RMM - Triangle Marathon 301 - 400

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301. **If in $\Delta\ ABC$ the orthocenter lies on the incircle then:**

$$\cos A + \cos B + \cos C - 1 = 2\sqrt{\cos A \cos B \cos C}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Si en un $\Delta\ ABC$ el ortocentro se encuentra en el círculo. Probar $\cos A + \cos B + \cos C - 1 = 2\sqrt{\cos A \cos B \cos C}$. De la condición se despernde: $IH = r$. Tener en cuenta las siguientes identidades en un

$\Delta\ ABC; \cos A \cos B \cos C = \frac{s^2-4R^2-4Rr-r^2}{4R^2}, \cos A + \cos B + \cos C - 1 = \frac{r}{R}$

Además $IH^2 = 4R^2 + 4Rr + 3r^2 - s^2 \Leftrightarrow r^2 = 4R^2 + 4Rr + 3r^2 - s^2$

$$s^2 - 4R^2 - 4Rr - r^2 = r^2 \Leftrightarrow \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2} = \frac{r^2}{4R^2}$$

$$\cos A \cos B \cos C = \frac{1}{4}(\cos A + \cos B + \cos C - 1)^2 \Leftrightarrow \sqrt{\cos A \cos B \cos C}$$

$$= \frac{1}{2}(\cos A + \cos B + \cos C - 1) \to \cos A + \cos B + \cos C - 1 = \sqrt{2 \cos A \cos B \cos C}$$

302. **In $\Delta\ ABC, O -$ circumcentre, $I -$ incentre, $m(\sphericalangle B) > 90°$:**

$$\frac{[AOB]}{[AIB]} + \frac{[BOC]}{[BIC]} - \frac{[COA]}{[CIA]} = \frac{R}{r} + 1$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Si $m(\sphericalangle B) > 90°$. Probar en un triángulo ABC $\frac{[AOB]}{[AIB]} + \frac{[BOC]}{[BIC]} - \frac{[COA]}{[CIA]} = \frac{R}{r} + 1$

Siendo $O -$ circuncentro, $I -$ incentro, se cumple lo siguiente

$$[AOB] = \frac{R^2|\sin 2C|}{2}, [BOC] = \frac{R^2|\sin 2A|}{2}, [COA] = \frac{R^2|\sin 2B|}{2}$$

$$[AIB] = \frac{cr}{2}, [BIC] = \frac{ar}{2}, [CIA] = \frac{br}{2}$$

Como $m(\sphericalangle B) > 90° \to m(\sphericalangle A), m(\sphericalangle B) < 90$

$$\Leftrightarrow [AOB] = \frac{R^2 \sin 2C}{2}, [BOC] = \frac{R^2 \sin 2A}{2}, [COA] = -\frac{R^2 \sin 2B}{2}$$

$$\frac{[AOB]}{[AIB]} + \frac{[BOC]}{[BIC]} - \frac{[COA]}{[CIA]} = \frac{R^2 \sin 2C}{2} \cdot \frac{2}{cr} + \frac{R^2 \sin 2A}{2} \cdot \frac{2}{ar} + \frac{R^2 \sin 2B}{2} \cdot \frac{2}{br}$$

$$\frac{[AOB]}{[AIB]} + \frac{[BOC]}{[BIC]} - \frac{[COA]}{[CIA]} = \frac{R^2}{r} \cdot \left(\frac{\sin 2C}{c} + \frac{\sin 2A}{a} + \frac{\sin 2B}{b}\right) =$$

$$= \frac{R}{r}\left(\frac{\sin 2C}{2\sin C} + \frac{\sin 2A}{2\sin A} + \frac{\sin 2B}{2\sin B}\right)$$

$$\frac{[AOB]}{[AIB]} + \frac{[BOC]}{[BIC]} - \frac{[COA]}{[CIA]} = \frac{R}{r}(\cos C + \cos A + \cos B) =$$

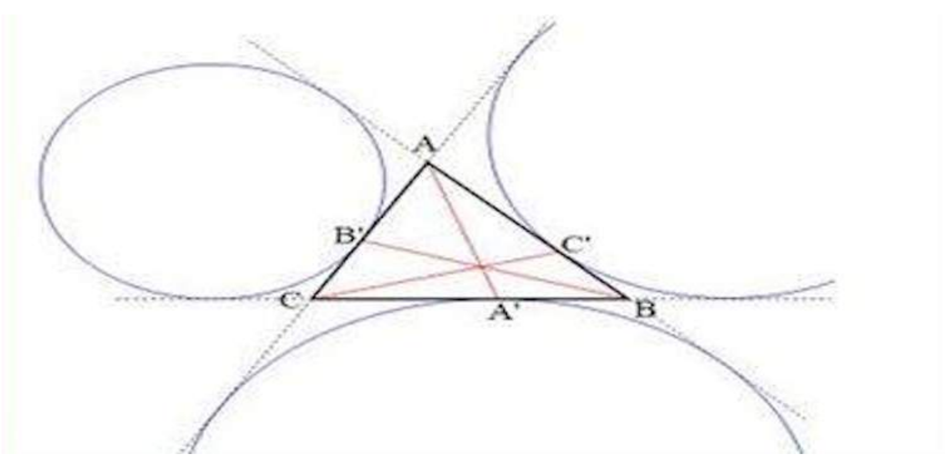
$$= (\cos C + \cos A + \cos B) = \frac{R}{r}\left(1 + \frac{r}{R}\right) = \frac{R}{r} + 1$$

303. **In ΔABC , AA', BB', CC' Nagel's Cevians. Prove that:**

$$a^2[AA']^2 + b^2[BB']^2 + c^2[CC']^2 \geq 4s^2r(2R - r)$$

$$a[AA']^2 + b[BB']^2 + c[CC']^2 \geq 2sr(8R - 7r)$$

Proposed by Abdilkadir Altintas-Afyonkarashisar-Turkey



Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo AA', BB', CC' Nagel's Cevians. Probar en un triángulo ABC

$$a^2(AA')^2+b^2(BB')^2+c^2(CC')^2\geq 4p^2r(2R-r)$$

Teniendo en cuenta las siguientes notaciones y desigualdad

$$a^2(AA')^2=-\frac{p}{2}a(a^2-2b^2-2c^2-ab-ac+4bc),$$

$$p^2+5r62\geq 16Rr \text{ (Gerretsen's inequality)}$$

$$b^2(BB')^2=-\frac{p}{2}b(b^2-2c^2-2a^2-bc-ba+4ca),$$

$$c^2(CC')^2=-\frac{p}{2}c(c^2-2a^2-2b^2-ca-cb+4ab)$$

EL LHS es equivalente

$$-\frac{p}{2}\left(\sum a^3-2\sum a(b^2+c^2)-\sum a^2(b+c)+12abc\right)=$$

$$=-\frac{p}{2}\left(\sum a^3-3\sum ab(a+b)+12abc\right)$$

$$=-\frac{p}{2}\left(2p(p^2-3r^2-6Rr)-6p(p^2-2Rr+r^2)+48pRr\right)$$

$$=-\frac{p}{2}(-4p^3-12pr^2+48pRr)=2p^2(p^2+3r^2-12Rr)$$

Es suficiente probar $2p^2(p^2+3r^2-12Rr)\geq 4s^2r(2R-r)\Leftrightarrow$

$\Leftrightarrow p^2+3r^2-12Rr\geq 4Rr-2r^2\Leftrightarrow p^2+5r^2\geq 16Rr$ **(LQQD)**

Siendo AA', BB', CC' Nagel's Cevians. Probar en un triángulo ABC

$$a(AA')^2+b(BB')^2+c(CC')^2\geq 2pr(8R-7r)$$

Teniendo en cuento las siguientes notaciones y desigualdad

$$a(AA')^2=-\frac{p}{2}(a^2-2b^2-2c^2-ab-ac+4bc),$$

$$p^2+5r^2\geq 16Rr \text{ (Gerretsen's Inequality)}$$

$$b(BB')^2 = -\frac{p}{2}(b^2 - 2c^2 - 2a^2 - bc - ba + 4ca),$$

$$c(CC')^2 = -\frac{p}{2}(c^2 - 2a^2 - 2b^2 - ca - cb + 4ab)$$

EL LHS es equivalente

$$-\frac{p}{2}(-3a^2 - 3b^2 - 3c^2 + 2ab + 2bc + 2ca) = \frac{p}{2}(3a^2 + 3b^2 + 3c^2 - 2ab - 2bc - 2ca)$$

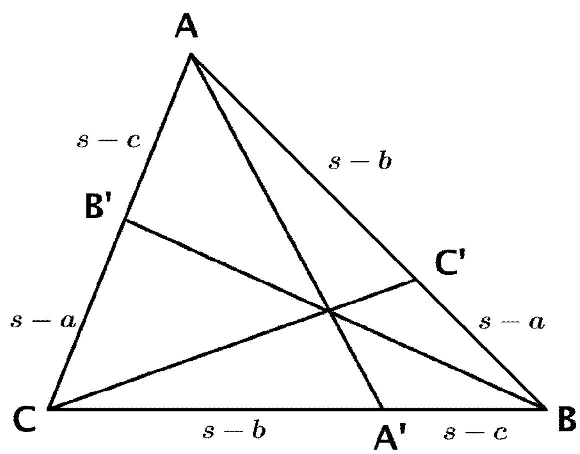
$$\Leftrightarrow \frac{p}{2}\left(6(p^2 - r^2 - 4Rr) - 2(p^2 + r^2 + 4Rr)\right) = \frac{p}{2}(4p^2 - 8r^2 - 32Rr) =$$

$= p(2p^2 - 4r^2 - 16Rr)$***. Es suficiente demostrar lo siguiente***

$$p(2p^2 - 4r^2 - 16Rr) \geq 2pr(8R - 7r) \Leftrightarrow 2p^2 - 4r^2 - 16Rr \geq 16Rr - 14r^2$$

$$\Leftrightarrow 2(p^2 + 5r^2) \geq 32Rr \Leftrightarrow p^2 + 5r^2 \geq 16Rr \text{ (LQQD)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India



Let $AA' = x, BB' = y, CC' = z$***. By Stewart's theorem,***

$$b^2(s-c) + c^2(s-b) = a\left(x^2 + (s-b)(s-c)\right)$$

$$\Rightarrow ax^2 = b^2(s-c) + c^2(s-b) - a(s-b)(s-c)$$

$$= s(b^2 + c^2) - (b^2c + bc^2) - a\{s^2 - s(2s-a) + bc\}$$

$$= s(b^2 + c^2) - bc(b+c) - s(-s^2 + as + bc)$$

$$= s(b^2 + c^2) - bc(b+c) + as^2 - a^2s - abc$$

$$\Rightarrow a^2x^2 = s(ab^2 + ac^2) - abc(b + c) + a^2s^2 - a^3s - a^2bc \quad \textbf{(1)}$$

Similarly, $b^2y^2 \overset{(2)}{=} s(bc^2 + ba^2) - abc(c + a) + b^2s^2 - b^3s - b^2ca,$

and, $c^2z^2 \overset{(2)}{=} s(ca^2 + cb^2) - abc(a + b) + c^2s^2 - c^3s - c^2ab$

