+ca}{9}-\frac{1}{12} \quad (13)$$

Then (10), (13):

$$\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\geq$$

$$\geq \frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12} \quad (14)$$

By AM-GM We have:

$$\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}\geq 2\sqrt{\frac{1}{ab+bc+ca}\cdot\frac{ab+bc+ca}{9}}=2\sqrt{\frac{1}{9}}=\frac{2}{3}$$

$$\Rightarrow\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}\geq 2\sqrt{\frac{1}{ab+bc+ca}\cdot\frac{ab+bc+ca}{9}}=2\sqrt{\frac{1}{9}}=\frac{2}{3}$$

$$\Rightarrow\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12}\geq\frac{2}{3}-\frac{1}{12}=\frac{7}{12}\Leftrightarrow\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12}\geq\frac{7}{12}\quad (15)$$

Then (14), (15):

$$\Rightarrow\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\geq\frac{7}{12}$$

⇒ Inequality (1) True and we get the result

Equality occurs if: $\begin{cases} a,b,c>0, a+b+c=3\\ a=b=c\\ \frac{1}{b^2+bc+c^2}=\frac{1}{c^2+ca+a^2}=\frac{1}{a^2+ab+b^2}\Leftrightarrow a=b=c=1\\ \sqrt[3]{a}=a^2;\ \sqrt[3]{b}=b^2;\ \sqrt[3]{c}=c^2\\ \frac{1}{ab+bc+ca}=\frac{ab+bc+ca}{9}\end{cases}$

175. If $x,y\in\mathbb{R}, xy+x+y=1, n\in\mathbb{N}$ then:

$$(1+x)^{2n}\left(\sqrt{\frac{1+y^2}{1+x^2}}\right)^n+(1+y)^{2n}\left(\sqrt{\frac{1+x^2}{1+y^2}}\right)^n\geq 2^{n+1}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si $x,y\in\mathbb{R}$, de tal manera que $xy+y+x=1, n\in\mathbb{N}$. Probar que

$$(1+x)^{2n}\left(\sqrt{\frac{1+y^2}{1+x^2}}\right)^n+(1+y)^{2n}\left(\sqrt{\frac{1+y^2}{1+x^2}}\right)^n\geq 2^{n+1}$$

De la condición

$$xy + y + x = 1 \Leftrightarrow (1+x)(1+y) = 2 \Leftrightarrow (1+x)^2(1+y)^2 = 4$$

Como $(1+x)^2, (1+y)^2, (1+x^2), (1+y^2) > 0$

Aplicando $MA \geq MG$

$$(1+x)^{2n}\left(\sqrt{\frac{1+y^2}{1+x^2}}\right)^n + (1+y)^{2n}\left(\sqrt{\frac{1+y^2}{1+x^2}}\right)^n \geq$$

$$\geq 2\sqrt[2]{\left((1+x)(1+y)\right)^{2n}} = 2\sqrt[2]{4^n} = 2\cdot 2^n = 2^{n+1}$$

(LQQD)

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\left((1+x)^n\cdot\left(\sqrt[4]{\frac{1+y^2}{1+x^2}}\right)^n\right)^2 - 2\cdot 2^n + \left((1+y)^n\left(\sqrt[4]{\frac{1+x^2}{1+y^2}}\right)^n\right)^2 =$$

$$= \left[(1+x)^n\cdot\left(\sqrt[4]{\frac{1+y^2}{1+x^2}}\right)^n - (1+y)^n\cdot\left(\sqrt[4]{\frac{1+x^2}{1+y^2}}\right)^n\right]^2 \geq 0$$

$$2^n = \left((1+x)(1+y)\right)^n\left(\sqrt[4]{\frac{1+y^2}{1+x^2}}\cdot\sqrt[4]{\frac{1+x^2}{1+y^2}}\right)^n =$$

$$= (1+x+y+xy)^n\cdot 1 = (1+1)^n = 2^n$$

$$x = y = \sqrt{2}-1$$

$$x = y = -\sqrt{2}-1$$

Solution 3 by Ravi Prakash-New Delhi-India

$$x + y + xy = 1 \Rightarrow 1 + x + y(1+x) = 2$$

$$\Rightarrow (1+x)(1+y) = 2 \Rightarrow 1 + x = \frac{2}{1+y}$$

Now,

$$(1+x)^{2n}\left(\sqrt{\frac{1+y^2}{1+x^2}}\right)^n+(1+y)^{2n}\left(\sqrt{\frac{1+x^2}{1+y^2}}\right)^n$$

$$\geq 2\left[(1+x)^{2n}\left(\sqrt{\frac{1+y^2}{1+x^2}}\right)^n(1+y)^{2n}\left(\sqrt{\frac{1+x^2}{1+y^2}}\right)^n\right]^{\frac{1}{2}}$$

$$=2\left[\frac{2^{2n}}{(1+y)^{2n}}\cdot(1+y)^{2n}\right]^{\frac{1}{2}}=2^{n+1}$$

Solution 4 by Abdallah El Farissi-Bechar-Algerie

We have $x+y+xy=1$ ***then*** $(1+x)(1+y)=2$ ***let***

$$A=(1+x)^{2n}\left(\sqrt{\frac{1+y^2}{1+x^2}}\right)$$

$$(1+x)^{2n}\left(\sqrt{\frac{1+y^2}{1+x^2}}\right)^n+(1+y)^{2n}\left(\sqrt{\frac{1+x^2}{1+y^2}}\right)^n=$$

$$=(1+x)^{2n}\left(\sqrt{\frac{1+y^2}{1+x^2}}\right)^n+\frac{2^{2n}}{(1+x)^{2n}}\left(\sqrt{\frac{1+x^2}{1+y^2}}\right)^n$$

$$=A+2^{2n}\frac{1}{A}\geq 2\sqrt{A\cdot 2^{2n}\frac{1}{A}}=2^{n+1}$$

176. **If $a,b,c>0, a+b+c=1$ then:**

$$\sum\left(\frac{a^2+bc}{b+c}+\frac{1}{b+2a}\right)\ge 4+\sum\frac{(a-b)(a-c)}{b+c}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $a,b,c>0$ de tal manera que $a+b+c=1$. Probar que

$$\sum\left(\frac{a^2+bc}{b+c}+\frac{1}{b+2a}\right)\ge 4+\sum\frac{(a-b)(a-c)}{b+c}$$

$$\Leftrightarrow\sum\left(\frac{a^2+bc}{b+c}+\frac{1}{b+2a}\right)\ge 4+\sum\frac{a^2+bc}{b+c}-\sum a$$

$$\Leftrightarrow\sum a+\sum\frac{1}{b+2a}\ge 4\Leftrightarrow\sum\frac{1}{b+2a}\ge 3$$

Por la desigualdad de Cauchy

$$\sum\frac{1}{b+2a}=\frac{1}{b+2a}+\frac{1}{c+2b}+\frac{1}{a+2c}\ge\frac{9}{3(a+b+c)}=\frac{3}{a+b+c}=3$$

(LQQD)

Solution 2 by Mohammed Jamal-Oujda-Morocco

$$\sum\left(\frac{a^2+bc}{b+c}+\frac{1}{b+2a}\right)\ge 4+\sum\frac{(a-b)(a-c)}{b+c}$$

ie $\sum\frac{a^2+bc}{b+c}+\sum\frac{1}{b+2a}\ge 4+\sum\frac{a^2+bc}{b+c}-(a+b+c)$

ie $\sum\frac{1}{b+2a}\ge 3$

or $\sum\frac{1}{b+2a}\ge\frac{9}{3(a+b+c)}=3$ ***done***

Solution 3 by Sanong Hauerai-Nakon Pathom-Thailand

We have to prove that

$$\frac{a^2+bc}{b+c}+\frac{1}{b+2a}+\frac{b^2+ca}{c+a}+\frac{1}{c+ab}+\frac{c^2+ab}{a+b}+\frac{1}{a+2c}\geq$$

$$\geq 9+\frac{(a-b)(b-c)}{(b+c)}+\frac{(b-a)(b-c)}{(a+c)}+\frac{(c-a)(c-b)}{(a+b)}$$

consider Right side

$$4+\frac{(a-b)(a-c)}{(b+c)}+\frac{(b-a)(b+c)}{(a+c)}+\frac{(c-a)(c-b)}{(a+b)}=$$

$$=4\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}-(ab+c)$$

$$=3+\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}$$

Hence we have to show that

$\frac{1}{a+2c}+\frac{1}{c+ab}+\frac{1}{b+ca}\geq 3$ ***only***

Because $a+b+c=1$***, we get***

$$(a+2c)(c+2b)(b+ca)=3$$

$$\frac{1}{a+2c}+\frac{1}{c+2b}+\frac{1}{b+ca}\geq 3$$

Therefore it is to be true.