(1)+(2)+(3) $\Rightarrow$ *LHS of* $\overset{(1)}{=} s(\sum a^2b + \sum ab^2) - abc(4s) +$

$$+2s^2(s^2 - 4Rr - r^2) - s\left\{3abc + 2s\left(\sum a^2 - \sum ab\right)\right\} - abc(2s)$$

$$= s\left[\sum\{ab(2s - c)\}\right] - 24s^2Rr + 2s^2(s^2 - 4Rr - r^2) -$$

$$-s\{12Rrs + 2s(s^2 - 12Rr - 3r^2)\}$$

$$= s\{2s(s^2 + 4Rr + r^2) - 12Rrs\} - 24s^2Rr + 2s^2(s^2 - 4Rr - r^2) -$$

$$-2s^2(6Rr + s^2 - 12Rr - 3r^2)$$

$$= 2s^2(s^2 + r^2 - 2Rr) - 24s^2Rr + 2s^2(s^2 - 4Rr - r^2) -$$

$$-2s^2(s^2 - 6Rr - 3r^2)$$

$$= 2s^2(s^2 + r^2 - 2Rr - 12Rr + s^2 - 4Rr - r^2 - s^2 + 6Rr + 3r^2)$$

$$= 2s^2(s^2 - 12Rr + 3r^2) \overset{?}{\geq} 4s^2r(2R - r)$$

$$\Leftrightarrow s^2 - 12Rr + 3r^2 \overset{?}{\geq} 4Rr - 2r^2$$

$\Leftrightarrow s^2 \overset{?}{\geq} 16Rr - 5r^2 \rightarrow$ ***true by Gerretsen*** $\Rightarrow$ ***(1) is proved. We earlier proved that:*** $ax^2 = s(b^2 + c^2) - bc(b + c) + as^2 - a^2s - abc$ ***(4)***

Similarly, $by^2 \overset{(5)}{=} s(c^2 + a^2) - ca(c + a) + bs^2 - b^2s - abc,$

and, $cz^2 \overset{(6)}{=} s(a^2 + b^2) - ab(a + b) + cs^2 - c^2s - abc$

(4)+(5)+(6) $\Rightarrow$ *LHS of (2)*

$$= 2s\left(\sum a^2\right) - \left\{\sum ab\,(2s - c)\right\} + s^2(2s) - s\left(\sum a^2\right) - 12Rrs$$

$$= 2s(s^2 - 4Rr - r^2) - 2s(s^2 + 4Rr + r^2) + 12Rrs + 2s^2 - 12Rrs$$

$$= 2s(s^2 - 8Rr - 2r^2) \overset{?}{\geq} 2sr(8R - 7r)$$

$$\Leftrightarrow s^2 \overset{?}{\geq} 16Rr - 5r^2 \rightarrow \textit{true by Gerretsen} \Rightarrow \textit{(2) is true}$$

304. Prove that in any triangle ABC

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \geq \sqrt{\frac{3}{2} \cdot \frac{4R + r}{R}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \geq \sqrt{\frac{3}{2} \cdot \left(\frac{4R+r}{R}\right)}$

Es suficiente demostrar lo siguiente $\frac{r_a}{a} \cdot \frac{r_b}{b} + \frac{r_b}{b} \cdot \frac{r_c}{c} + \frac{r_c}{c} \cdot \frac{r_a}{a} = \frac{4R+r}{2r}$

Tener en cuenta lo siguiente

$$r_a = 4R \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}, r_b = 4R \sin\frac{B}{2} \cos\frac{A}{2} \cos\frac{C}{2}, r_c = 4R \sin\frac{C}{2} \cos\frac{A}{2} \cos\frac{B}{2}$$

$$r_a r_b = (2R \sin A)(2R \sin B)\left(\cos^2\frac{C}{2}\right) \Leftrightarrow r_a r_b = \frac{ab}{2}(1 + \cos C),$$

$$r_b r_c = \frac{bc}{2}(1 + \cos a), r_c r_a = \frac{ca}{2}(1 + \cos B)$$

$$\Leftrightarrow \frac{r_a}{a} \cdot \frac{r_b}{b} + \frac{r_b}{b} \cdot \frac{r_c}{c} + \frac{r_c}{c} \cdot \frac{r_a}{a} = \frac{1 + \cos C}{2} + \frac{1 + \cos A}{2} + \frac{1 + \cos B}{2} =$$

$= \frac{4 + \frac{r}{R}}{2} = \frac{4R+r}{2R}$. *Aplicando la siguiente desigualdad para todo* R^+ x, y, z *se cumple la desigualdad propuesta*

$$x + y + z \geq \sqrt{3(xy + yz + zx)}, x = \frac{r_a}{a}, y = \frac{r_b}{b}, z = \frac{r_c}{c}$$

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305. In $\Delta\ ABC$:

$$m_a w_a + m_b w_b + m_c w_c \geq \frac{h_a h_b h_c}{r}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

Lemma 1. $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}, w_a = \frac{2bc}{b+c}\cos\frac{A}{2}, h_a h_b h_c = \frac{2r^2p^2}{R}$.

Lemma 2. $4p^2 \leq 27R^2, \cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2} = \frac{p}{4R}$

$$LHS \geq bc\cdot\cos^2\frac{A}{2} + ca\cdot\cos^2\frac{B}{2} + ab\cdot\cos^2\frac{C}{2} \geq$$

$$\geq 3\cdot\sqrt[3]{(abc)^2\cdot\left(\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}\right)^2} =$$

$$= 3\cdot\sqrt[3]{16R^2r^2p^2\cdot\frac{p^2}{16R^2}} = 3\cdot\sqrt[3]{r^2p^4} \geq \frac{2rp^2}{R} \leftrightarrow 27\cdot r^2p^4 \geq \frac{8r^3p^6}{R^3} \leftrightarrow$$

$$\leftrightarrow 4p^2 \leq \frac{27R^3}{2r}.\ 4p^2 \leq 27R^2 \leq \frac{27R^3}{2r} \leftrightarrow R \geq 2r \text{ (Euler)}$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ P = m_a\cdot w_a + m_b\cdot w_b + m_c\cdot w_c \geq \frac{h_a h_b h_c}{r}$

Tener en cuenta las siguientes desigualdades e identidad en un triángulo $ABC\ m_a \geq h_a, m_b \geq h_b, m_c \geq h_c, \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$

$w_a \geq h_a, w_b \geq h_b, w_c \geq h_c$. ***Por la tanto***

$$P \geq h_a^2 + h_b^2 + h_c^2 \geq h_b h_c + h_c h_a + h_a h_b = h_a h_b h_c\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) = \frac{h_a h_b h_c}{r}$$

(LQQD)

306. Let ABC be a triangle and m_a, m_b, m_c are the medians and h_a, h_b, h_c

Prove that

$$\frac{m_a^2}{h_a}+\frac{m_b^2}{h_b}+\frac{m_c^2}{h_c}\geq\frac{63r^2-s^2}{4r}$$

where r and s inradius and semiperimeter of ABC respectively.

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Daniel Sitaru – Romania

$$\sum\frac{m_a^2}{h_a}\overset{BERGSTROM}{\geq}\frac{\left(\sum\sqrt{s(s-a)}\right)^2}{\sum\frac{2S}{a}}=\frac{\left(\sum\frac{S}{\sqrt{(s-b)(s-c)}}\right)^2}{2S\sum\frac{1}{a}}\overset{AM-GM}{\geq}$$

$$\geq\frac{S}{2}\cdot\frac{\left(\sum\frac{2}{a}\right)^2}{\sum\frac{1}{a}}=2S\cdot\sum\frac{1}{a}=2S\cdot\frac{s^2+r^2+4Rr}{4SR}\geq\frac{63r^2-s^2}{4r}\text{ (to prove)}$$

$$2r(s^2+r^2+4Rr)\geq R(63r^2-s^2)\leftrightarrow$$

$$s^2(R+2r)\geq 63Rr^2-2r^2-8Rr^2\text{ (to prove)}$$

$$s^2(R+2r)\overset{GERRETSEN}{\geq}(R+2r)(16Rr-5r^2)\geq 63Rr^2-2r^3-8Rr^2\text{ (to prove)}$$

$$4R^2-7Rr-2r^2\geq 0\leftrightarrow(R-2r)(4R+r)\geq 0\text{ (Euler)}$$

307. Prove that in any triangle ABC

$$(b+c)r_a l_a+(c+a)r_b l_b+(a+b)r_c l_c\leq\frac{9}{2}abc$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC la siguiente desigualdad

$$(b+c)r_a l_a+(c+a)r_b l_b+(a+b)r_c l_c \le \frac{9}{2}abc$$

Teniendo en cuenta las siguientes notaciones y desigualdades en un

$$\Delta\, ABC;\ r_a = p\tan\frac{A}{2}, r_b = \tan\frac{B}{2}, r_c = p\tan\frac{C}{2}$$

$$l_a = \frac{2bc}{b+c}\cos\frac{A}{2}, l_b = \frac{2ca}{a+b}\cos\frac{B}{2}, l_c = \frac{2ab}{a+b}\cos\frac{C}{2}$$

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\frac{3}{2}, \sin\frac{A}{2}\le\frac{a}{b+c}, \sin\frac{B}{2}\le\frac{b}{c+a}, \sin\frac{C}{2}\le\frac{c}{a+b}$$

La desigualdad propuesta es equivalente

$$\Leftrightarrow p\sin\frac{A}{2}\cdot 2bc+p\sin\frac{B}{2}\cdot 2ca+p\sin\frac{C}{2}\cdot 2ab\le\frac{9}{2}abc$$

$$\Leftrightarrow (a+b+c)\sin\frac{A}{2}\cdot\frac{2}{a}+(b+c+a)\sin\frac{B}{2}\cdot\frac{2}{b}+(c+a+b)\sin\frac{C}{2}\cdot\frac{2}{c}\le 9$$

$$\Leftrightarrow 2\sin\frac{A}{2}+2\sin\frac{B}{2}+2\sin\frac{C}{2}+2\sin\frac{A}{2}\cdot\frac{b+c}{a}+2\sin\frac{B}{2}\cdot\frac{c+a}{b}+2\sin\frac{C}{2}\cdot\frac{a+b}{c}\le$$

$$\le 2\cdot\frac{3}{2}+2\cdot 3 = 9$$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$s\tan\frac{A}{2} = s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a} = \frac{\Delta}{s-a} = r_a$$

$$\therefore (b+c)r_a l_a = (b+c)\left(s\tan\frac{A}{2}\right)\frac{2bc\cos\frac{A}{2}}{(b+c)}$$

$$= 2s\,bc\sin\frac{A}{2} = 2s\,bc\sqrt{\frac{(s-b)(s-c)}{bc}} \overset{(1)}{=} 2s\sqrt{bc(s-b)(s-c)}$$

Similarly, $(c+a)r_b l_b \overset{(2)}{=} 2s\sqrt{ca(s-c)(s-ca)}$**, and,**

$$(a+b)r_c l_c \overset{(3)}{=} 2s\sqrt{ab(s-a)(s-b)}$$

$$\textbf{(1)+(2)+(3)} \Rightarrow LHS$$

$$= 2s\left\{\sqrt{bc(s-b)(s-c)} + \sqrt{ca(s-c)(s-a)} + \sqrt{ab(s-a)(s-b)}\right\}$$

$$= 2s\left\{\sqrt{b(s-b)}\sqrt{c(s-c)} + \sqrt{c(s-c)}\sqrt{a(s-a)} + \sqrt{a(s-a)}\sqrt{b(s-b)}\right\}$$

$$\overset{C-B-S}{\leq} 2s\sqrt{b(s-b)+c(s-c)+a(s-a)}\sqrt{c(s-c)+a(s-a)+b(s-b)}$$

$$= 2s\sqrt{s(2s)-\sum a^2}\sqrt{s(2s)-\sum a^2} = 2s\{2s^2-2(s^2-4Rr-r^2)\}$$

$$= 2s(8Rr+2r^2) \overset{?}{\leq} \frac{9}{2}\cdot abc = 18Rrs$$

$$\Leftrightarrow 8R+2r \overset{?}{\leq} 9R \Leftrightarrow 2r \overset{?}{\leq} R \rightarrow \textit{true (Euler) (Proved)}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\textbf{1)}\ \sum(b+c)\cdot\frac{S}{p-a}\cdot\frac{2\sqrt{bc}\sqrt{p\cdot(p-a)}}{b+c} =$$

$$= \sum\frac{2\cdot\sqrt{bc}\cdot p(p-a)\cdot\sqrt{(p-b)\cdot(p-c)}}{p-a} =$$

$$= \sum_{\Delta} 2\sqrt{bc}\cdot p\sqrt{(p-b)(p-c)} = \begin{Bmatrix} p-a=x \\ p-b=y \\ p-c=z \end{Bmatrix} =$$

$$= \sum 2\cdot\sqrt{(x+z)(x+y)}\cdot(x+y+z)\cdot\sqrt{yz} =$$

$$= 2(x+y+z)\cdot\sum\sqrt{y(x+z)}\cdot\sqrt{z(x+y)} \overset{CBS}{\leq}$$

$$= 2\cdot(x+y+z)\cdot\sqrt{\big(x(y+z)+y(z+x)+z(x+y)\big)^2} =$$

$$= 2(x+y+z)\cdot 2(xy+yz+zx) =$$

$$= 4\cdot\left(\sum(x^2y+xy^2)+3xyz\right) = LHS$$

$$\textbf{2)}\ \frac{9}{2}abc = \frac{9}{2}\cdot(x+y)(y+z)(z+x) = RHS$$

$$LHS \leq RHS \quad \text{(ASSURE)}$$

$$4\cdot\left(\sum(x^2y+xy^2)+3xyz\right)\leq\frac{9}{2}(x+y)(y+z)(z+x)$$

$$8\cdot\left(\sum(x^2y+xy^2)+3xyz\right)\leq 9\cdot\left(\sum(x^2y+xy^2)+2xyz\right)$$

$$6xyz\leq\sum(x^2y+xy^2);\ GM\leq AM$$

308. In $\Delta\ ABC$:

$$m_a^2\tan^2\frac{A}{2}+m_b^2\tan^2\frac{B}{2}+m_c^2\tan^2\frac{C}{2}\leq\frac{s^2-r^2-4Rr}{2}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo no obtusángulo $ABC\ \frac{a}{2m_a}\geq\tan\frac{A}{2}$

Elevando al cuadrado la expresión se tiene lo siguiente

$$\frac{a^2}{2b^2+2c^2-a^2}\geq\frac{(s-b)(s-c)}{s(s-a)}\Leftrightarrow$$

$$\Leftrightarrow a^2((b+c)^2-a^2)\geq(2(b^2+c^2)-a^2)(a^2-(b-c)^2)$$

Como es un triángulo no obtusángulo $\rightarrow\cos A\geq 0$

$$\Leftrightarrow a^2(b+c)^2-a^4\geq 2a^2(b^2+c^2)-a^4-2(b^2+c^2)(b-c)^2+a^2(b-c)^2$$

$$\Leftrightarrow 0\geq 2a^2(b^2+c^2-2bc)-2(b^2+c^2)(b-c)^2\Leftrightarrow -2(b^2+c^2-a^2)(b-c)^2\leq 0$$

$\Leftrightarrow 4bc\cos A\,(b-c)^2\geq 0.$ ***Análogamente para los siguientes términos***

$\frac{b}{2m_a}\geq\tan\frac{B}{2},\frac{c}{m_c}\geq\tan\frac{C}{2}$. ***Probar en un triágulo*** ABC

$$P=m_a^2\tan^2\frac{A}{2}+m_b^2\tan^2\frac{B}{2}+m_c^2\tan^2\frac{C}{2}\leq\frac{s^2-r^2-4Rr}{2}$$

$$P\leq\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}=\frac{s^2-r^2-4Rr}{2}\ \text{(LQQD)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Case 1. ΔABC ***is acute-angled.***

$$m_a \leq R(1+\cos A) = 2R\cos^2\frac{A}{2} \Rightarrow m_a^2 \leq 4R^2\cos^4\frac{A}{2}$$

$$\therefore m_a^2\tan^2\frac{A}{2} \overset{(1)}{\leq} 4R^2\cos^4\frac{A}{2}\tan^2\frac{A}{2} = 4R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}$$

$$= \left(2R\sin\frac{A}{2}\cos\frac{A}{2}\right)^2 = R^2\sin^2 A = \frac{1}{4}(2R\sin A)^2 = \frac{a^2}{4}$$

Similarly, $m_b^2\tan^2\frac{B}{2} \overset{(2)}{\leq} \frac{b^2}{4}$ ***and*** $m_c^2\tan^2\frac{C}{2} \overset{(3)}{\leq} \frac{c^2}{4}$

(1)+(2)+(3)⇒ LHS $\leq \frac{\sum a^2}{4} = \frac{s^2-4Rr-r^2}{2}$ ***(Proved)***

Case 2: $\Delta\ ABC$ ***is right-angled. WLOG, we may assume*** $m(\angle BAC) = 90°$

$$m_a^2 = \frac{2b^2+2c^2-a^2}{4} = \frac{2a^2-a^2}{4} = \frac{a^2}{4} \Rightarrow m_a = \frac{a}{2}$$

$$\therefore m_a^2\tan^2\frac{A}{2} = \frac{a^2}{4} \quad (4)$$

Also, $\because \angle LB$ ***and*** $\angle LC$ ***are acute,*** $\therefore m_b \leq R(1+\cos B)$ ***etc, which will ultimately yield*** $m_b^2\tan^2\frac{B}{2} \leq \frac{b^2}{4}$ ***(5) and*** $m_c^2\tan^2\frac{C}{2} \leq \frac{c^2}{4}$ ***(6)***

(4)+(5)+(6)⇒ $\sum m_a^2\tan^2\frac{A}{2} \leq \frac{\sum a^2}{4} = \frac{s^2-4Rr-r^2}{2}$ ***(Proved)***

309. Prove that in any triangle ABC the following inequality holds

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \leq 5+\frac{2R}{r}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC ***la siguiente desigualdad***

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\le 5+\frac{2R}{r}$$

$$\Leftrightarrow 3+\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\le 5+\frac{2R}{r}\Leftrightarrow \frac{2R}{r}+2\ge \frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$$

$$\Leftrightarrow \frac{abc}{2(s-a)(s-b)(s-c)}+2\ge \frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\quad (A)$$

Siendo** a,b,c **los lados de un triángulo realizamos los siguientes cambios** $x=s-a>0, y=s-b>0, z=s-c>0,$ $x+y=c>0, y+z=a>0, z+x=b>0$. **Lo cual tenemos en (A)

$$\Leftrightarrow \frac{(x+y)(y+z)(z+x)}{2xyz}+2\ge \frac{y+z+2x}{y+z}+\frac{z+x+2y}{z+x}+\frac{x+y+2z}{x+y}$$

$\Leftrightarrow \frac{x+y}{2z}+\frac{y+z}{2x}+\frac{z+x}{2y}+3\ge \frac{2x}{y+z}+\frac{2y}{z+z}+\frac{2z}{x+y}+3$. ***Por la desigualdad de Cauchy*** $\frac{x}{2z}+\frac{x}{2y}\ge \frac{2x}{y+z}$ ***(M),*** $\frac{y}{2x}+\frac{y}{2z}\ge \frac{2y}{z+x}$ ***(N),*** $\frac{z}{2y}+\frac{z}{2x}\ge \frac{2z}{x+y}$ ***(P)***

Sumando (M) + (N) + (P) $\rightarrow \frac{x+y}{2z}+\frac{y+z}{2x}+\frac{z+x}{2y}\ge \frac{2x}{y+z}+\frac{2y}{z+z}+\frac{2z}{x+y}$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=(2s)\frac{(\sum ab)}{4Rrs}=\frac{(\sum ab)}{2Rr}\le\frac{2R+5r}{r}$$

$$\Leftrightarrow \sum ab\le 2R(2R+5r)=4R^2+10Rr\Leftrightarrow s^2+4Rr+r^2\le$$

$\le 4R^2+10Rr\Leftrightarrow s^2\le 4R^2+6Rr-r^2$. ***Now, Gerretsen*** $\Rightarrow s^2\le 4R^2+$ $+4Rr+3r^2$ $\therefore$ ***it suffices to prove:*** $4R^2+4Rr+3r^2\le 4R^2+6Rr-r^2$

$\Leftrightarrow 2Rr\ge 4r^2\Leftrightarrow R\ge 2r\rightarrow$ ***true, by Euler (Proved)***

310. In $\Delta\,ABC, G-$ centroid, $I-$ incentre

$$AG\cdot BG\cdot CG+AI\cdot BI\cdot CI\ge\frac{4R}{27r}(h_ah_bh_c+27r^3)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo G - Centroid, I – Incentre. Probar en un $\Delta\ ABC$ la siguiente desigualdad $GA\cdot GB\cdot GC+IA\cdot IB\cdot IC\geq\frac{4R}{27r}(h_ah_bh_c+27r^3)$

Tener en cuenta las siguientes identidades y desigualdades en un $\Delta\ ABC$

$$GA=\frac{2}{3}m_a, GB=\frac{2}{3}m_b, GC=\frac{2}{3}m_c, IA=r\csc\frac{A}{2}, IB=r\csc\frac{B}{2}, IC=r\csc\frac{C}{2}$$

$$h_a=\frac{bc}{2R}, h_b=\frac{ca}{2R}, h_c=\frac{ab}{2R}, h_ah_bh_c=\frac{a^2b^2c^2}{8R^3}=\frac{16s^2r^2R^2}{8R^3}=\frac{2s^2r^2}{R}$$

$$m_a\geq\sqrt{s(s-a)}, m_b\geq\sqrt{s(s-b)}, m_c\geq\sqrt{s(s-c)}\Leftrightarrow m_am_bm_c\geq sS=s^2r$$

Luego $GA\cdot GB\cdot GC=\frac{8}{27}m_am_bm_c\geq\frac{8}{27}s^2r=\frac{4R}{27r}h_ah_bh_c$ ***(A)***

$IA\cdot IB\cdot IC=r\csc\frac{A}{2}r\csc\frac{B}{2}r\csc\frac{C}{2}=r^3\cdot\frac{4R}{r}=\frac{4R}{27r}\cdot 27r^3$ ***(B)***

Sumando (A) + (B)$\Rightarrow GA\cdot GB\cdot GC+IA\cdot IB\cdot IC\geq\frac{4R}{27r}(h_ah_bh_c+27r^3)$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS=\frac{8}{27}m_am_bm_c+r^3\csc\frac{A}{2}\csc\frac{B}{2}\csc\frac{C}{2}$$

$$\underset{(1)}{\overset{m_a\geq\sqrt{s(s-a)}}{\geq}}\frac{8}{27}\sqrt{s^3(s-a)(s-b)(s-c)}+r^3\sqrt{\frac{bc}{(s-b)(s-c)}}\sqrt{\frac{ca}{(s-c)(s-a)}}\sqrt{\frac{ab}{(s-a)(s-b)}}$$

$$=\frac{8s\cdot rs}{27}+r^3\frac{abc}{(s-a)(s-b)(s-c)}=\frac{8s^2r}{27}+r^3\left(\frac{4Rrs^2}{r^2S^2}\right)=\frac{8s^2r}{27}+4Rr^2$$

$$RHS=\frac{4R}{27r}\left(\frac{a^2b^2c^2}{8R^3}+27r^3\right)=\frac{4R\cdot 16R^2r^2s^2}{27\times 8R^3r}+4Rr^2\overset{(2)}{=}\frac{8s^2r}{27}+4Rr^2$$

(1), (2) $\Rightarrow LHS\geq RHS$ ***(Proved)***

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$x=p-a$$
$$y=p-b$$
$$z=p-c$$