Solution 4 by Nguyen Ngoc Tu-Ha Giang-Vietnam

$$\sum\left(\frac{a^2+bc}{b+c}+\frac{1}{b+2a}\right)\geq 4\sum\frac{(a-b)(a-c)}{b+c}\Leftrightarrow\sum\frac{1}{b+2a}\geq 3$$

It's true by

$$\sum\frac{1}{b+2a}\geq\frac{9}{3(a+b+c)}=3$$

177. If $a,b,c\in\mathbb{R}, x,y,z\in(0,\infty)$ then:

$$(x+y+z)a^2+\left(\frac{1}{x}+\frac{1}{z}\right)b^2+\left(\frac{1}{y}+\frac{1}{z}\right)c^2+2ab+2ac+\frac{2bc}{z}\ge 0$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Dehi-India

$$(x+y+z)a^2+\left(\frac{1}{x}+\frac{1}{z}\right)b^2+\left(\frac{1}{y}+\frac{1}{z}\right)c^2+2ab+2ac+\frac{2bc}{z}=$$

$$=\left(xa^2+\frac{1}{x}b^2+2ab\right)+za^2+\left(ya^2+\frac{1}{y}c^2+2ac\right)+\left(\frac{1}{z}b^2+\frac{1}{z}c^2+\frac{2bc}{z}\right)$$

$$=\left(\sqrt{x}a+\frac{1}{\sqrt{x}}b\right)^2+za^2+\left(\sqrt{y}a+\frac{1}{\sqrt{y}}c\right)^2+\frac{1}{z}(b+c)^2\ge 0$$

178. If $a,b,c,x,y,z\in\mathbb{R}, xyz\neq 0$ then:

$$(a^2+b^2+c^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)+\frac{2(ab+bc+ca)(x+y+z)}{xyz}\ge 0$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$(a^2+b^2+c^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)+\frac{2(ab+bc+ca)(x+y+z)}{xyz}=$$

$$=\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}+\frac{2ab}{xy}+\frac{2bc}{yz}+\frac{2ca}{xz}+$$

$$+\frac{a^2}{y^2}+\frac{b^2}{z^2}+\frac{c^2}{x^2}+\frac{2ab}{yz}+\frac{2bc}{zx}+\frac{2ca}{xy}+$$

$$+\frac{a^2}{z^2}+\frac{b^2}{x^2}+\frac{c^2}{y^2}+\frac{2ab}{zx}+\frac{2bc}{xy}+\frac{2ca}{yz}$$

$$=\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)^2+\left(\frac{a}{y}+\frac{b}{z}+\frac{c}{x}\right)^2+\left(\frac{a}{z}+\frac{b}{x}+\frac{c}{y}\right)^2\geq 0$$

179. If $x,y,z>0$ then:

$$\frac{1}{x+y+z}\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 2$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo $x,y,z>0$. Probar que

$$\frac{1}{x+y+z}\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 2$$

LEMMA

Siendo $a,b,c\geq 0$ se cumple la siguiente desigualdad

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{8abc}{(a+b)(b+c)(c+a)}\geq 2\quad (A)$$

Sustituyendo

$$\to a^2=\frac{xy}{z},b^2=\frac{yz}{x},c^2=\frac{zx}{y}\Leftrightarrow ab=y,bc=z,ca=x\Leftrightarrow x,y,z>0$$

Por lo tanto tenemos en (A)

$$\Rightarrow\frac{1}{x+y+z}\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 2$$

(LQQD)

180. If $a,b,c>0,abc=1$ then:

$$\frac{a^4}{b^4\sqrt{a^4+4}}+\frac{b^4}{c^4\sqrt{b^4+4}}+\frac{c^4}{a^4\sqrt{c^4+4}}\geq\sqrt{\frac{3(a+b+c)}{5}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Hoang Le Nhat Tung – Hanoi – Vietnam

By Inequality Cauchy – Schwarz and AM-GM. We have:

$$\sum \frac{a^4}{b^4\sqrt{5(a^4+4)}} = \sum \frac{\left(\frac{a^2}{b^2}\right)^2}{\sqrt{5(a^2-2a+2)(a^2+2a+2)}} \geq$$

$$\geq \sum \frac{\left(\frac{a^2}{b^2}\right)^2}{\frac{5(a^2-2a+2)+(a^2+2a+2)}{2}}$$

$$\Rightarrow \sum \frac{a^4}{b^4\sqrt{5(a^4+4)}} \geq \sum \frac{\left(\frac{a^2}{b^2}\right)^2}{3a^2-4a+6} \geq \frac{\left(\sum\frac{a^2}{b^2}\right)^2}{\sum(3a^2-4a+6)} = \frac{\left(\sum\frac{a^2}{b^2}\right)^2}{3\sum a^2-4\sum a+18} \quad (1)$$

Other, by AM-GM: $\sum \frac{a^2}{b^2} = \sum \frac{\frac{a^2}{b^2}+\frac{a^2}{b^2}+\frac{b^2}{c^2}}{3} \geq \sum \frac{3\cdot\sqrt[3]{\frac{a^2}{b^2}\cdot\frac{a^2}{b^2}\cdot\frac{b^2}{c^2}}}{3} = \sum \sqrt[3]{\frac{a^4}{b^2c^2}} = \sum a^4$

(because $abc = 1$***)***

Therefore (1):

$$\Rightarrow \sum \frac{a^4}{b^4\sqrt{5(a^4+4)}} \geq \frac{(\sum a^2)^2}{3\sum a^2-4\sum a+18} \geq \frac{(\sum a^2)^2}{3\sum a^2-4\cdot 3\sqrt[3]{abc}+18} \geq$$

$$\geq \frac{(\sum a^2)^2}{3\sum a^2+2\sum a^2} = \frac{\sum a^2}{5} \quad (2)$$

We have: $\frac{\sum a^2}{5} \geq \frac{(\sum a)^2}{3\cdot 5} = \frac{(\sum a)^2}{15} = \frac{\sqrt{\sum a}\cdot\sqrt{(\sum a)^3}}{15} \geq \frac{\sqrt{\sum a}\cdot\sqrt{27abc}}{15} = \frac{\sqrt{27abc}}{15} = \frac{\sqrt{3\sum a}}{5} \quad (3)$

Then (2), (3): $\Rightarrow \frac{a^4}{b^4\sqrt{a^4+4}} + \frac{b^4}{c^4\sqrt{b^4+4}} + \frac{c^4}{a^4\sqrt{c^4+4}} \geq \sqrt{\frac{3(a+b+c)}{5}} \Rightarrow$ ***QED.***

181. Let a, b, c be positive real numbers such that $abc \leq 1$. Prove that

$$\frac{3}{a+b+c} + \frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq 2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Hurmey – Peru

Siendo a, b, c números R^+ de tal manera que $abc \le 1$. Probar que

$$\frac{3}{a+b+c}+\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\ge 2$$

Como $a, b, c > 0$

Aplicando MA ≥ MG

$$\frac{a}{b}+\frac{a}{b}+\frac{b}{c}\ge 3\sqrt[3]{\frac{a^2}{bc}}=3\sqrt[3]{\frac{a^3}{abc}}=\frac{3a}{\sqrt[3]{abc}}\ge 3a \quad \textbf{(A)}$$

De forma análoga

$$\frac{b}{c}+\frac{b}{c}+\frac{c}{a}\ge 3b \quad \textbf{(B)}$$

$$\frac{c}{a}+\frac{c}{a}+\frac{a}{b}\ge 3c \quad \textbf{(C)}$$

Sumando (A) + (B) + (C)

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\ge 3(a+b+c)\Leftrightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a+b+c$$

Por último, nuevamente por MA ≥ MG

$$\frac{3}{a+b+c}+\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\ge \frac{3}{a+b+c}+\frac{1}{3}(a+b+c)\ge 2\sqrt{\frac{3}{a+b+c}\cdot\frac{a+b+c}{3}}=2$$

(LQQD)

182. If $a, b, c > 0$ then:

$$\frac{1}{(2a^2+bc)^2}+\frac{1}{(2b^2+ca)^2}+\frac{1}{(2c^2+ab)^2}\ge\frac{(a+b+c)^2}{9(a^6+b^6+c^6)}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

We have:

$$3(a^6+b^6+c^6) \ge (a^4+b^4+c^4)(a^2+b^2+c^2) \ge (a^4+b^4+c^4)\cdot\frac{(a+b+c)^2}{3}$$

$$\Rightarrow \frac{(a+b+c)^2}{9(a^6+b^6+c^6)} \le \frac{1}{a^4+b^4+c^4} \quad \textbf{(1)}$$

By Cauchy – Scwarz:

$$\sum\frac{1}{(2a^2+bc)^2} \ge \frac{\left(\sum\frac{1}{2a^2+bc}\right)^2}{3} \ge \frac{\left(\frac{9}{\sum(2a^2+bc)}\right)^2}{3}$$

$$\Rightarrow \sum\frac{1}{(2a^2+bc)^2} \ge \frac{27}{(2\sum a^2+\sum bc)^2} \ge \frac{27}{(2\sum a^2+\sum a^2)^2} = \frac{27}{9(\sum a^2)^2}$$

$$\Rightarrow \sum\frac{1}{(2a^2+bc)^2} \ge \frac{3}{(\sum a^2)^2} \ge \frac{3}{3\sum a^4} = \frac{1}{\sum a^4} \quad \textbf{(2)}$$

Then (1), (2) $\Rightarrow \sum\frac{1}{(2a^2+bc)^2} \ge \frac{(\sum a)^2}{9\cdot\sum a^6}$

$$\Rightarrow Q.E.D$$

Solution 2 by Sanong Hauerai-Nakon Pathom-Thailand

Because $(a^3+b^3+c^3)(a+b+c) \ge 2(a^2b^2+b^2c^2+c^2a^2)+a^3b+b^3c+c^2a$ ***is to be true***

Imply

$$(a^3+b^3+c^3)^2(a+b+c)^2 \ge (2(a^2b^2)+(b^2c^2)+(c^2a^2)+a^3b+b^3c+c^3a)^2$$

Imply

$$9(a^3+b^3+c^3)^2(a+b+c)^2 \ge 9(2(a^2b^2+b^2c^2+c^2a^2)+a^3b+b^3c+c^3a)^2$$

Imply

$$9(a^6+b^6+c^6)(a+b+c)^2 \ge 3(2(a^2b^2+b^2c^2+c^2a^2)+a^3b+b^3c+c^3a)^2$$

$$\textbf{\textit{Imply}}\ \frac{\frac{(a+b+c)^2}{\left(2(a^2b^2+b^2c^2+c^2a^2)+a^3b+b^3c+c^3a\right)^2}}{3} \ge \frac{1}{9(a^6+b^6+c^6)}$$

$$\textbf{\textit{Imply}}\ \frac{\left(\frac{(a+b+c)^2}{2(a^2b^2+b^2c^2+c^2a^2+a^3b+b^3c+c^3a)}\right)^2}{3} \ge \frac{(a+b+c)^2}{9(a^6+b^6+c^6)}$$

$$\textbf{Imply } \frac{\left(\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\right)^2}{3} \geq \frac{(a+b+c)^2}{9(a^6+b^6+c^6)}$$

$$\textbf{Imply } \frac{1}{(2a^2+bc)^2}+\frac{1}{(2b^2+ca)^2}+\frac{1}{(2c^2+ab)^2} \geq \frac{(a+b+c)^2}{9(a^6+b^6+c^6)}$$

There for it is to be true

Solution 3 by Nguyen Thanh Nho-Tra Vinh-Vietnam

$$LHS=\frac{1}{(2a^2+bc)^2}+\frac{1}{(2b^2+ca)^2}+\frac{1}{(2c^2+ab)^2} \geq \frac{1}{3}\left(\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\right)^2$$

$$\geq \frac{1}{3}\left(\frac{9}{2(a^2+b^2+c^2)+ab+bc+ca}\right)^2 \geq \frac{1}{3}\left(\frac{9}{3(a^2+b^2+c^2)}\right)^2=\frac{3}{(a^2+b^2+c^2)^2} \quad (*)$$

$$a^6+b^6+c^6 \geq 3\left(\frac{a^2+b^2+c^2}{3}\right)^3=\frac{(a^2+b^2+c^2)^2}{9}\cdot(a^2+b^2+c^2)$$

$$\geq \frac{(a^2+b^2+c^2)}{9}\cdot\frac{1}{3}(a+b+c)^2$$

$$\Rightarrow \frac{3}{(a^2+b^2+c^2)^2} \geq \frac{(a+b+c)^2}{9(a^6+b^6+c^6)}=RHS \quad (**)$$

$$(*)\ \&\ (**) \Rightarrow LHS=\frac{3}{(a^2+b^2+c^2)} \geq RHS \Rightarrow LHS \geq RHS$$

$$\text{“} = \text{“} \Leftrightarrow a=b=c$$

Solution 4 by SK Rejuan-West Bengal-India

$$a,b,c>0$$

$$\textbf{by } AM \geq GM \textbf{ we get, } 2a^2+\frac{b^2+c^2}{2} \geq 2a^2+bc$$

$$\Rightarrow \frac{4a^2+b^2+c^2}{2} \geq 2a^2+bc \Rightarrow \sum\frac{4a^2+b^2+c^2}{2} \geq \sum(2a^2+bc)$$

$$\Rightarrow 3\sum a^2 \geq \sum(2a^2+bc) \Rightarrow \frac{1}{\sum(2a^2+bc)} \geq \frac{1}{3\sum a^2} \quad (1)$$

$$LHS=\sum\left(\frac{1}{2a^2+bc}\right)^2=P \ \text{(say)}$$

$$\Rightarrow P \ge \frac{3}{9}\left(\sum \frac{1}{2a^2+bc}\right)^2$$ ***[by mth power theorem]***

$$\ge \frac{1}{3}\left\{\frac{9}{\sum(2a^2+bc)}\right\}^2$$ ***[by AM ≥ HM]*** $$\ge \frac{1}{3}\left\{\frac{9}{3\sum a^2}\right\}^2$$ ***[from (1)]***

$$\Rightarrow P \ge \frac{3}{(\sum a^2)^2} \ge \frac{1}{\sum a^4}$$ ***[by Cauchy inequality]*** $$\Rightarrow P \ge \frac{1}{\sum a^4} \quad (2)$$

$$RHS = \frac{1}{9}\frac{(\sum a)^2}{\sum a^6} = 9 \quad \textit{(say)}$$

$$\Rightarrow 9 = \frac{1}{9}\frac{(\sum a)^2}{\sum a^6} \le \frac{1}{9}\cdot\frac{3\sum a^2}{\sum a^6}$$ ***[by Cauchy inequality]*** $$\Rightarrow 9 \le \frac{1}{3}\cdot\frac{\sum a^2}{\sum a^6} \quad (3)$$

We now have to prove that,

$$\frac{1}{3}\cdot\frac{\sum a^2}{\sum a^6} \le \frac{1}{\sum a^4} \Leftrightarrow \frac{\sum a^2}{3}\cdot\frac{\sum a^4}{3} \le \frac{\sum a^6}{3} = \frac{\sum a^{2+9}}{3}$$

$$\Leftrightarrow \frac{\sum a^2}{3}\cdot\frac{\sum a^4}{3} \le \frac{\sum a^6}{3} = \frac{\sum a^{2+4}}{3}$$ ***which is true i.e.,*** $$\frac{1}{3}\cdot\frac{\sum a^2}{\sum a^6} \le \frac{1}{\sum a^4} \quad (4)$$

Combining (2), (3) & (4) we get $9 \le \frac{1}{3}\cdot\frac{\sum a^2}{\sum a^6} \le \frac{1}{\sum a^4} \le P \Rightarrow P \ge 9$

$$\Rightarrow \sum\left(\frac{1}{2a^2+bc}\right)^2 \ge \frac{1}{9}\cdot\frac{(\sum a)^2}{\sum a^6}$$

183. If $x, y, z > 0$ then:

$$\frac{x}{(x+\sqrt{y})(x+\sqrt[3]{z})} + \frac{\sqrt{y}}{(\sqrt{y}+\sqrt[3]{z})(\sqrt{y}+x)} + \frac{\sqrt[3]{z}}{(\sqrt[3]{z}+x)(\sqrt[3]{z}+\sqrt{y})} \le \frac{3}{4\sqrt[18]{x^6y^3z^2}}$$

Proposed by Daniel Sitaru – Romania

Solution by Nguyen Minh Tri-Ho Chi Minh-Vietnam

Suppose $x = a, \sqrt{y} = b, \sqrt[3]{2} = c$***. So we need to prove that***

$$\frac{a}{(a+b)(a+c)} + \frac{b}{(b+a)(b+c)} + \frac{c}{(c+a)(c+b)} \le \frac{3}{4\sqrt[18]{a^6b^6c^6}}$$

$$\Leftrightarrow \frac{a(b+c)+b(a+c)+c(a+b)}{(a+b)(b+c)(a+c)} \le \frac{3}{4\sqrt[3]{abc}}$$

$$\Leftrightarrow \frac{2(ab+bc+ac)}{(a+b)(b+c)(a+c)} \le \frac{4}{4\sqrt[3]{abc}} \quad (*)$$

We use this inequality:

$$8(ab+bc+ac)(a+b+c) \le 9(a+b)(b+c)(a+c)$$

$$\Leftrightarrow \frac{2(ab+bc+ac)}{(a+b)(b+c)(a+c)} \le \frac{9}{4(a+b+c)} \quad (1)$$

Use Cauchy for 3 numbers: $a+b+c \ge 3\sqrt[3]{abc}$

$$\Leftrightarrow \frac{9}{4(a+b+c)} \le \frac{3}{4\sqrt[3]{abc}} \quad (2)$$

From (1), (2) $\Rightarrow \frac{2(ab+bc+ac)}{(a+b)(b+c)(a+c)} \le \frac{3}{4\sqrt[3]{abc}} \Rightarrow$ ***(*) true*** $\Rightarrow$ ***Q.E.D.***

184. For $a, b, c > 0$. Prove:

$$\frac{(a+b)a^3}{a^2+ab+b^2} + \frac{(b+c)b^3}{b^2+bc+c^2} + \frac{(c+a)c^3}{c^2+ca+a^2} \ge \frac{2(a+b+c)^2}{9}$$

Proposed by Nho Nguyen Van-Nghe An-Vietnam

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

We have: $\sum \frac{(a+b)a^3}{a^2+ab+b^2} - \sum \frac{b^3(a+b)}{a^2+ab+b^2} = \sum \frac{a^3(a+b)-b^3(a+b)}{a^2+ab+b^2} = \sum \frac{a^4-b^4+ab(a^2-b^2)}{a^2+ab+b^2}$

$$= \sum \frac{(a^2-b^2)(a^2+ab+b^2)}{a^2+b^2+ab} = \sum (a^2-b^2) = 0 \Rightarrow$$

$$\Rightarrow \sum \frac{(a+b)a^3}{a^2+ab+b^2} = \sum \frac{b^3(a+b)}{a^2+ab+b^2} \Rightarrow \sum \frac{(a+b)a^3}{a^2+ab+b^2} =$$

$$= \frac{1}{2}\sum \frac{a^3(a+b)+b^3(a+b)}{a^2+ab+b^2} = \frac{1}{2}\sum \frac{(a^3+b^3)(a+b)}{a^2+ab+b^2} \Rightarrow$$

$$\Rightarrow \sum \frac{(a+b)a^3}{a^2+ab+b^2} \ge \frac{1}{2}\sum \frac{(a^2+b^2)^2}{a^2+ab+b^2} \ge \frac{1}{2}\sum \frac{(a^2+b^2)^2}{a^2+\frac{a^2+b^2}{2}+b^2} = \sum \frac{a^2+b^2}{3}$$

$$\Rightarrow \sum \frac{(a+b)a^3}{a^2+ab+b^2} \ge \frac{2}{3}\sum a^2 \ge \frac{2}{3}\cdot\frac{(\sum a)^2}{3} = \frac{2(\sum a)^2}{9} \Rightarrow$$

$$\Rightarrow \frac{(a+b)a^3}{a^2+ab+b^2} + \frac{(b+c)b^3}{b^2+bc+c^2} + \frac{(c+a)c^3}{c^2+ca+a^2} \ge \frac{2(a+b+c)^2}{9} \Rightarrow \textbf{Q.E.D.}$$