(I) $\prod AG=\prod\frac{2}{3}m_a\geq\frac{8}{27}\cdot\prod\sqrt{p(p-a)}=$

$$=\frac{8}{27}\cdot\sqrt{p^3(p-a)(p-b)(p-c)}=\frac{8}{27}\cdot(x+y+z)\cdot\sqrt{xyz(x+y+z)}$$

$$\textit{(II)}\ \prod_\Delta AI=\prod\frac{r}{\sin\frac{A}{2}}=\frac{r^3}{\prod\sin\frac{A}{2}}=\begin{bmatrix} r=\sqrt{\frac{xyz}{x+y+z}} \\ \prod_\Delta \sin\frac{A}{2}=\frac{xyz}{\prod(x-y)}\end{bmatrix}$$

$$\prod_\Delta AI=\frac{\prod(x+y)}{x+y+z}\cdot\sqrt{\frac{xyz}{x+y+z}}$$

$$\textit{(III)}\ LHS:\prod_\Delta AG+\prod_\Delta AI=\frac{[8\cdot(x+y+z)^3+27\cdot\prod(x+y)]}{27\cdot(x+y+z)\sqrt{(x+y+z)}}\cdot\sqrt{xyz}\quad (*)$$

$$\textit{(IV)}\ RHS\Rightarrow\frac{4}{27}\cdot\left(\frac{R}{r}\cdot\prod h_a+27Rr^2\right)$$

$$\textit{(V)}\ \frac{R}{2}h_ah_bh_c=\frac{R}{2}\cdot\frac{8\Delta^3}{abc}=\frac{\prod(x+y)}{4xyz}\cdot\frac{8xyz(x+y+z)\sqrt{xyz\cdot(x+y+z)}}{\prod(x+y)}=$$

$$=2\cdot(x+y+z)\cdot\sqrt{xyz(x+y+z)}$$

$$\textit{(VI)}\ 27Rr^2=\frac{27}{4}\cdot\frac{\prod(x+y)}{\sqrt{xyz(x+y+z)}}\cdot\frac{xyz}{x+y+z}=\frac{27}{4}\cdot\frac{\prod(x+y)\sqrt{xyz}}{(x+y+z)\sqrt{x+y+z}}$$

(VII), (IV); (V), (VI) $\Rightarrow$

$$\Rightarrow RHS=\frac{4}{27}\cdot\left(2\cdot(x+y+z)\sqrt{xyz(x+y+z)}+\frac{27}{4}\cdot\frac{\prod(x+y)\sqrt{xyz}}{(x+y+z)\sqrt{x+y+z}}\right)=$$

$$=\frac{1}{27}\cdot\left(\frac{8(\sum x)^3+27\prod(x+y)}{\sum x\sqrt{\sum x}}\right)\cdot\sqrt{xyz}\quad (**)$$

$$LHS\overset{m_a\geq\sqrt{p(p-a)}}{\geq}LHS\ (*)=RHS\ (**)$$

$$(*)=(**)$$

311. In $\Delta\ ABC$:

$$\frac{4}{9}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2\leq\frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \frac{4}{9}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2\le\frac{R}{2r}$

Recordar las siguientes identidad y desigualdad en un $\Delta\ ABC$

$a^2+b^2+c^2\le 9R^2$ ***(Leibniz),*** $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}$***. Lo cual implica, por***

$MA\ge MG$***:*** $\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\le\frac{1}{4ab}+\frac{1}{4bc}+\frac{1}{4ca}=\frac{1}{8Rr}$

Aplicando la desigualdad de Cauchy en la desigualdad propuesta

$$\frac{4}{9}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2\le\frac{4}{9}(a^2+b^2+c^2)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\le\frac{4}{9}\cdot 9R^2\cdot\frac{1}{8Rr}=\frac{R}{2r}$$

Por transitividad $\Rightarrow\frac{4}{9}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2\le\frac{R}{2r}$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{\sum\{a(c+a)(a+b)\}}{(b+c)(c+a)(a+b)}=\frac{\sum\{a(\sum ab+a^2)\}}{2abc+\sum a^2b+\sum ab^2}$$

$$=\frac{(\sum ab)(\sum a)+\sum a^3}{2abc+\sum ab\,(2s-c)}=\frac{2s(\sum ab)+3abc+2s(\sum a^2-\sum ab)}{2abc+2s(\sum ab)-3abc}$$

$$=\frac{12Rrs+4s(s^2-4Rr-r^2)}{2s(s^2+4Rr+r^2)-4Rrs}=\frac{4s(s^2-Rr-r^2)}{2s(s^2+r^2+2Rr)}\overset{(1)}{=}\frac{2(s^2-Rr-r^2)}{s^2+2Rr+r^2}$$

(1) $\Rightarrow$ ***it suffices to prove:*** $\frac{R}{2r}\ge\frac{4}{9}\cdot\frac{4(s^2-Rr-r^2)^2}{(s^2+2Rr+r^2)^2}$

$$\Leftrightarrow 9R(s^2+2Rr+r^2)^2\ge 32r(s^2-Rr-r^2)^2\quad\textbf{(2)}$$

LHS of (2) $\underset{(3)}{\overset{Gerretsen}{\le}} 9R(s^2+2Rr+r^2)(18Rr-4r^2)$

Also RHS of (2) $\underset{(4)}{\overset{Gerretsen}{\le}} 32r(s^2-Rr-r^2)(4R^2+3Rr+2r^2)$

(3), (4) $\Rightarrow$ ***it suffices to prove:***

$$9R(9R-2r)(s^2+2Rr+r^2)\ge 16(4R^2+3Rr+2r^2)\cdot(s^2-Rr-r^2)$$

$$\Leftrightarrow s^2(17R^2-66Rr-32r^2)+r(226R^3+157R^2r+62Rr^2+32r^3)\ge 0 \quad (5)$$

LHS of (5) $\ge (16Rr-5r^2)(17R^2-66Rr-32r^2)+r(226R^3+157R^2r+62Rr^2+32r^3)\overset{?}{\ge} 0$

$$\Leftrightarrow 498R^3-984R^2r-120Rr^2+192r^3\overset{?}{\ge} 0$$

$$\Leftrightarrow 249t^3-492t^2-60t+96\overset{?}{\ge} 0 \text{ (where } t=\frac{R}{r}\text{)}$$

$$\Leftrightarrow (t-2)\{(t-2)(249t+504)+960\}\overset{?}{\ge} 0 \to \text{true}, \because t=\frac{R}{r}\ge 2 \text{ (Euler)}$$

312. Prove that in any triangle ABC

$$\frac{h_a}{r_b+r_c}+\frac{h_b}{r_c+r_a}\le\frac{2h_ah_b}{(r_b+r_c)(r_c+r_a)}+\frac{1}{2}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \frac{h_a}{r_b+r_c}+\frac{h_b}{r_c+r_a}\le\frac{2h_ah_b}{(r_b+r_c)(r_c+r_a)}+\frac{1}{2}.$

La desigualdad es equivalente

$$\frac{\frac{2S}{a}}{\frac{2S}{a+c-b}+\frac{2S}{a+b-c}}+\frac{\frac{2S}{b}}{\frac{2S}{a+b-c}+\frac{2S}{b+c-a}}\le\frac{\frac{2S}{a}}{\frac{2S}{a+c-b}+\frac{2S}{a+b-c}}\cdot\frac{\frac{2S}{b}}{\frac{2S}{a+b-c}+\frac{2S}{b+c-a}}+\frac{1}{2}$$

$$\frac{(a+c-b)(a+b-c)}{2a^2}+\frac{(a+b-c)(b+c-a)}{2b^2}\le\frac{(a+b-c)^2(a+c-b)(b+c-a)}{2a^2b^2}+\frac{1}{2}\quad (A)$$

Como a,b,c son lados de un triángulo, defínase lo siguiente

$$x=b+c-a\ge 0, y=a+c-b\ge 0, z=a+b-c\ge 0$$

$$\Leftrightarrow x+y=2x, y+z=2a, z+x=2b. \text{ Tenemos en (A)}$$

$$\Leftrightarrow \frac{2yz}{(y+z)^2}+\frac{2zx}{(z+x)^2}\le\frac{8z^2xy}{(y+z)^2(z+x)^2}+\frac{1}{2}\Leftrightarrow$$

$$\Leftrightarrow 4yz(z+x)^2+4zx(y+z)^2-16z^2xy\le(z+y)^2(z+x)^2$$

$$\Leftrightarrow 4z^3y+4x^2yz+4y^2zx+4z^3x=4(z^2+xy)(zy+zx)\le(z^2+xy+zy+zx)^2=$$

$$= (z+y)^2(z+x)^2 \Leftrightarrow (MA \geq MG)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{h_a}{r_b+r_c} + \frac{h_b}{r_c+r_a} \overset{(1)}{\leq} \frac{2h_a h_b}{(r_b+r_c)(r_c+r_a)} + \frac{1}{2}.$$

$$\frac{h_a}{r_b+r_c} = \frac{\frac{2\Delta}{a}}{\frac{\Delta}{s-b}+\frac{\Delta}{s-c}} = \frac{\frac{2}{a}}{\frac{a}{(s-b)(s-c)}} \overset{(2)}{=} \frac{2(s-b)(s-c)}{a^2}. \textit{ Similarly, } \frac{h_b}{r_c+r_a} = \frac{2(s-c)(s-a)}{b^2} \quad (3)$$

$$(2),(3) \Rightarrow (1) \Leftrightarrow \frac{2(s-b)(s-c)}{a^2} + \frac{2(s-c)(s-a)}{b^2} \leq \frac{8(s-a)(s-b)(s-c)^2}{a^2b^2} + \frac{1}{2}$$

$$\Leftrightarrow 4b^2(s-a)(s-c) + 4a^2(s-c)(s-a) \overset{(4)}{\leq} 16(s-a)(s-b)(s-c)^2 + a^2b^2$$

$$\text{Let } \left.\begin{array}{r} s-a=x \\ s-b=y \\ s-c=z \end{array}\right\} \therefore S = x+y+z \Rightarrow \begin{array}{l} a=y+z, \\ b=z+x, \\ c=x+y \end{array}$$

$$\therefore (4) \Leftrightarrow 16xyz^2 + (y+z)^2(z+x)^2 - 4(z+x)^2yz - 4zx(y+z)^2 \geq 0$$

$$\Leftrightarrow x^2y^2 + y^2z^2 + z^2x^2 + z^4 - 2y^2zx + 2z^2xy -$$

$$-2x^2yz + 2z^2xy - 2z^3y - 2z^3x \geq 0$$

$$\Leftrightarrow (xy - yz - zx + z^2)^2 \geq 0 \rightarrow \textit{true (Proved)}$$

313. In $\Delta\, ABC, I$ - incenter:

$$\frac{1}{AI^2} + \frac{1}{BI^2} + \frac{1}{CI^2} \geq \frac{3}{2rR}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

$$\textit{Lemma 1. } \frac{\sin\frac{A}{2}}{r} = \frac{1}{AI}$$

$$\textit{Lemma 2. } \cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

$$LHS = \frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}}{r^2} = \frac{1}{r^2}\cdot\frac{3-(\cos A+\cos B+\cos C)}{2} = \frac{2-\frac{r}{R}}{2r^2} \geq \frac{3}{2Rr} \leftrightarrow R \geq 2r \ \ldots \textit{(EULER)}$$

Solution 2 by Kevin Soto Palacios - Huarmey – Peru

Siendo I – incentro. Probar en triángulo ABC $\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\geq\frac{3}{2rR}$

Tener en cuenta las siguientes identidades y desigualdades en un ΔABC

$$IA=r\csc\frac{A}{2}, IB=r\csc\frac{B}{2}, IC=r\csc\frac{C}{2}, R\geq 2r \text{ (Inequality Euler)}$$

$$\textbf{Coma}\rightarrow\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$$

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq 1-\frac{1}{4}=\frac{3}{4}$$