Solution 2 by Do Quoc Chinh-Ho Chi Minh-Vietnam

By the Cauchy-Schwarz inequality, we have

$$\left(\sum \frac{a^3(a+b)}{a^2+ab+b^2}\right)\left(\sum \frac{a(a^2+ab+b^2)}{a+b}\right) \ge (a^2+b^2+c^2)^2$$

We have: $\sum \frac{a(a^2+ab+b^2)}{a+b} = \sum \frac{a[(a+b)^2-ab]}{a+b} = \sum a(a+b) - \sum \frac{a^2b}{a+b}$

By the Cauchy – Schwarz inequality, we have:

$$\sum \frac{a^2b}{a+b} = \sum \frac{a^2b^2}{ab+b^2} \ge \frac{(\sum ab)^2}{\sum a^2 + \sum ab} \Rightarrow \sum \frac{a(a^2+ab+b^2)}{a+b} \le$$

$$\le \sum a(a+b) - \frac{(\sum ab)^2}{\sum a^2+\sum ab} = \sum a^2 + \sum ab - \frac{(\sum ab)^2}{\sum a^2+\sum ab} =$$

$$= \frac{(\sum a^2+\sum ab)^2 - (\sum ab)^2}{\sum a^2+\sum ab} = \frac{(\sum a^2)^2 + 2(\sum a^2)(\sum ab)}{\sum a^2+\sum ab} = \frac{(\sum a^2)(\sum a)^2}{\sum a^2+\sum ab}$$

$$\Rightarrow LHS \ge \frac{(a^2+b^2+c^2)^2}{\sum \frac{a(a^2+ab+b^2)}{a+b}} \ge \frac{(\sum a^2+\sum ab)(\sum a^2)}{(\sum a)^2} \ge \frac{(\sum a^2+\sum ab)(\sum a)^2}{3(\sum a)^2} =$$

$= \frac{\sum(a+b)^2}{6} \ge \frac{4(\sum a)^2}{18} = \frac{2(\sum a)^2}{9}$. ***The equality holds for*** $a=b=c$.

Solution 3 by Le Khanh Sy-Long An-Vietnam

$$\frac{a^3}{a^2+ab+b^2} - \frac{2a-b}{3} = \frac{(a-b)^2(a+b)}{3(a^2+ab+b^2)}$$

$$\Rightarrow \sum_{cyc}(a+b)f(a,b) \ge \sum_{cyc}\frac{(a+b)(2ab-b)}{3} = \sum_{cyc}\frac{a^2+ab}{3} \ge \frac{2}{9}\left(\sum a\right)^2$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\textbf{LHS} = \sum \frac{a^2(a^2+ab+b^2-b^2)}{a^2+ab+b^2} = \sum a^2 - \sum \frac{a^2b^2}{a^2+ab+b^2} \overset{Chebyshev}{\ge} \frac{(\sum a)^2}{3} - \sum \frac{a^2b^2}{a^2+ab+b^2} \overset{?}{\ge} \frac{2(\sum a)^2}{9} \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{a^2b^2}{a^2+ab+b^2} \le \frac{1}{9}(\sum a)^2 \to \textbf{(1)}$$

$$\because a^2+ab+b^2\overset{A-G}{\geq}3ab\therefore\sum\frac{a^2b^2}{a^2+ab+b^2}\leq\frac{1}{3}\sum ab\leq\frac{1}{9}\left(\sum a\right)^2$$

$(\because (\sum a)^2\geq 3\sum ab)\Rightarrow$ ***(1) is true (proved)***

Solution 5 by Nguyen Thanh Nho-Tra Vinh-Vietnam

$$LHS=\sum\left(\frac{(a+b)a^3}{a^2+ab+b^2}-a^2\right)+\sum a^2=$$

$$=-\sum\frac{a^2b^2}{a^2+ab+b^2}+\sum a^2\geq-\sum\frac{ab}{3}+\sum a^2\geq$$

$$\geq-\frac{(a+b+c)^2}{9}+\frac{(a+b+c)^2}{3}=\frac{2(a+b+c)^2}{9}=RHS$$

185. For $a,b,c>0$. Prove:

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{a\sqrt[3]{ab^2}}{a+b}+\frac{b\sqrt[3]{bc^2}}{b+c}+\frac{c\sqrt[3]{ca^2}}{c+a}$$

Proposed by Nho Nguyen Van-Nghe An-Vietnam

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

$$\sum\frac{a^2}{a+b}\geq\sum\frac{a\sqrt[3]{ab^2}}{a+b}$$

We have: $\sum\frac{a^2}{a+b}-\sum\frac{a\cdot\sqrt[3]{ab^2}}{a+b}=\sum\frac{a\left(a-\sqrt[3]{ab^2}\right)}{a+b}\geq\sum\frac{a\left(a-\frac{a+b+b}{3}\right)}{a+b}\Rightarrow$

$\Rightarrow\sum\frac{a^2}{a+b}-\sum\frac{a\cdot\sqrt[3]{ab^2}}{a+b}\geq\frac{1}{3}\sum\frac{2a(a-b)}{a+b}$ ***(1)***

We prove: $\sum\frac{a(a-b)}{a+b}\geq 0\leftrightarrow\sum\frac{a(a+b-2b)}{a+b}\geq 0\leftrightarrow\sum a\geq 2\sum\frac{ab}{a+b}$ ***(2)***

Other: $\sum\frac{ab}{a+b}\leq\sum\frac{(a+b)^2}{4(a+b)}=\sum\frac{a+b}{4}=\frac{\sum a}{2}\to\sum a\geq 2\sum\frac{ab}{a+b}$

$\to$ ***(2) true*** $\to$ ***(1)*** $\Rightarrow\sum\frac{a^2}{a+b}-\sum\frac{a\left(\sqrt[3]{ab^2}\right)}{a+b}\geq 0\Rightarrow\sum\frac{a^2}{a+b}\geq\sum\frac{a\cdot\sqrt[3]{ab^2}}{a+b}$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{ab^2}\overset{GM\leq AM}{\leq}\frac{a+2b}{3}\therefore\frac{a\sqrt[3]{ab^2}}{a+b}\leq\frac{a(a+2b)}{3(a+b)}=\frac{a(a+b+b)}{3(a+b)}$$

$= \frac{a}{3} + \frac{ab}{3(a+b)} \overset{(1)}{\leq} \frac{a}{3} + \frac{\sqrt{ab}}{6}$ $\left(\because a + b \overset{A-G}{\geq} 2\sqrt{ab}\right)$. ***Similarly,*** $\frac{b\sqrt[3]{bc^2}}{b+c} \overset{(2)}{\leq} \frac{b}{3} + \frac{\sqrt{bc}}{6}$ ***and,***

$$\frac{c\sqrt[3]{ca^2}}{c+a} \overset{(3)}{\leq} \frac{c}{3} + \frac{\sqrt{ca}}{6}$$

(1)+(2)+(3)$\Rightarrow RHS \leq \frac{\sum a}{3} + \frac{\sum\sqrt{ab}}{6} \overset{C-B-S}{\leq} \frac{\sum a}{3} + \frac{\sum a}{6} = \frac{\sum a}{2} \to$ ***(a)***

Again, $LHS \overset{Bergstrom}{\geq} \frac{(\sum a)^2}{2\sum a} = \frac{\sum a}{2} \to$ ***(b)***

(a), (b) $\Rightarrow LHS \geq RHS$ ***(Proved)***

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{a^2}{a+b} = \sum_{cyc} a - \sum_{cyc}\frac{ab}{a+b} \overset{AM\geq GM}{\geq} \sum_{cyc} a - \frac{1}{2}\sum_{cyc}\sqrt{ab} \geq \sum_{cyc} a - \frac{1}{2}\sum_{cyc} a$$

$$= \frac{a+b+c}{2}.$$

$$\sum_{cyc}\frac{a\sqrt[3]{ab^2}}{a+b} \overset{AM\geq GM}{\leq} \frac{1}{3}\sum_{cyc}\frac{a(2b+a)}{a+b} = \frac{1}{3}\sum_{cyc}\frac{ab}{a+b} + \frac{a+b+c}{3} \leq \sum_{cyc}\frac{\sqrt{ab}}{6} + \frac{a+b+c}{3}$$

We need to show that, $\frac{a+b+c}{2} \geq \frac{a+b+c}{3} + \frac{1}{6}\sum_{cyc}\sqrt{ab} \Leftrightarrow a + b + c \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$***,***

which is true. $\therefore \sum_{cyc}\frac{a^2}{a+b} \geq \sum_{cyc}\frac{a\sqrt[3]{ab^2}}{a+b}$ ***(proved)***

186. If a, b, c are positive real number such that $ab + bc + ca = 3$ then

$$\frac{1}{a^3 + b^2 + c} + \frac{1}{b^3 + c^2 + a} + \frac{1}{c^3 + a^2 + b} \leq 1$$