LHS es equivalente $\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}=\frac{\sin^2\frac{A}{2}}{r^2}+\frac{\sin^2\frac{B}{2}}{r^2}+\frac{\sin^2\frac{C}{2}}{r^2}\geq\frac{3}{4r^2}\geq\frac{3}{2r\cdot 2r}\geq\frac{3}{2rR}$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS=\frac{1}{r^2}\sum\sin^2\frac{A}{2}=\frac{1}{r^2}\sum\frac{(s-b)(s-c)}{bc}=\frac{1}{r^2}\frac{\sum\{a(s^2-s(b+c)+bc)\}}{abc}$$

$$=\frac{1}{4Rr^3s}\{s^2(2s)-2s(\textstyle\sum ab)+12Rrs\}=\frac{2s^3-2s(s^2+4Rr+r^2)+12Rrs}{4Rr^3s}$$

$$=\frac{12Rrs-2s(4Rr+r^2)}{4Rr^3s}=\frac{2rs(6R-4R-r)}{4Rr^3s}$$

$$=\frac{2R-r}{2Rr^2}\overset{?}{\geq}\frac{3}{2Rr}\Leftrightarrow 2R-r\overset{?}{\geq}3r\Leftrightarrow R\overset{?}{\geq}2r\rightarrow \textbf{true (Euler) (Proved)}$$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sin\frac{A}{2}=\sqrt{\frac{yz}{(x+y)\cdot(x+z)}};\ldots$$

$$r=\sqrt{\frac{xyz}{x+y+z}};R=\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz\cdot(x+y+z)}}$$

$$AI=\sqrt{\frac{x(x+z)(x+y)}{x+y+z}};\ldots$$

$$\sum_{\Delta}\frac{1}{AI^2}=(x+y+z)\cdot\sum_{\Delta}\frac{1}{x(x+z)(x+y)}=LHS$$

$$RHS:\frac{3}{2R\cdot r}=\frac{6\cdot(x+y+z)}{(x+y)(y+z)(z+x)}\Rightarrow LHS\ge RHS\ \textbf{(ASSURE)}$$

$$\sum\frac{1}{AI^2}=(x+y+z)\cdot\sum\frac{(y+z)yz}{xyz(x+y)(y+z)(z+x)}=$$

$$=(x+y+z)\cdot\frac{\sum(y+z)\cdot zy}{xyz(x+y)(y+z)(z+x)}=$$

$$=(x+y+z)\cdot\frac{[(x^2y+xy^2)+(y^2z+yz^2)+(z^2x+zx^2)]}{(xyz)(x+y)(y+z)(z+x)}\overset{Cauchy}{\ge}$$

$$=\frac{(x+y+z)\cdot 6\sqrt[6]{x^6y^6z^6}}{(xyz)(x+y)(y+z)(z+x)}=\frac{6\cdot(x+y+z)}{(x+y)(y+z)(z+x)}=RHS$$

Solution 5 by Martin Lukarevski-Stip

In $\Delta\ ABC$: $AI^2=\frac{bc(s-a)}{s}$. Hence by $\sum a(s-b)(s-c)=2rs(2R-r)$

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}=\frac{s}{abc}\sum\frac{a}{s-a}=\frac{1}{2Rr}\left(\frac{2R}{r}-1\right)\ge\frac{3}{2Rr}$$

314. In $\Delta\ ABC$:

$$\frac{9r^2}{4}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\ge S^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c los lados de un triángulo $ABC\wedge x,y,z$ números R, se cumple $(ya^2+zb^2+xc^2)(za^2+xb^2+yc^2)\ge(xy+yz+zx)(a^2b^2+b^2c^2+c^2a^2)$

Solución

$$yza^4+yxa^2b^2+y^2a^2c^2+z^2a^2b^2+zxb^4+zyb^2c^2+xza^2c^2+x^2b^2c^2+xyc^4\ge$$

$$\geq (xy + yz + zx)(a^2b^2 + b^2c^2 + c^2a^2)$$

$$\Leftrightarrow x^2b^2c^2 + y^2c^2a^2 + z^2a^2b^2 \geq yza^2(b^2 + c^2 - a^2) + zxb^2(c^2 + a^2 - b^2) + xyc^2(a^2 + b^2 - c^2)$$

$$\Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \geq \frac{2yzbc\cos A}{b^2c^2} + \frac{2zxca\cos B}{c^2a^2} + \frac{2xyab\cos C}{a^2b^2}$$

$$\Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \geq \frac{2yz\cos A}{bc} + \frac{2zx\cos B}{ca} + \frac{2xy\cos C}{ab}$$ ***(Lo cual demostraremos)***

Partimos de la siguiente expresión

$$\left(\frac{x}{a} - \frac{y}{b}\cos C - \frac{z}{c}\cos B\right)^2 + \left(\frac{y}{b}\sin C - \frac{z}{c}\sin B\right)^2 \geq 0$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2}(\sin^2 C + \cos^2 C) + \frac{z^2}{c^2}(\sin^2 B + \cos^2 B) - \frac{2xy\cos C}{ab} - \frac{2zx\cos B}{ca} -$$

$$-\frac{2yz(\cos B\cos C - \sin B\sin C)}{bc} \geq 0$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \geq \frac{2yz\cos A}{bc} + \frac{2zx\cos B}{ca} + \frac{2xy\cos C}{ab}$$ ***(LQQD)***

Siendo $x = \frac{1}{c^2}, y = \frac{1}{a^2}, z = \frac{1}{b^2}$ ***obtenemos Desigualdad de WALKER***

$$\Rightarrow 3\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \geq (a^2 + b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

$$\Rightarrow 9\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \geq 3(a^2 + b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq$$

$$\geq (a + b + c)^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$ ***Por transitividad***

$$\Rightarrow 9\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \geq (a + b + c)^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \frac{4S^2}{r^2}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

$$\Leftrightarrow \frac{9r^2}{4}\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \geq S^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

(LQQD)

315. In $\Delta\ ABC$:

$$h_a^2 \cdot \tan\frac{A}{2} + h_b^2 \cdot \tan\frac{B}{2} + h_c^2 \cdot \tan\frac{C}{2} \le \frac{9\sqrt{3}}{4} R^2$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en triángulo $ABC\ h_a^2 \tan\frac{A}{2} + h_b^2 \tan\frac{B}{2} + h_c^2 \tan\frac{C}{2} \le \frac{9\sqrt{3}R^2}{4}$ ***(A)***

Teniendo en cuenta la siguientes desigualdades en un triángulo ABC

$$2r \le R,\ 2p \le 3\sqrt{3}R,\ \tan\frac{A}{2} = \frac{(a+(b-c))(a-(b+c))}{4S} \le \frac{a^2-(b-c)^2}{4S} = \frac{a^2}{4S}$$

Análogamente para los siguientes términos $\tan\frac{B}{2} \le \frac{b^2}{4S},\ \tan\frac{C}{2} \le \frac{c^2}{4S}$

Utilizando las desigualdades previas en (A)

$$h_a^2 \tan\frac{A}{2} + h_b^2 \tan\frac{B}{2} + h_c^2 \tan\frac{C}{2} = \frac{4S^2}{a^2} \cdot \frac{a^2}{4S} + \frac{4S^2}{b^2} \cdot \frac{b^2}{4S} + \frac{4S^2}{c^2} \cdot \frac{c^2}{4S} = 3S =$$

$$= 3pr \le 3 \cdot \frac{3\sqrt{3}R}{2} \cdot \frac{R}{2} = \frac{9\sqrt{3}R^2}{4}$$ ***(LQQD)***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum h_a^2 \cdot \sqrt{\frac{(p-b)\cdot(p-c)}{p\cdot(p-a)}} = \sum h_a^2 \cdot \sqrt{\frac{(p-b)(p-c)(p-b)(p-c)}{p(p-a)(p-b)(p-c)}} =$$

$$= \sum \frac{h_a^2}{S} \cdot (p-b)(p-c) \overset{Cauchy}{\le} \sum \frac{h_a^2}{S} \cdot \left(\frac{p-b+p-c}{2}\right)^2 = \sum \frac{h_a^2}{S} \cdot \frac{a^2}{4} =$$

$$= \sum \left(\frac{2S}{a}\right)^2 \cdot \frac{1}{4S} \cdot a^2 = \sum \frac{S}{a^2} \cdot a^2 = 3S =$$

$$= 3 \cdot p \cdot r \overset{Euler}{\le} 3p \cdot \frac{R}{2} = \frac{3R}{2} \cdot \left(\frac{a+b+c}{2}\right) =$$

$$= \frac{3R}{2} \cdot \left(\frac{2R\cdot(\sin A + \sin B + \sin C)}{2}\right) \le \frac{9\sqrt{3}}{4} \cdot R^2$$

$$f''(x) = (\sin x)'' \le 0$$

316. In $\Delta\, ABC$:

$$\frac{\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}{\left(1-\sin\frac{A}{2}\right)\left(1-\sin\frac{B}{2}\right)\left(1-\sin\frac{C}{2}\right)}\geq 3\sqrt{3}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Daniel Sitaru – Romania

$$f(x)=\log\left(\frac{\cos x}{1-\sin x}\right), f'(x)=-\frac{1}{\cos x}, f''(x)=-\frac{\sin x}{\cos^2 x}<0$$

$$\sum f\left(\frac{A}{2}\right)\overset{JENSEN}{\geq} 3f\left(\frac{A+B+C}{6}\right)=3\log\sqrt{3}$$

$$\sum\log\left(\frac{\cos\frac{A}{2}}{1-\sin\frac{A}{2}}\right)\geq\log\left(\sqrt{3}\right)^3;\ \prod\left(\frac{\cos\frac{A}{2}}{1-\sin\frac{A}{2}}\right)\geq 3\sqrt{3}$$

317. In $\Delta\, ABC$:

$$\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}\geq\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Probar en un triángulo $ABC\ \frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}\geq\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$

Teniendo en cuenta las relaciones de las medians y simedians

$\frac{m_a}{s_a}=\frac{b^2+c^2}{2bc}, m_b=\frac{c^2+a^2}{2ca}, m_c=\frac{a^2+b^2}{2ab}$***. Por lo tanto***

$$\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)+\frac{1}{2}\left(\frac{c}{a}+\frac{a}{c}\right)+\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)=$$

$$=\frac{1}{2}\left(\frac{b+c}{a}+1\right)+\frac{1}{2}\left(\frac{a+c}{b}+1\right)+\frac{1}{2}\left(\frac{a+b}{c}+1\right)-\frac{3}{2}$$

$\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}=\frac{1}{2}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{3}{2}$***. Es necesario demostrar lo siguiente*** $\frac{1}{2}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{3}{2}\geq\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$

donde $x=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$ ***(MA ≥ MG)***

$x-3\geq 2\sqrt{x}\Leftrightarrow(\sqrt{x}-3)(\sqrt{x}+1)\geq 0$***, lo cual es cierto ya que***

$\sqrt{x}-3\geq 0\wedge\sqrt{x}+1\geq 4>0$ ***(LQQD)***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$s_a=\frac{2bc\cdot m_a}{b^2+c^2};\dots$$

$$\sum\frac{m_a}{s_a}=\sum\frac{b^2+c^2}{2bc}=\sum\frac{a^2+b^2}{2ab}=\frac{1}{2abc}\cdot\sum a^2c+b^2c=$$

$$=\frac{1}{2abc}\cdot\sum(a^2b+ab^2)=\frac{1}{2}\left(\frac{(a+b+c)(ab+bc+ca)-3abc}{abc}\right)=$$

$$=\frac{1}{2}\cdot\underbrace{\left((a+b+c)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-3\right)}_{LHS}\geq\underbrace{\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}}_{RHS}$$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=t\Leftrightarrow t\geq 9;\frac{1}{2}(t-3)\geq\sqrt{z}\Rightarrow t^2-6t+9\geq 4t$$

$$t^2-10t+9\geq 0;\underbrace{(t-9)}_{\geq 0}\cdot\underbrace{(t-1)}_{0}\geq 0$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$s_a=\frac{2bc}{b^2+c^2}\cdot m_a\ \textbf{etc}\Rightarrow\frac{m_a}{s_a}=\frac{b^2+c^2}{2bc}\ \textbf{etc}\ \therefore\sum\frac{m_a}{s_a}=\sum\frac{b^2+c^2}{2bc}\overset{(1)}{=}\frac{\sum a^2b+\sum ab^2}{2abc}$$

(1)⇒ it suffices to prove: $\frac{\left(\sum a^2b+\sum ab^2\right)^2}{4a^2b^2c^2}\geq\frac{(\sum a)(\sum ab)}{abc}$

$$\Leftrightarrow \left(\sum a^2b + \sum ab^2\right)^2 \ge 4abc\left(\sum a\right)\left(\sum ab\right)$$

$$\Leftrightarrow \sum a^4b^2 + \sum a^2b^4 + 2\sum a^3b^3 + 2abc\left(\sum a^3\right) \ge$$

$$\ge 2a^3b^2c + 2a^3bc^2 + 2a^2b^3c + 2a^2bc^3 + 3ab^3c^2 + 2ab^2c^3 + 6a^2b^2c^2$$

Now, $\sum x^3 + 3xyz \overset{Schur}{\ge} \sum x^2y + \sum xy^2 \rightarrow$ ***(i)***

$\sum x^3 \overset{A-G}{\ge} 3xyz \rightarrow$ ***(ii)***

(i) + (ii) $\Rightarrow 2\sum x^3 \ge \sum x^2y = \sum xy^2 \rightarrow$ ***(iii)***

(iii) $\rightarrow 2\sum a^3b^3 \ge a^3b^2c + a^3bc^2 + a^2b^3c + a^2bc^3 + ab^3c^2 + ab^2c^3$, ***and***

also $2\sum a^3 \ge \sum a^2b + \sum ab^2$

$$\Rightarrow abc\left(2\sum a^3\right) \ge a^3b^2c + a^3bc^2 + a^2b^3c + a^2bc^3 + ab^3c^2 + ab^2c^3$$

Also, AM – GM $\Rightarrow \sum a^4b^2 + \sum a^2b^4 \ge 6a^2b^2c^2$ ***(5)***

(3) + ***(4)*** + ***(5)*** $\Rightarrow$ ***(2) is true (Proved)***

318. **In acute $\Delta\, ABC$:**

$$\tan A \cdot \tan B \cdot \tan C > \frac{2S}{s^2}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo ABC

$$\tan A \cdot \tan B \cdot \tan C > \frac{2S}{s^2} = \frac{2sr}{s^2} = \frac{2r}{s} = 2\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$$

$$\Leftrightarrow \tan A \cdot \tan B \cdot \tan C \cdot \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} > 2$$

Dado que es un triángulo acutángulo $\cos A, \cos B, \cos C > 0$

Además $x+y+z=xyz$***, donde*** $x=\tan A>0, y=\tan B>0, z=\tan C>0$

Aplicando MA ≥ MG

$x+y+z\ge 3\sqrt[3]{xyz}\Leftrightarrow (xyz)^3\ge 27xyz\Leftrightarrow xyz\ge 3\sqrt{3}$***. De manera análoga*** $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\Leftrightarrow \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\ge 3\sqrt{3}$

Por lo tanto $\tan A\cdot\tan B\cdot\tan C\cdot\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\ge 27>2$

Solution 2 by Ravi Prakash-New Delhi-India

For $0<\theta<\frac{\pi}{2}$; $\tan\theta=\frac{2\tan\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}\Rightarrow 2\tan\left(\frac{\theta}{2}\right)=\tan\theta\left[1-\tan^2\frac{\theta}{2}\right]<\tan\theta$

$$\therefore \tan A\cdot\tan B\cdot\tan C>8\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)$$

$$=8\left(\frac{\Delta}{s(s-a)}\right)\left(\frac{\Delta}{s(s-b)}\right)\left(\frac{\Delta}{s(s-c)}\right)=\frac{8\Delta^3}{s^2\Delta}=\frac{8\Delta}{s^2}>\frac{2\Delta}{s^2}$$

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\tan A+\tan B+\tan C\ge 3\sqrt{3};\ \tan A\cdot\tan B\cdot\tan C\ge 3\sqrt{3}$$

$$3\sqrt{3}>\frac{2S}{s^2};\ 3\sqrt{3}S^2>2Sr;\ a=3\sqrt{3}S>2r\to 2s\ge 3\sqrt{3}r$$

$$a\ge 3\sqrt{3}\cdot\frac{3\sqrt{3}r}{2}=\frac{27r}{2};\ 27>4$$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{\prod\sin A}{\prod\cos A}=\frac{\frac{2S}{4R^2}}{\frac{p^2-(2R+r)^2}{4R^2}}=\frac{2S}{p^2-(2R+r)^2};\ \prod\tan A=\frac{2S}{p^2-(2R+r)^2}$$

$$LHS=\frac{2S}{p^2-(2R+r)^2}>\frac{2S}{p^2};\ p^2>p^2-(2R+r)^2$$

Solution 5 by Soumava Chakraborty-Kolkata-India

$$\tan A\tan B\tan C\overset{(1)}{>}\frac{2S}{s^2}$$

$$(1) \Leftrightarrow \frac{\prod \sin A}{\prod \cos A} > \frac{2 \cdot 2R^2 (\prod \sin A)}{s^2}$$

$$\Leftrightarrow \frac{1}{\prod \cos A} > \frac{4R^2}{s^2} \Leftrightarrow \prod \cos A < \frac{s^2}{4R^2} \text{ (2) } (\because \prod \cos A > 0)$$

Now, $\prod \cos A \overset{(3)}{\leq} \frac{1}{8}$***; (2), (3)*** $\Rightarrow$ ***it suffices to show:*** $\frac{1}{8} < \frac{s^2}{4R^2} \Leftrightarrow 2s^2 > R^2$ ***(4)***

In an acute-angled triangle, $s > 2R + r$

$$\Rightarrow 2s^2 > 2(2R + r)^2 > R^2 \Rightarrow \text{(4) is true (Proved)}$$

Solution 6 by Adil Abdullayev-Baku-Azerbaidian

Lemma 1. In acute $ABC \tan A \cdot \tan B \cdot \tan C \geq \frac{s}{r}$.

Lemma 2. $s^2 \geq 27r^2$.

$$LHS \geq \frac{s}{r} > \frac{2S}{s^2} \leftrightarrow S^2 > 2r^2 \cdot s^2 \geq 27r^2 > 2r^2.$$

319. In $\Delta\ ABC$:

$$\frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \geq 3 \cdot \sqrt[3]{\frac{2r}{R}}$$

Proposed by Adil Abdullayev – Baku – Azerbaidjian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \geq 3 \cdot \sqrt[3]{\frac{2r}{R}}$

Teniendo en cuenta las siguientes notaciones

$$h_a = \frac{bc}{2R}, h_b = \frac{ca}{2R}, h_c = \frac{ab}{2R}, p = 4R \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$$

$$w_a = \frac{2bc}{b + c} \cos\frac{A}{2}, w_b = \frac{2ca}{c + a} \cos\frac{B}{2}, w_c = \frac{2ab}{a + b} \cos\frac{C}{2}$$

Por lo tanto la desigualdad es equivalente

$$\frac{b+c}{4R\cos\frac{A}{2}}+\frac{c+a}{4R\cos\frac{A}{2}}+\frac{a+b}{4R\cos\frac{C}{2}}\geq 3\sqrt[3]{\frac{r}{2R}}.\text{ Aplicando } MA\geq MG$$

$$\frac{b+c}{4R\cos\frac{A}{2}}+\frac{c+a}{4\cos\frac{B}{2}}+\frac{a+b}{4R\cos\frac{C}{2}}\geq 3\sqrt[3]{\frac{\prod(a+b)}{16R^2p}}\geq 3\sqrt[3]{\frac{8abc}{16R^2p}}=3\sqrt[3]{\frac{32pRr}{16R^2p}}=3\sqrt[3]{\frac{2r}{R}}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{h_a}{w_a}=\frac{\frac{bc}{2R}}{\frac{2bc\cos\frac{A}{2}}{b+c}}=\frac{b+c}{4R\cos\frac{A}{2}}=\frac{(b+c)\sqrt{bc}}{4R\sqrt{s(s-a)}}\overset{A-G}{\underset{(1)}{\geq}}\frac{bc}{2R\sqrt{s(s-a)}}$$

$$\text{Similarly, }\frac{h_b}{w_b}\overset{(2)}{\geq}\frac{ca}{2R\sqrt{s(s-b)}}\text{ and, }\frac{h_c}{w_c}\overset{(3)}{\geq}\frac{ab}{2R\sqrt{s(s-c)}}$$

$$\therefore\sum\frac{h_a}{w_a}\overset{A-G}{\geq}\frac{3}{2R}\sqrt[3]{\frac{a^2b^2c^2}{s\sqrt{s(s-a)(s-b)(s-c)}}}$$

$$\text{(using (1), (2), (3))}=\left(\frac{3}{2R}\right)\sqrt[3]{\frac{16R^2r^2s^2}{rs^2}}=\left(\frac{3}{R}\right)\sqrt[3]{2R^2r}=3\sqrt[3]{\frac{2R^2r}{R^3}}=3\sqrt[3]{\frac{2r}{R}}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\text{1) }\prod_\Delta\sin\frac{A}{2}=\frac{r}{4R}$$

$$\text{2) }\prod_\Delta\left(\frac{b+c}{a}\right)=\frac{p^2+2Rr+r^2}{2Rr}$$

$$\text{3) }p^2\geq 16Rr-5r^2\text{ (GERRETSEN)}$$

$$\sum\frac{h_a}{w_a}=\sum\frac{2S}{a}\cdot\frac{b+c}{2\sqrt{bc}\cdot\sqrt{p(p-a)}}=\sum\sqrt{\frac{(p-b)(p-c)}{bc}}\cdot\left(\frac{b+c}{a}\right)=$$

$$=\sum\sin\frac{A}{2}\left(\frac{b+c}{2}\right)\overset{Cauchy}{\geq}3\cdot\sqrt[3]{\prod_\Delta\sin\frac{A}{2}\cdot\prod\left(\frac{b+c}{a}\right)}=$$

$$=3\cdot\sqrt[3]{\frac{r}{4R}\cdot\frac{p^2+2Rr+r^2}{2Rr}}=3\cdot\sqrt[3]{\frac{p^2+2Rr+r^2}{8R^2}}\geq 3\sqrt[3]{\frac{2r}{R}}\quad\text{(ASSURE)}$$

$$\frac{p^2+2Rr+r^2}{8R^2}\ge\frac{2r}{R}\Rightarrow p^2\ge 14Rr-r^2\overset{GERRETSEN}{\Rightarrow}$$

$$p^2\ge 16Rr-5r^2\ge 14Rr-r^2\Leftrightarrow R\ge 2r\ \textit{(EULER)}$$

320. Let ABC be a triangle G_e is Gergonne point of ABC. Contact points of incircle are D, E and F. $D\in[BC], E\in[AC], F\in[AB]$

Prove that

$$|AD|^2+|BE|^2+|CF|^2\ge\frac{72r^3-rR(r_a+r_b+r_c)}{R}$$

where r and R are inradius and circumradius, r_a, r_b, r_c are exradii of ABC.