Proposed by Pham Quoc Sang-Ho Chi Minh-Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{1}{a^3 + b^2 + c} \overset{Holder's\ Inequality}{\leq} \sum_{cyc}\frac{3(1 + b + c^2)}{(a+b+c)^3}$$

we need to prove, $(a+b+c)^3 \geq 3\left(3\sum_{cyc} a^2 + \sum_{cyc} a\right) = 3(p^2 + p - 3)$

$$\left[where\ p = a + b + c\ and \sum_{cyc} a^2 = p^2 - 6\right]$$

$p^3 \geq 3(p^2 + p - 3) \Leftrightarrow (p - 3)(p^2 - 3) \geq 0$, **which is true** $\because p \geq 3$

$\therefore \sum_{cyc} \frac{1}{a^3+b^2+c} \leq$ ***(proved)***

187. If a, b, c be positive real number such that $a + b + c = 3$ then

$$\frac{ab}{(2a + bc)(2b + ca)} + \frac{bc}{(2b + ca)(2c + ab)} + \frac{ca}{(2c + ab)(2a + bc)} \leq \frac{1}{3}$$

Proposed by Pham Quoc Sang-Ho Chi Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$LHS = \frac{ab(2c+ab)+bc(2a+bc)+ca(2b+ca)}{(2a+bc)(2b+ca)(2c+ab)} \leq \frac{1}{3} \Leftrightarrow 3\left(6abc + \sum a^2b^2\right) \leq$$

$$\leq 4\sum a^2b^2 + 8abc + 2abc\left(\sum a^2\right) + a^2b^2c^2 \Leftrightarrow \sum a^2b^2 + 2abc\left(\sum a^2\right) + a^2b^2c^2 \geq$$

$$\geq 10abc \Leftrightarrow 3\sum a^2b^2 + 6abc\left(\sum a^2\right) + 3a^2b^2c^2 \geq 30abc \quad \textit{(1). Now,}$$

$a^2b^2 + a^2b^2c^2 = a^2b^2(1 + c^2) \overset{A-G}{\geq} 2ca^2b^2$. ***Similarly,*** $b^2c^2 + a^2b^2c^2 \geq 2ab^2c^2$ ***and*** $c^2a^2 + a^2b^2c^2 \geq 2bc^2a^2$. ***Adding the last 3 inequalities, we get***

$$\sum a^2b^2 + 3a^2b^2c^2 \geq 2abc\left(\sum ab\right) \quad \textit{(2)}$$

(2) $\Rightarrow$ in order to prove (1), it suffices to prove:

$$2\sum a^2b^2 + 6abc\left(\sum a^2\right) + 2abc\left(\sum ab\right) \geq$$

$$\geq 30abc \Leftrightarrow \sum a^2b^2 + 3abc\left(\sum a^2\right) + abc\left(\sum ab\right) \geq 15abc \quad \textit{(3). Now, LHS of (3)} \geq$$

$$\geq abc\left(\sum a\right) + 3abc\left(\sum a^2\right) + abc\left(\sum ab\right) \overset{?}{\geq} 15abc$$

$$\Leftrightarrow \sum a + 3\sum a^2 + \sum ab \overset{?}{\geq} 15$$

$$\Leftrightarrow \frac{1}{3}\left(\sum a\right)^2 + 3\sum a^2 + \sum ab \overset{?}{\geq} \frac{15}{9}\left(\sum a\right)^2 \left(\because \sum a = 3\right) \Leftrightarrow 9\sum a^2 + 3\sum ab \overset{?}{\geq}$$

$\geq 4\left(\sum a\right)^2 = 4\sum a^2 + 8\sum ab \Leftrightarrow 5\sum a^2 \overset{?}{\geq} 5\sum ab \rightarrow$ ***true $\Rightarrow$ (3) is true (proved)***

188. Let $a, b, c \geq -1$ be real numbers with $a^3 + b^3 + c^3 = 1$. Prove that:

$$a + b + c + a^2 + b^2 + c^2 \leq 4.$$

When does equality holds?

Sweden NMO

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

$$a, b, c \geq -1 \Rightarrow (a+1)(a-1)^2 + (b+1)(b-1)^2 + (c+1)(c-1)^2 \geq 0$$

$$\Leftrightarrow (a+1)(a^2-2a+1) + (b+1)(b^2-2b+1) + (c+1)(c^2-2c+1) \geq 0$$

$$\Leftrightarrow a^3 + b^3 + c^3 + 3 \geq a^2 + b^2 + c^2 + a + b + c \Rightarrow a^2 + b^2 + c^2 + a + b + c \leq 4$$

(because $a^3 + b^3 + c^3 = 1$) $\Leftrightarrow a = 1; b = 1; c = -1.$

Solution 2 by Sarah El-Kenitra-Morocco

We have $a^3 + 1 - a^2 - a = (a+1)(a-1)^2 \geq 0$. Then $a^2 + a \leq a^3 + 1$

Using the same think we get $b^2 + b \leq b^3 + 1$ and $c^2 + c \leq c^3 + 1$. Therefore

$a + b + c + a^2 + b^2 + c^2 \leq a^3 + b^3 + c^3 + 3 = 4$. Equality holds when

$a = b = 1, c = -1$ or $a = c = 1, b = -1$ or $b = c = 1, a = -1$

Solution 3 by Soumava Chakraborty-Kolkata-India

$\forall a, b, c \geq -1 | \sum a^3 = 1$, we have: $\sum a + \sum a^2 \leq 4$

Let $a + 1 = x, b + 1 = y, c + 1 = z$. Now, $\sum a^3 = 1 \Rightarrow \sum (x-1)^3 = 1 \Rightarrow$

$\Rightarrow \sum x^3 - 3\sum x^2 + 3\sum x = 4 \rightarrow$ (1). Now, $\sum a + \sum a^2 \leq 4 \Leftrightarrow \sum (x-1) + \sum (x-1)^2 \leq$

$\leq \sum x^3 - 3\sum x^2 + 3\sum x$ (using (1)) $\Leftrightarrow \sum x - 3 + \sum x^2 - 2\sum x + 3 \leq \sum x^3 - 3\sum x^2 +$

$$+3\sum x \Leftrightarrow \sum x^3 - 4\sum x^2 + 4\sum x \geq 0 \Leftrightarrow \sum x(x^2 - 4x + 4) \geq 0 \Leftrightarrow$$

$\Leftrightarrow \sum x(x-2)^2 \geq 0 \rightarrow$ true $\because x = a + 1 \geq 0$, etc. (Proved)

Equality occurs when $(a = 1, b = 1, c = -1)$ or $(a = -1, b = 1, c = 1)$ or

$(a = 1, b = -1, c = 1)$

189. If $a_1, a_2, \dots, a_n \in (0,1)$ then:

$$\sqrt[2016]{\prod_{k=1}^{2017} a_k} + \sqrt[2016]{\prod_{k=1}^{2017} (1-a_k)} < 1$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Daniel Sitaru-Romania

$$x = \prod_{k=1}^{2017} a_k, y = \prod_{k=1}^{2017} (1-a_k), x, y \in (0,1) \rightarrow$$

$$\rightarrow \sqrt[2016]{x} + \sqrt[2016]{y} < \sqrt[2017]{x} + \sqrt[2017]{y} = \sqrt[2017]{\prod_{k=1}^{2017} a_k} + \sqrt[2017]{\prod_{k=1}^{2017} (1-a_k)} \le$$

$$\overset{MAHLER}{\leq} \sqrt[2017]{\prod_{k=1}^{2017} (a_k + 1 - a_k)} = 1$$

190. If $x, y, z > 0, x + y + z = 1$ then:

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} + 2\left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right) \ge 3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mehmet Sahin-Ankara-Turkey

$$x, y, z > 0, x + y + z = 1$$

$$L = \frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} + 2\left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right) \overset{?}{\ge} 3$$

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} \ge \frac{(x+y+z)^2}{x+y+z} = (x+y+z) = 1 \quad (*)$$

$$2\left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right) = 2\left(\frac{(xy)^2}{xyz} + \frac{(yz)^2}{xyz} + \frac{(zx)^2}{xyz}\right) \ge 2\frac{(xy+yz+zx)^2}{3xyz} \quad (**)$$

$$(xy + yz + zx)^2 = x^2y^2 + y^2z^2 + z^2x^2 + 2xyz\underbrace{(x + y + z)}_{1}$$

$$(xy)^2 + (yz)^2 + (zx)^2 \ge (xy)(yz) + (yz)(zx) + (zx)(xy) \ge (xyz)\underbrace{(x + y + z)}_{1} = xyz \quad (***)$$

From (*), (**) and (***)

$L = \frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} + 2\left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right)$, $L \ge 1 + 2\left(\frac{xyz+2xyz}{xyz}\right)$, $L \ge 3$ ***as desired.***

Solution 2 by Kunihiko Chikaya-Tokyo-Japan

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} \ge \frac{(x+y+z)^2}{x+y+z} = x + y + z \quad (1)$$

$$\frac{yz}{x} + \frac{zx}{y} \ge 2\sqrt{\frac{yz}{x} \cdot \frac{zx}{y}} = 2z$$

$$\frac{zx}{y} + \frac{xy}{z} \ge 2x, \frac{xy}{z} + \frac{yz}{x} \ge 2y, \ldots\ldots\ldots\ldots (+)$$

$$2\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right) \ge 2(x + y + z) \quad (2)$$

$$\therefore (1) \text{ and } (2) \ge 3(x + y + z) = 3$$

Solution 3 by Uche Eliezer Okeke-Anambra-Nigeria

$$x, y, z > 0 \wedge x + y + z = 1$$

$$\textit{prove} \sum \frac{x^2}{z} + 2\sum \frac{xy}{z} \ge 3$$

$$\textbf{Known: } \left(\sum xy\right)^2 \ge 3xyz(x + y + z) \quad (a)$$

$$\sum \frac{x^2}{z} \cdot \sum z \ge \left(\sum x\right)^2 \to \sum \frac{x^2}{z} \ge 1 \quad (1)$$

$$2\sum \frac{xy}{z} = 2xyz\sum \frac{1}{z^2} \ge 2xyz \cdot \frac{1}{3}\left(\sum \frac{1}{x}\right)^2 = \frac{2}{3} \cdot \frac{(\sum xy)^2}{xyz} \overset{(a)}{\ge} 2\sum x = 2 \to 2\sum \frac{xy}{z} \ge 2 \quad (2)$$

(1)+(2) completes the proof!

191. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that:

$$\frac{ab}{\sqrt[4]{a(a^2 + 2b + 3)}} + \frac{bc}{\sqrt[4]{b(b^2 + 2c + 3)}} + \frac{ca}{\sqrt[4]{c(c^2 + 2a + 3)}} \le \sqrt[4]{\frac{9(ab + bc + ca)}{2}}$$

Proposed by Do Quoc Chinh-Vietnam

Solution by proposer

Applying the Hölder's inequality, we have:

$LHS^4 \leq (ab+bc+ca)\left(b\sqrt{ab}+c\sqrt{bc}+a\sqrt{ca}\right)^2\left(\frac{a}{a^2+2b+3}+\frac{b}{b^2+2c+3}+\frac{c}{c^2+2a+3}\right)$. *Therefore, we need to prove that:*

$\left(b\sqrt{ab}+c\sqrt{bc}+a\sqrt{ca}\right)^2\left(\frac{a}{a^2+2b+3}+\frac{b}{b^2+2c+3}+\frac{c}{c^2+2a+2}\right) \leq \frac{9}{2}$. *By the AM-GM inequality, we have:* $\frac{a}{a^2+2b+3}+\frac{b}{b^2+2c+3}+\frac{c}{c^2+2a+3}=\frac{a}{(a^2+1)+2b+2}+\frac{b}{(b^2+1)+2c+2}+\frac{c}{(c^2+1)+2a+2}$

$\leq \frac{1}{2}\left(\frac{a}{a+b+1}+\frac{b}{b+c+1}+\frac{c}{c+a+1}\right)=\frac{1}{2}\left(\frac{a}{4-c}+\frac{b}{4-a}+\frac{c}{4-b}\right)$. *We will prove that:*

$\frac{a}{4-c}+\frac{b}{4-a}+\frac{c}{4-b} \leq 1$ *which is equivalent to:* $32(a+b+c)+\left(a^2b+b^2c+c^2a+abc\right) \leq$
$\leq 4\left(a^2+b^2+c^2\right)+8(ab+bc+ca)+64 \Leftrightarrow a^2b+b^2c+c^2a+abc \leq 4$. *Without loss of generality, we assume that b is the number between c and a. Then, we have:*

$$c(b-a)(b-c) \leq 0 \Leftrightarrow b^2c+c^2a \leq abc+bc^2$$

Applying the AM-GM inequality, we have:

$$a^2b+b^2c+c^2a+abc \leq a^2b+abc+bc^2+abc=b(c+a)^2=\frac{1}{2}\cdot 2b(c+a)(c+a)$$

$\leq \frac{1}{2}\cdot\frac{(2b+c+a+c+a)^3}{27}=4$. *Therefore, we need to prove that:* $b\sqrt{ab}+c\sqrt{bc}+a\sqrt{ca} \leq 3 \Leftrightarrow$

$$\Leftrightarrow 3\left(\sqrt{ab^3}+\sqrt{bc^3}+\sqrt{ca^3}\right) \leq (a+b+c)^2 \Leftrightarrow \frac{1}{2}\sum\left(a-b+2\sqrt{bc}-\sqrt{ab}-\sqrt{ca}\right)^2 \geq 0$$

The equality holds for $a=b=c=1$.

192. If $a,b,c>0$ then:

$$\sum\left(\frac{b+c}{a}\right)^2\cdot\sum\left(\frac{c+a}{b}\right)^3\cdot\sum\left(\frac{a+b}{c}\right)^4 \geq 13824$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania

$$\sum\left(\frac{b+c}{a}\right)^2 \overset{M_a\geq M_g}{\geq} 3\sqrt[3]{\frac{\prod(b+c)^2}{a^2b^2c^2}} \overset{M_a\geq M_g}{\geq} 3\sqrt[3]{\frac{\prod 4bc}{a^2b^2c^2}}=3\cdot 2^2$$

$$\sum\left(\frac{c+a}{b}\right)^3 \overset{M_a\geq M_g}{\geq} 3\sqrt[3]{\frac{\prod(c+a)^3}{a^3b^3c^3}} \overset{M_a\geq M_g}{\geq} 3\sqrt[3]{\frac{\prod 8\sqrt{ac}^3}{a^3b^3c^3}}=3\cdot 8=3\cdot 2^3$$

$$\sum\left(\frac{a+b}{c}\right)^4 \overset{M_a\geq M_g}{\geq} 3\sqrt[3]{\frac{\prod(a+b)^4}{a^4b^4c^4}} \overset{M_a\geq M_g}{\geq} 3\sqrt[3]{\frac{\prod 2^4\sqrt{ab}^4}{a^4b^4c^4}}=3\cdot 16=3\cdot 2^4$$

$$\Rightarrow \sum\left(\frac{b+c}{a}\right)^2\cdot\sum\left(\frac{c+a}{b}\right)^3\cdot\sum\left(\frac{a+b}{c}\right)^4 \ge 3\cdot 2^2\cdot 3\cdot 2^3\cdot 3\cdot 2^4 = 3^3\cdot 2^9 = 13824$$

Solution 2 by Lazaros Zachariadis-Thessaloniki-Greece

Let $f(x) = x^2, g(x) = x^3, h(x) = x^4, x > 0$

f, g, h ***convexe functions***

$$\text{so } LHS = \left[f\left(\frac{b+c}{a}\right)+f\left(\frac{c+a}{b}\right)+f\left(\frac{a+b}{c}\right)\right]\cdot\left[g\left(\frac{b+c}{a}\right)+g\left(\frac{c+a}{b}\right)+g\left(\frac{a+b}{c}\right)\right]\cdot$$

$$\cdot\left[h\left(\frac{b+c}{a}\right)+h\left(\frac{c+a}{b}\right)+h\left(\frac{a+b}{c}\right)\right]$$

$$\overset{Jensen}{\ge} 3\cdot f\left(\frac{\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}}{3}\right)\cdot 3\cdot g\left(\frac{\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}}{3}\right)\cdot 3\cdot h\left(\frac{\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}}{3}\right)$$

$$\overset{AM-GM}{\ge} 3^3\cdot f\left(\frac{3\sqrt[3]{\frac{(b+c)(c+a)(a+b)}{abc}}}{3}\right)\cdot g\left(\frac{3\cdot\sqrt[3]{\frac{(b+c)(c+a)(a+b)}{abc}}}{3}\right)\cdot$$

$$\cdot h\left(\frac{3\cdot\sqrt[3]{\frac{(b+c)(c+a)(a+b)}{abc}}}{3}\right)$$

$f, g, h \nearrow$ ***strictly increasing for*** $x > 0$

$$\overset{AM-GM}{\ge} 27\cdot f\left(1\cdot\sqrt[3]{\frac{8abc}{abc}}\right)\cdot g\left(1\cdot\sqrt[3]{\frac{8abc}{abc}}\right)\cdot h\cdot\left(1\cdot\sqrt[3]{\frac{8abc}{abc}}\right)$$

$$= 27\cdot f(2)\cdot g(2)\cdot h(2) = 27\cdot 2^2\cdot 2^3\cdot 2^4 = 27\cdot 2^9 = 13824 \ \textit{(proved)}$$

Solution 3 by Dimitris Kastriotis-Greece

Put $x = \frac{b+c}{a}, y = \frac{c+a}{b}, z = \frac{a+b}{c}$

$$x+y+z = \frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c} \overset{AM-GM}{\ge} 6 \ :(1)$$

$$\sum x^2 \overset{C-S}{\ge} \frac{(x+y+z)^2}{3} \overset{(1)}{\ge} \frac{6^2}{3} = 12 \ :(2)$$

$$\sum y^3 \overset{Holder}{\ge} \frac{(x+y+z)^3}{9} \overset{(1)}{\ge} \frac{6^3}{9} = 24 \ :(3)$$

$$\sum(z^2)^2 \overset{C-S}{\geq} \frac{(x^2+y^2+z^2)^2}{3} \overset{(2)}{\geq} \frac{12^2}{3} = 48 \quad \boldsymbol{:(4)}$$

$$\left.\begin{matrix}(2)\\(3)\\(4)\end{matrix}\right\} \to \left(\sum x^2\right)\left(\sum y^2\right)\left(\sum z^4\right) \geq 12 \cdot 24 \cdot 48 = 13824$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\sum\left(\frac{a+b}{c}\right) = \sum\left(\frac{a}{b}+\frac{b}{a}\right) \overset{A-G}{\geq} 3 \cdot 2 = 6 \to \ \textbf{(a)}$$

$$\sum x^2 \underset{(1)}{\overset{Chebyshev}{\geq}} \frac{(\sum x)^2}{3}, \sum x^3 \underset{(2)}{\overset{Chebyshev}{\geq}} \frac{(\sum x)^3}{9}, \sum x^4 \underset{(3)}{\overset{Chebyshev}{\geq}} \frac{(\sum x)^4}{27}$$

$$\therefore \textbf{(1).(2).(3)} \Rightarrow LHS = \sum x^2 \cdot \sum x^3 \cdot \sum x^4 \left(x = \frac{a+b}{c}\right) \geq \frac{(\sum x)^9}{3^6} \overset{by\ (a)}{\geq} \frac{6^9}{3^6}$$

$$= \frac{3^9 \cdot 2^9}{3^6} = 3^3 \cdot 8^3 = 24^3 = 13824 \ \textbf{(proved)}$$

193. If $a, b, c \geq 0$ then:

$$(a^3+b^3)^4 + (b^4+c^4)^5 + (c^5+a^5)^6 \geq (a^4+b^4)^3 + (b^5+c^5)^4 + (c^6+a^6)^5$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mihalcea Andrei Stefan-Romania

$$(a^2 \cdot a \cdot + b^2 \cdot b \cdot b)^3 \overset{Hölder}{\leq} (a^6+b^6)(a^3+b^3)^2 \leq (a^3+b^3)^4 \Leftrightarrow a^6+b^6 \leq (a^3+b^3)^2$$

$$\Leftrightarrow 2a^3b^3 \geq 0; (b^2 \cdot b \cdot b \cdot b + c^2 \cdot c \cdot c \cdot c)^4 \leq (b^8+c^8)(b^4+c^4)^3 \leq (b^4+c^4)^5 \Leftrightarrow$$

$$b^8+c^8 \leq (b^4+c^4)^2 \Leftrightarrow 2b^4c^4 \geq 0$$

$$(c^2 \cdot c \cdot c \cdot c \cdot c + a^2 \cdot a \cdot a \cdot a \cdot a)^5 \leq (c^{10}+a^{10})(c^5+a^5)^4 \leq (c^5+a^5)^6 \Leftrightarrow$$

$$\Leftrightarrow 2a^5c^5 \geq 0. \textbf{ Equality for } a = b = c = 0.$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\textbf{For } 0 < a \leq 1, x > 1, \textbf{ let } f(x) = (1+a^x)^{\frac{1}{x}},\ \ln f(x) = \frac{1}{x}\ln(1+a^x)$$

$$\frac{1}{f(x)}f'(x) = -\frac{1}{x^2}\ln(1+a^x) + \frac{1}{x} \cdot \frac{a^x \ln a}{1+a^x} < 0 \textbf{ as } 0 < a \leq 1, x > 1$$

$$\Rightarrow f(x) \textbf{ is a decreasing function on } [1,\infty) \therefore (1+a^x)^{\frac{1}{x}} \geq (1+a^{x+1})^{\frac{1}{x+1}}$$

Suppose $0 < \alpha \le \beta$, *then* $\left(1 + \left(\frac{\alpha}{\beta}\right)^x\right)^{\frac{1}{x}} \ge \left(1 + \left(\frac{\alpha}{\beta}\right)^{x+1}\right)^{\frac{1}{x+1}}$

$$\Rightarrow (a^x + \beta^x)^{x+1} \ge (a^{x+1} + \beta^{x+1})^x$$

Thus, $(a^3 + b^3)^4 \ge (a^4 + b^4)^3$, $(b^4 + c^4)^5 \ge \left(b^5 + c^5\right)^4$, $\left(c^5 + a^5\right)^6 \ge (c^6 + a^6)^5$

Adding, we get desired inequality.

194. If $x, y, z > 0, xy + yz + zx = 3$ then:

$$9 + \frac{(x^2 - yz)^2 + (y^2 - zx)^2 + (z^2 - xy)^2}{3 + x^2 + y^2 + z^2} \ge 9\sqrt[3]{x^2y^2z^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Serban George Florin-Romania

$$(x^2 - yz)^2 = (x^2 + xy + xz - 3)^2 = [x(x + y + z) - 3]^2 =$$
$$= x^2(x + y + z)^2 - 6x(x + y + z) + 9$$
$$\sum(x^2 - yz)^2 = (x^2 + y^2 + z^2)(x + y + z)^2 - 6(x + y + z)^2 + 27$$
$$\frac{\sum(x^2 - yz)^2}{3 + \sum x^2} = \frac{(x + y + z)^2[(x^2 + y^2 + z^2) - 6] + 27}{3 + \sum x^2} =$$
$$= \frac{(x + y + z)^2(x^2 + y^2 + z^2 + 3 - 9) + 27}{3 + \sum x^2}$$
$$= (x + y + z)^2 \cdot \frac{3+\sum x^2}{3+\sum x^2} + \frac{-9(x+y+z)^2+27}{3+\sum x^2} = (x + y + z)^2 + \frac{(-9)(x+y+z)^2+27}{3+\sum x^2}$$
$$\frac{\sum(x^2 - yz)^2}{3 + \sum x^2} + 9 = (x + y + z)^2 + \frac{(-9)(x + y + z)^2 + 27}{3 + \sum x^2} + 9 =$$
$$= (x + y + z)^2 + \frac{-9\left(x^2 + y^2 + z^2 + 2xy + 2yz + 2xz\right) + 27 + 27 + 9\left(x^2 + y^2 + z^2\right)}{3 + \sum x^2} =$$
$$= (x + y + z)^2 + \frac{-18(xy + yz + zx) + 54}{3 + \sum x^2} = (x + y + z)^2 + \frac{-18 \cdot 3 + 54}{3 + \sum x^2} =$$
$$= (x + y + z)^2 + \frac{-54 + 54}{3 + \sum x^2} = (x + y + z)^2 \overset{AM-GM}{\ge} 9\sqrt[3]{x^2y^2z^2}$$

195. If $a_1, a_2, \dots, a_{n+2}, n \in \mathbb{N}^*$ then:

$$\sum_{k=1}^{n} \frac{a_k}{a_k^2 + a_{k+1}a_{k+2}} \le \frac{1}{2}\sum_{k=1}^{n} \frac{1}{a_k}, a_{n+1} = a_1, a_{n+2} = a_2.$$

Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania

Solution by Rajsekhar Azaad-India

$$\frac{a_k}{a_k^2 + a_{k+1} \cdot a_{k+2}} \le \frac{a_k}{2a_k\sqrt{a_{k+1}a_{k+2}}} \quad (AM \ge GM)$$

$$= \frac{1}{2\sqrt{a_{k+1}a_{k+2}}} \le \frac{1}{2} \times \frac{1}{2}\left(\frac{1}{a_{k+1}} + \frac{1}{a_{k+2}}\right) = \frac{1}{4}\left(\frac{1}{a_{k+1}} + \frac{1}{a_{k+2}}\right)$$

$$\therefore \sum_{k=1}^{n} \frac{a_k}{a_k^2 + a_{k+1} \cdot a_{k+2}} \le \frac{1}{4} \cdot 2 \cdot \sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{2}\sum_{k=1}^{n} \frac{1}{a_k}$$

(Proved)

196. Let a, b, c be positive real numbers. Prove that:

$$2\sqrt[4]{\frac{xy(x+y) + yz(y+z) + zx(z+x)}{6xyz}} \ge \frac{x^2 + yz}{x^2 + 2yz} + \frac{y^2 + zx}{y^2 + 2zx} + \frac{z^2 + xy}{z^2 + 2xy}$$

Proposed by Do Quoc Chinh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x + y = a, y + z = b, z + x = c$. Then, $a + b > c, b + c > a, c + a > b$

$\Rightarrow a, b, c$ are sides of a triangle with semiperimeter, circumradius, inradius $= s, R, r$

resepctively (say) $\sum x = \frac{\sum a}{2} = s \therefore z = s - a, x = s - b, y = s - c$

$$\therefore \sum xy = \sum (s-b)(s-c) = \sum \{s^2 - s(b+c) + bc\}$$

$$= 3s^2 - s(4s) + s^2 + 4Rr + r^2 \overset{(1)}{=} 4Rr + r^2$$

$$\therefore \sum x^2y + \sum xy^2 = \sum xy\,(s - z) \overset{by(1)}{=} s(4Rr + r^2) - 3xyz$$

$$= s(4Rr + r^2) - 3r^2 s \overset{(2)}{=} s(4Rr - 2r^2)$$

$$\therefore LHS = 2\sqrt[4]{\frac{s(4Rr - 2r^2)}{6r^2 s}} \overset{(3)}{=} 2\sqrt[4]{\frac{2R - r}{3r}}$$

Now, $RHS = \sum\left(\frac{x^2+2yz}{x^2+2yz} - \frac{yz}{x^2+2yz}\right) \overset{(4)}{=} 3 - \sum\frac{yz}{x^2+2yz}$

(3), (4) $\Rightarrow$ ***given inequality*** $\Leftrightarrow \sum\frac{y^2z^2}{x^2yz+2y^2z^2} + 2\sqrt[4]{\frac{2R-r}{3r}} \overset{(5)}{\geq} 3$

Now, $\sum\frac{y^2z^2}{x^2yz+2y^2z^2} \underset{(6)}{\overset{Bergström}{\geq}} \frac{(\sum yz)^2}{xyz(\sum x)+2\sum y^2z^2} \overset{by\ (1)}{=} \frac{(4R+r)^2r^2}{xyz(\sum x)+2[(\sum xy)^2-2xyz(\sum x)]}$

$$= \frac{r^2(4R + r)^2}{2(\sum xy)^2 - 3xyz(\sum x)} \overset{by\ (1)}{=} \frac{r^2(4R + r)^2}{2r^2(4R + r)^2 - 3r^2s^2} = \frac{(4R + r)^2}{2(4R + r)^2 - 3s^2}$$

(5), (6) $\Rightarrow$ ***it suffices to prove:*** $\frac{(4R+r)^2}{2(4R+r)^2-3s^2} + 2\sqrt[4]{\frac{2R-r}{3r}} \geq 3 \Leftrightarrow 2\sqrt[4]{\frac{2R-r}{3r}} \overset{(7)}{\geq} 3 - \frac{(4R+r)^2}{2(4R+r)^2-3s^2}$