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Daniel Sitaru – Romania

$$\sum AD^2\ge\sum h_a^2=4S^2\sum\frac{1}{a^2}=4S^2\cdot\frac{s^4-2s^2r(4R-r)+r^2(4R+r)^2}{16R^2S^2}=$$

$$=\frac{s^4-2s^2r(4R-r)+r^2(4R+r)^2}{4R^2}\ge\frac{72r^3-rR(4R+r)}{R}\ \textit{(to prove)}$$

$$s^2\left(s^2-2r(4R-r)\right)\overset{GERRETSEN}{\geqq}(16Rr-5r^2)(16Rr-5r^2-8Rr+2r^2)\ge$$

$$\ge 288Rr^3-4rR^2(4r+r)-r^2(4Rr+r)^2\ \ \textit{(to prove)}$$

$$\frac{R}{r}=t\ge 2;\ (16t-5)(8t-3)\ge 288t-4t^2(4t+1)-(4t+1)^2$$

$$4t^3+37t^2-92t+4\ge 0;\ (t-2)(4t^2+45t-2)\ge 0,\ t\ge 2$$

321. In acute $\Delta\ ABC$:

$$\frac{1}{\sqrt{\tan\frac{A}{2}}}+\frac{1}{\sqrt{\tan\frac{B}{2}}}+\frac{1}{\sqrt{\tan\frac{C}{2}}}\ge 3\sqrt[4]{4-\frac{2r}{R}}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution by Soumava Chakraborty-Kolkata-India

$$LHS \overset{A-G}{\underset{(1)}{\geq}} 3\sqrt[3]{\frac{1}{\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}}}$$

$$= 3\sqrt[3]{\frac{1}{\sqrt{\sqrt{\frac{(s-b)(s-c)(s-c)(s-a)(s-a)(s-b)}{s(s-a)\cdot s(s-b)\cdot s(s-c)}}}}}$$

$$= 3\sqrt[3]{\frac{1}{\sqrt{\frac{(s-a)(s-b)(s-c)}{s\sqrt{s(s-a)(s-b)(s-c)}}}}} = 3\sqrt[3]{\frac{1}{\sqrt{\frac{r^2s^2}{s^2rs}}}} = 3\sqrt[3]{\sqrt{\frac{s}{r}}}$$

(1)⇒it suffices to prove: $\left(\frac{s}{r}\right)^{\frac{1}{6}} \geq \left(\frac{4R-2r}{R}\right)^{\frac{1}{4}}$

$$\Leftrightarrow \left(\frac{s}{r}\right)^2 \geq \left(\frac{4R-2r}{R}\right)^3 \Leftrightarrow s^2R^3 \geq r^2(4R-2r)^3 \quad \textbf{(2)}$$

LHS of (2) $\overset{Gerretsen}{\underset{(3)}{\geq}} (16Rr-5r^2)R^3$

(2), (3) ⇒ ***it suffices to prove:*** $(16R-5r)R^3 \geq 8r(2R-r)^3$

$$\Leftrightarrow 16r^4 - 69t^3 + 96t^2 - 48t + 8 \geq 0 \quad \left(\textit{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\{(t-2)^2(16t+27) + 66(t-2) + 20\} \geq 0$$

→ ***true,*** ∵ $t = \frac{R}{r} \geq 2$ ***(Euler) (Proved)***

322. In $\Delta\ ABC$:

$$\frac{3\sqrt{3}}{2} \leq \frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \leq \frac{3\sqrt{3}}{4}\cdot\frac{R}{r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{3\sqrt{3}}{2} \le \frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \le \frac{3\sqrt{3}R}{4r}$

Aplicando $MA \ge MG$

$$4m_a^2 + 3a^2 \ge 4\sqrt{3}am_a \Leftrightarrow 2(a^2 + b^2 + c^2) \ge 4\sqrt{3}am_a \Leftrightarrow a^2 + b^2 + c^2 \ge 2\sqrt{3}am_a$$

$$4m_b^2 + 3b^2 \ge 4\sqrt{3}bm_b \Leftrightarrow a^2 + b^2 + c^2 \ge 2\sqrt{3}bm_b$$

$$4m_c^2 + 3c^2 \ge 4\sqrt{3}cm_c \Leftrightarrow a^2 + b^2 + c^2 \ge 2\sqrt{3}cm_c$$

Por lo tanto

$$\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} = \frac{m_a^2}{am_a} + \frac{m_b^2}{bm_b} + \frac{m_c^2}{cm_c} \ge \frac{2\sqrt{3}m_a^2}{a^2 + b^2 + c^2} + \frac{2\sqrt{3}m_b^2}{a^2 + b^2 + c^2} + \frac{2\sqrt{3}m_c^2}{a^2 + b^2 + c^2}$$

$$\Leftrightarrow \frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \ge \frac{2\sqrt{3}(m_a^2 + m_b^2 + m_c^2)}{a^2 + b^2 + c^2} = 2\sqrt{3} \cdot \frac{3}{4} = \frac{3\sqrt{3}}{2}$$

Teniendo en cuenta las siguientes desigualdades

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2) \le \frac{3}{4} \cdot 9R^2 = \frac{27R^2}{4} \wedge \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \le \frac{1}{4r^2}$$

Por la desigualdad de Cauchy

$$\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \le \sqrt{(m_a^2 + m_b^2 + m_c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)} \le \sqrt{\frac{27R^2}{4} \cdot \frac{1}{4r^2}} = \frac{3\sqrt{3}R}{4r}$$

323. In $\Delta\, ABC$:

$$\frac{m_a m_b m_c}{w_a w_b w_c} \ge \frac{(a+b)(b+c)(c+a)}{8abc}$$

Proposed by Adil Abdullayev – Baku – Azerbaidjian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC $\frac{m_a m_b m_c}{l_a l_b l_c} \ge \frac{(a+b)(b+c)(c+a)}{8abc}$

Teniendo en cuento las siguientes identidades y desigualdades

$$l_a = \frac{2bc}{b+c}\cos\frac{A}{2}, l_b = \frac{2ca}{c+a}\cos\frac{B}{2}, l_c = \frac{2ab}{a+b}\cos\frac{C}{2}$$

$$m_a \geq \sqrt{p(p-a)}, m_b \geq \sqrt{p(p-b)}, m_c \geq \sqrt{p(p-c)}$$

$$\Rightarrow m_am_bm_c \geq \sqrt{p^2 \cdot p(p-a)(p-b)(p-c)} = pS = abc\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Por lo tanto $\Rightarrow \frac{m_am_bm_c}{l_al_bl_c} \geq \frac{abc\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\frac{2bc}{b+c}\cos\frac{A}{2}\cdot\frac{2ca}{c+a}\cos\frac{B}{2}\cdot\frac{2ab}{a+b}\cos\frac{C}{2}} = \frac{(a+b)(b+c)(c+a)}{8abc}$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\because m_a \geq \sqrt{r_br_c}, \text{ etc}, \therefore \prod m_a \overset{(1)}{\geq} r_ar_br_c = rs^2.$$

Again, $\frac{1}{w_aw_aw_c} = \frac{b+c}{2bc\cos\frac{A}{2}}\cdot\frac{c+a}{2ca\cos\frac{B}{2}}\cdot\frac{a+b}{2ab\cos\frac{C}{2}}$

$$= \frac{\pi(a+b)}{8a^2b^2c^2\cdot s\frac{S}{abc}} = \frac{\prod(a+b)}{8abc\cdot s\cdot rs} \overset{(2)}{=} \frac{\prod(a+b)}{8abc\,rs^2}$$

(1), (2) $\Rightarrow \frac{\prod m_a}{\prod w_a} \geq \frac{rs^2(\prod(a+b))}{8abc(rs^2)} = \frac{\prod(a+b)}{8abc}$ ***(Proved)***

Solution 3 by Martin Lukarevski-Stip

Tsintsifas inequality: $\frac{m_a}{w_a} \geq \frac{(b+c)^2}{4bc}$. ***Hence***

$$\frac{m_am_bm_c}{w_aw_bw_c} \geq \frac{(a+b)^2(b+c)^2(c+a)^2}{64a^2b^2c^2} \geq \frac{(a+b)(b+c)(c+a)}{8abc}$$

324. In $\Delta\ ABC$:

$$\frac{\sin^2 A\cdot\sin^2 B}{\sin^2 C} + \frac{\sin^2 B\cdot\sin^2 C}{\sin^2 A} + \frac{\sin^2 C\cdot\sin^2 A}{\sin^2 B} \geq \frac{9}{4}$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo ABC

$$\frac{\sin^2 A \sin^2 B}{\sin^2 C} + \frac{\sin^2 B \sin^2 C}{\sin^2 A} + \frac{\sin^2 C \sin^2 A}{\sin^2 B} \geq \frac{9}{4}$$

Iran Inequality 1996. Siendo $x, y, z > 0$***. Se cumple la siguiente desigualdad*** $(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}$ ***(A)***

Donde $x = \cot A > 0,\ \ y = \cot B,\ \ \ z = \cot C > 0 \Rightarrow xy + yz + zx = 1$

$$x + y = \cot A + \cot B = \frac{\sin C}{\sin A \sin B}, y + z = \frac{\sin A}{\sin B \sin C}, z + x = \frac{\sin C}{\sin A \sin B}$$

Por la tanto tenemos en (A) $\Rightarrow \frac{\sin^2 A \sin^2 B}{\sin^2 C} + \frac{\sin^2 B \sin^2 C}{\sin^2 A} + \frac{\sin^2 C \sin^2 A}{\sin^2 B} \geq \frac{9}{4}$

Solution 2 by Sanong Hauerai-Nakon Pathom-Thailand

Give $x = \sin A; y = \sin B;\ z = \sin C$; ***we obtain*** $x + y + z \leq \frac{(3\sqrt{3})}{2}$

$$xy + xy + zx \leq \frac{9}{4} \qquad (xy)^4 + \cdots + (zx)^4 \leq \frac{3^5}{4^4}$$

$$(xyz) \leq \frac{(3\sqrt{3})}{8} \qquad (xyz)^2 \leq \frac{3^3}{4^3} \qquad \frac{1}{(xyz)^2} \geq \frac{4^3}{3^3}$$

and since $\frac{1}{(xyz)^2} \geq (xy)^4 + \cdots + (zx)^4$***, hence*** $\left(\frac{xy}{z}\right)^2 + \left(\frac{yz}{x}\right)^2 + \left(\frac{zx}{y}\right)^2$

$= \frac{\left((xy)^4 + (yz)^4 + (zx)^4\right)^2}{(xyz)^2} \geq \left(\frac{3^5}{4^4}\right)\left(\frac{4^3}{3^3}\right) = \frac{3^2}{4} = \frac{9}{4}$***. Because*** $0 < x, y, z \leq 1$

325. In $\Delta\ ABC$:

$$(a + b)(b + c)(c + a) \geq \left(\frac{36r^2}{s}\right)^2$$

Proposed by Mehmet Șahin – Ankara – Turkey

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\prod(a + b) = \sum a \cdot \sum ab - abc =$$

$$= 2p \cdot \left(\sum ab - 2Rr\right) = \frac{2p^4(p^2 + 2Rr + r^2)}{p^3}$$

$$\geq \left.\begin{matrix} p \geq 3\sqrt{3}r \\ R \geq 2r \end{matrix}\right\} \geq \frac{2 \cdot 27^2 \cdot r^4 \cdot (27r^2 + 4r^2 + r^2)}{p^3} =$$

$$= \frac{27^2 \cdot 64 \cdot r^6}{p^3} = \frac{36^3 \cdot r^6}{p^3} = \left(\frac{36r^2}{p}\right)^3$$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$LHS \geq 8abc = 32RS = 32Rrs; 32Rrs^4 \geq 6^6r^6 \Rightarrow R \geq 2r$$

$$64s^4 \geq 6^6r^4; s^4 \geq 3^6r^4; s \geq 3\sqrt{3}r \text{ (Proved)}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

We have, $ab + bc + ca = p^2 + r^2 + 4Rr$, $abc = 4Rrp$, $R \geq 2r$ ***and*** $p \geq 3\sqrt{3}r$

$$\prod_{cyc}(a+b) = \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - abc = 2p(p^2 + r^2 + 4Rr) - 4Rrp$$

$$= \frac{2p^6 + 2p^4r^2 + 4Rrp^4}{p^3} \geq \frac{2\cdot 3^9r^6 + 2\cdot 3^6r^6 + 4\cdot 2r\cdot r\cdot 3^6}{p^3} = \left(\frac{36r^2}{p}\right)^3 \text{ (Proved)}$$

326. In $\Delta\, ABC$:

$$\sum \frac{r_a}{r_b} \geq \sum \frac{\cos^2 \frac{A}{2}}{\cos^2 \frac{B}{2}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si: $a, b, c > 0$***. Probar que:*** $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}$ ***(A)***

Realizamos los siguientes cambios de variables:

$P=\frac{1+\frac{a}{b}}{2}>0, Q=\frac{1+\frac{b}{c}}{2}>0, R=\frac{1+\frac{c}{a}}{2}>0.$ ***Por la desigualdad de Holder:***

$$8PQR=\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\geq\left(1+\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}\right)^3=8\Leftrightarrow PQR\geq 1$$

La desigualdad es equivalente:

$$\left(\frac{a}{b}-\frac{c+a}{c+b}\right)+\left(\frac{b}{c}-\frac{a+b}{a+c}\right)+\left(\frac{c}{a}-\frac{b+c}{b+a}\right)=\frac{c(a-b)}{b(c+b)}+\frac{a(b-c)}{c(c+a)}+\frac{b(c-a)}{a(a+b)}\geq 0$$

$$\Rightarrow\frac{\frac{a}{b}-1}{1+\frac{b}{c}}+\frac{\frac{b}{c}-1}{1+\frac{c}{a}}+\frac{\frac{c}{a}-1}{1+\frac{a}{b}}=\frac{2(P-1)}{2Q}+\frac{2(Q-1)}{2R}+\frac{2(R-1)}{2P}\geq 0$$

$\Rightarrow\frac{P}{Q}+\frac{Q}{R}+\frac{R}{P}\geq\frac{1}{Q}+\frac{1}{R}+\frac{1}{P}$ ***... (Lo cual demostraremos)***

Desde que: $P,Q,R>0$. ***Por: MA ≥ MG***

$$3\sum\frac{P}{Q}=\sum\left(\frac{P}{Q}+\frac{P}{Q}+\frac{R}{P}\right)\geq 3\sqrt[3]{\frac{PR}{Q^2}}=3\sqrt[3]{PQR}\sum\frac{1}{Q}=3\sum\frac{1}{Q}\ \textit{(LQQD)}$$

Probar en un triángulo ABC $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}+\frac{\cos^2\frac{B}{2}}{\cos^2\frac{C}{2}}+\frac{\cos^2\frac{C}{2}}{\cos^2\frac{A}{2}}$

La desigualdad es equivalente $\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\geq\frac{\sec^2\frac{B}{2}}{\sec^2\frac{A}{2}}+\frac{\sec^2\frac{C}{2}}{\sec^2\frac{B}{2}}+\frac{\sec^2\frac{A}{2}}{\sec^2\frac{C}{2}}$

$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\geq\frac{1+\tan^2\frac{B}{2}}{1+\tan^2\frac{A}{2}}+\frac{1+\tan^2\frac{C}{2}}{1+\tan^2\frac{B}{2}}+\frac{1+\tan^2\frac{A}{2}}{1+\tan^2\frac{C}{2}}\quad \textit{(A)}$$

Siendo $a=\tan\frac{A}{2}>0, y=\tan\frac{B}{2}>0, z=\tan\frac{C}{2}>0$

Lo cual implica $\rightarrow ab+bc+ca=1$

Además $\rightarrow 1+b^2=ab+bc+ca+b^2=(b+a)(b+c)$

Análogamente $\rightarrow 1+a^2=(a+b)(a+c)\wedge 1+c^2=(c+a)(c+b)$

En (A) $\rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{c+b}{c+a}+\frac{a+c}{a+b}+\frac{b+a}{b+c}$

Sea $x = \frac{a}{b} > 0, y = \frac{b}{c} > 0, z = \frac{c}{a} > 0 \Leftrightarrow xyz = 1$

La desigualdad es equivalente $x + y + z \geq \frac{z(y+1)}{z+1} + \frac{x(z+1)}{x+1} + \frac{y(x+1)}{y+1}$

$$\Leftrightarrow (x + y + z)(1 + x)(1 + y)(1 + z) \geq z(y + 1)^2(x + 1) +$$

$$+x(z + 1)^2(y + 1) + y(x + 1)^2(z + 1).$$ ***En el LHS***

$$(x + y + z)(1 + x)(1 + y)(1 + z) = (x + y + z)(1 + x + y + z + xy + yz + zx + xyz)$$

$$\Rightarrow (x + y + z)(1 + x)(1 + y)(1 + z) = x + y + z + (x + y + z)^2 +$$

$$+(x + y + z)(xy + yz + zx) + (x + y + z)xyz.$$ ***En el RHS***

$$\Rightarrow \sum z(y + 1)^2(x + 1) = \sum(zx + z)\left(y^2 + 2y + 1\right) = \sum\left(y^2zx + 2yzx + zx + y^2z + 2yz + z\right)$$

$$\Rightarrow \sum z(y + 1)^2(x + 1) = xyz(x + y + z) + 3(xy + yz + zx) + y^2z + z^2x + x^2y + 6xyz + x + y + z$$

Es suficiente probar

$$(x + y + z)^2 + (x + y + z)(xy + yz + zx) \geq 3(xy + yz + zx) + y^2z + z^2x + x^2y + 6xyz$$

Ahora $\rightarrow (x + y + z)^2 \geq 3(xy + yz + zx)$

$$(x + y + z)(xy + yz + zx) = y^2z + z^2x + x^2y + z^2y + x^2z + y^2x + 3xyz \geq$$

$$\geq y^2z + z^2x + x^2y + 6xyz$$ ***(Válido por MA ≥ MG)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{r_a}{r_b} \overset{(1)}{\geq} \sum \frac{\cos^2 \frac{A}{2}}{\cos^2 \frac{B}{2}}$$

$$(1) \Leftrightarrow \frac{s-b}{s-a} + \frac{s-c}{s-b} + \frac{s-a}{s-c}$$

$$\overset{(2)}{\geq} \frac{s(s-a)}{bc} \cdot \frac{ca}{s(s-b)} + \frac{s(s-b)}{ca} \cdot \frac{ab}{s(s-c)} + \frac{s(s-c)}{ab} \cdot \frac{bc}{s(s-a)}$$

$$= \frac{a(s-a)}{b(s-b)} + \frac{b(s-b)}{c(s-c)} + \frac{c(s-c)}{a(s-a)}$$

Let $s - a = x, s - b = y, s - c = z \therefore s = x + y + z$

$$\Rightarrow a = y + z, b = z + x, c = x + y$$

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$$\therefore (2) \Leftrightarrow \frac{y}{x}+\frac{z}{y}+\frac{x}{z} \geq \frac{(y+z)x}{(z+x)y}+\frac{(z+x)y}{(x+y)z}+\frac{(x+y)z}{(y+z)x}$$

$$\Leftrightarrow \frac{y^2z+z^2x+x^2y}{xyz} \geq \frac{zx^2(y+z)^2(x+y)+xy^2(z+x)^2(y+z)}{xyz(x+y)(y+z)(z+x)}+$$

$$+\frac{yz^2(x+y)^2(z+x)}{xyz(x+y)(y+z)(z+x)} \Leftrightarrow (x^2y+y^2z+z^2x)(x+y)(y+z)(z+x)$$

$$\geq xy^2(y+z)(z+x)^2+yz^2(z+x)(x+y)^2+zx^2(x+y)(y+z)^2$$

$$\Leftrightarrow x^4y^2+y^4z^2+z^4x^2+xyz\left(\sum x^3\right) \geq 3x^2y^2z^2+xyz\left(\sum xy^2\right) \quad \textbf{(3)}$$

Now, $x^4y^2+y^4z^2+z^4x^2 \underbrace{\overset{A-G}{\geq}}_{(4)} 3x^2y^2z^2$

Also,
$$\left.\begin{array}{r} x^3+y^3+y^3 \overset{A-G}{\geq} 3xy^2 \\ y^3+z^3+z^3 \overset{A-G}{\geq} 3yz^2 \\ z^3+x^3+x^3 \overset{A-G}{\geq} 3zx^2 \end{array}\right\} \begin{array}{c} \Rightarrow 3\left(\sum x^3\right) \geq 3\left(\sum xy^2\right) \\ \Rightarrow \sum x^3 \geq \sum xy^2 \\ \Rightarrow xyz\left(\sum x^3\right) \overset{(5)}{\geq} xyz\left(\sum xy^2\right) \end{array}$$

(4) + (5) ⇒ (3) is true (Proved)

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$r_a = \frac{\Delta}{p-a}$, $r_b = \frac{\Delta}{p-b}$ ***and*** $r_c = \frac{\Delta}{p-c}$

$$\therefore \sum_{cyc}\frac{r_a}{r_b} = \sum_{cyc}\frac{\frac{\Delta}{p-a}}{\frac{\Delta}{p-b}} = \sum_{cyc}\frac{p-b}{p-a} = \sum_{cyc}\frac{c+a-b}{b+c-a}.$$

Again, $\sum_{cyc}\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}} = \sum_{cyc}\frac{1+\cos A}{1+\cos B} = \sum_{cyc}\frac{1+\frac{c^2+b^2-a^2}{2bc}}{1+\frac{a^2+c^2-b^2}{2ca}} = \sum_{cyc}\frac{a}{b}\left(\frac{b+c-a}{c+a-b}\right)$

Applying RAVI TRANSFORMATION, $a = x+y$, $b = y+z$ ***and*** $c = z+x$

$\therefore a+b-c = 2y$, $b+c-a = 2z$ ***and*** $c+a-b = 2x$. ***We need to prove,***

$$\sum_{cyc}\frac{x}{z}=\sum_{cyc}\left(\frac{x+y}{y+z}\right)\frac{z}{x}\Leftrightarrow\sum_{cyc}\left(\frac{x}{z}-\frac{xz+yz}{xy+zx}\right)\geq 0\Leftrightarrow$$

$$\Leftrightarrow (xy+yz+zx)\sum_{cyc}\frac{x-z}{zx(y+z)}\geq 0$$

$$\Leftrightarrow\frac{xy+yz+zx}{xyz(x+y)(y+z)(z+x)}\left(\sum_{cyc}y(x-z)(x+z)(x+y)\right)\geq 0$$

$$\Leftrightarrow\frac{xy+yz+zx}{xyz(x+y)(y+z)(z+x)}\left(\sum_{cyc}x^3y-xyz\sum_{cyc}x\right)\geq 0$$

$$\Leftrightarrow\frac{xy+yz+zx}{(x+y)(y+z)(z+x)}\left(\sum_{cyc}\frac{x^2}{z}-x-y-z\right)\geq 0,$$

which is true $\left[\because \frac{x^2}{z}+\frac{y^2}{x}+\frac{z^2}{y}\overset{BERGSTROM}{\geq} x+y+z\right]$

$$\therefore \sum_{cyc}\frac{r_a}{r_b}\geq\sum_{cyc}\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}$$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$RHS=\sum\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}=\sum\frac{\frac{p(p-a)}{bc}}{\frac{p(p-b)}{ac}}$$

$$=\sum\frac{a(p-a)}{b(p-b)}=\sum\frac{\frac{(p-a)(p-b+p-c)}{S^2}}{\frac{(p-b)(p-a+p-c)}{S^2}}=\sum\frac{\frac{1}{r_a}\cdot\left(\frac{1}{r_b}+\frac{1}{r_c}\right)}{\frac{1}{r_b}\cdot\left(\frac{1}{r_a}+\frac{1}{r_c}\right)}=\sum\frac{r_b+r_c}{r_a+r_c}.$$

LHS; RHS $\Rightarrow r_a=x, r_b=y, r_c=z$

$$\sum\frac{x}{y}\geq\sum\frac{y+z}{x+z}\Leftrightarrow\sum\left(\frac{x}{y}+1\right)\geq\sum\frac{y+z}{x+z}+\frac{x+y}{y}+\frac{y+z}{z}+\frac{z+x}{x}\geq$$

$$\geq \frac{x+y}{z+y}+\frac{y+z}{x+z}+\frac{z+x}{y+x}+\sum(x+y)\cdot\left(