Now, Gerretsen $\Rightarrow$

$$2(4R + r)^2 - 3s^2 \leq 2(4R + r)^2 - 48Rr + 15r^2 = 32R^2 - 32Rr + 17r^2$$

$$\Rightarrow \frac{-(4r + r)^2}{2(4R + r)^2 - 3s^2} \leq \frac{-(4R + r)^2}{32R^2 - 32Rr + 17r^2}$$

$$\Rightarrow 3 - \frac{(4R + r)^2}{2(4R + r)^2 - 3s^2} \overset{(8)}{\leq} \frac{80R^2 - 104Rr + 50r^2}{32R^2 - 32Rr + 17r^2}$$

(7), (8) $\Rightarrow$ ***it suffices to prove:***

$$\frac{2R - r}{3r} \geq \frac{(40R^2 - 52Rr + 25r^2)^4}{(32R^2 - 32Rr + 17r^2)^4}$$

$$\Leftrightarrow (t-2)\left\{(t-2)\begin{pmatrix} 2,097,152t^7 - 8,728,576t^6 + 17,866,752t^5 - \\ -20,971,520t^4 + 16,915,456t^3 - 6,798,720t^2 + \\ +4,714,112t + 4,469,120 \end{pmatrix} + 9,565,938\right\} \geq 0$$

$\therefore$ ***it suffices to prove*** $p > 0 \because t \geq 2$ ***(Euler). Now,***

$$p = (t-2)\left[\left\{(t-2)\begin{pmatrix} 2,097,152t^5 - 339,968t^4 + 8,118,272t^3 + 12,861,440t^2 + \\ +35,888,128t + 85,308,032 \end{pmatrix} + 202393728\right\}\right] +$$

$$+68024448$$

is of course > 0 ***as*** $t \geq 2 \Rightarrow$ ***(7) is true (Proved)***

197. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$ then

$$\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \ge \frac{9}{2(a + b + c)}$$

Proposed by Pham Quoc Sang-Ho Chi Minh-Vietnam

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

$$\frac{a^2}{b^2+c^2} = \frac{a^2}{3-a^2} \ge \frac{a^3}{2} \Leftrightarrow a^2\left(\frac{1}{3-a^2} - \frac{a}{2}\right) \ge 0 \Leftrightarrow \frac{a^2(a-1)^2(a+2)}{2(3-a^2)} \ge 0 \quad \textit{(true)}$$

$$\Rightarrow \frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \ge \frac{a^3 + b^3 + c^3}{2} \ge \frac{9}{2(a + b + c)}$$

$$\Leftrightarrow (a^3 + b^3 + c^3)(a + b + c) \ge 9 = (a^2 + b^2 + c^2)^2$$

(true because: $(a^3 + b^3 + c^3)(a + b + c) \ge \left(\sqrt{a^3a} + \sqrt{b^3b} + \sqrt{c^3c}\right)^2 = (a^2 + b^2 + c^2)$

198. If $a, b, c, \alpha > 0, abc = \alpha$ then:

$$\frac{(a + b)^4}{a^2 + b^2} + \frac{(b + c)^4}{b^2 + c^2} + \frac{(c + a)^4}{c^2 + a^2} \ge 24\sqrt[3]{\alpha^2}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum \frac{(a + b)^4}{a^2 + b^2} = \sum \frac{\left(a^2 + b^2 + 2ab\right)^2}{a^2 + b^2} = \sum \frac{\left(a^2 + b^2\right)^2}{a^2 + b^2} + 4\sum \frac{ab\left(a^2 + b^2\right)}{a^2 + b^2} + 4\sum \frac{a^2b^2}{a^2 + b^2} =$$

$$= \sum (a^2 + b^2) + 4\sum ab + 2\sum \frac{2}{\frac{1}{a^2} + \frac{1}{b^2}} \overset{AM-GM}{\gtrless} 2 \cdot 3\sqrt[3]{\alpha^2} + 4 \cdot 3\sqrt[3]{\alpha^2} + 2\sum ab \overset{AM-GM}{\gtrless}$$

$$\ge 18\sqrt[3]{\alpha^2} + 2 \cdot 3\sqrt[3]{\alpha^2} = 24\sqrt[3]{\alpha^2}$$

199. Let x, y, z be positive real numbers satisfying $x + y + z = 1$. Prove that:

$$\frac{(1 + xy + yz + zx)(1 + 3x^3 + 3y^3 + 3z^3)}{9(x + y)(y + z)(z + x)} \ge \left(\frac{x\sqrt{1 + x}}{\sqrt[4]{3 + 9x^2}} + \frac{y\sqrt{1 + y}}{\sqrt[4]{3 + 9y^2}} + \frac{z\sqrt{1 + z}}{\sqrt[4]{3 + 9z^2}}\right)^2$$

Proposed as subject at Korea NMO-2017

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

By Cauchy-Schwarz

$$3+9x^2=(3x)^2+1+1+1\ge\frac{(3x+1+1+1)^2}{4}=\frac{9(x+1)^2}{4}\Rightarrow\sqrt[4]{3+9x^2}\ge\sqrt{\frac{3(x+1)}{2}}$$

$$\Rightarrow\sum\frac{x\sqrt{1+x}}{\sqrt[4]{3+9x^2}}\le\sum\frac{x\sqrt{1+x}}{\sqrt{\frac{3(x+1)}{2}}}=\sum\sqrt{\frac{2}{3}}\Rightarrow\left(\sum\frac{x\sqrt{1+x}}{\sqrt[4]{3+9x^2}}\right)^2\le\frac{2}{3}\left(\sum x\right)^2=\frac{2}{3}\cdot 1=\frac{2}{3}\quad \textbf{(1)}$$

We have: $\frac{(1+\sum xy)(1+3\sum x^2)}{9(x+y)(y+z)(z+x)}=\frac{1+3\sum x^2+\sum xy+3(\sum x^3)(\sum xy)}{9(x+y)(y+z)(z+x)}$

Other: $3(\sum x^3)(\sum xy)\ge 3\cdot\frac{(\sum x)^3}{9}\cdot(\sum xy)=3\cdot\frac{1}{9}(\sum xy)=\frac{(\sum xy)(\sum x)}{3}$, *and* $\sum x=1$

$$\Rightarrow 1+3\sum x^3+\sum xy+3\left(\sum x^3\right)\left(\sum xy\right)\ge\left(\sum x\right)^3+3\sum x^3+\left(\sum x\right)\left(\sum xy\right)+\frac{(\sum xy)(\sum x)}{3}$$

$$\Rightarrow\frac{(1+\sum xy)(1+3\sum x^3)}{9(x+y)(y+z)(z+x)}\ge\frac{(\sum x)^3+3\sum x^3+\frac{4(\sum xy)(\sum x)}{3}}{9(x+y)(y+z)(z+x)}=$$

$$=\frac{3(\sum x)^3+9\sum x^3+4(\sum xy)(\sum x)}{27(x+y)(y+z)(z+x)}\ge\frac{2}{3}$$

$$\Leftrightarrow 3\left(\sum x\right)^3+9\sum x^3+4\left(\sum xy\right)\left(\sum x\right)\ge 18(x+y)(y+z)(z+x)$$

$$\Leftrightarrow 12\sum x^3\ge 5\sum xy(x+y)+6xyz\Leftrightarrow$$

$$5\sum\left(x^3+y^3-xy(x+y)\right)+2\left(\sum x^3-3xyz\right)\ge 0$$

$$\Leftrightarrow 5\sum(x+y)(x-y)^2+\left(\sum x\right)\left(\sum(x-y)^2\right)\ge 0\quad \textit{(True)}$$

Then (1): $\Rightarrow\frac{(1+\sum xy)(1+3\sum x^3)}{9(x+y)(y+z)(z+x)}\ge\frac{2}{3}\ge\left(\sum\frac{x\sqrt{1+x}}{\sqrt[4]{3+9x^2}}\right)^2\Rightarrow\left(\sum\frac{x\sqrt{1+x}}{\sqrt[4]{3+9x^2}}\right)^2\le\frac{(1+\sum xy)(1+3\sum x^3)}{9(x+y)(y+z)(z+x)}$

200. If a,b,c are positive real number such that $(a+b)(b+c)(c+a)=8$ then:

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\ge\frac{ab+bc+ca}{a+b+c}$$

Proposed by Pham Quoc Sang-Ho Chi Minh-Vietnam

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Solution by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{1}{a^2+ab+b^2}\ge\frac{ab+bc+ca}{a+b+c}$$
$$\Leftrightarrow\sum_{cyc}\frac{1}{(a^2+ab+b^2)(ab+bc+ca)}\ge\frac{1}{a+b+c}$$
$$\Leftrightarrow\sum_{cyc}\frac{4}{(a^2+ab+b^2+ab+bc+ca)^2}\ge\frac{1}{a+b+c}$$
$$\left[\because (a^2+ab+b^2)(ab+bc+ca)\le\frac{(a+b)^2(a+b+c)^2}{4}\right]$$
$$\Leftrightarrow\sum_{cyc}\frac{4}{(a+b)^2(a+b+c)^2}\ge\frac{1}{a+b+c}\Leftrightarrow\sum_{cyc}\frac{4}{(a+b)^2}\ge a+b+c$$
$$\therefore\sum_{cyc}\frac{4}{(a+b)^2}\ge\sum_{cyc}\frac{4}{(a+b)(b+c)}=\frac{8(a+b+c)}{\prod_{cyc}(a+b)}=a+b+c$$

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Its nice to be important but more important its to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru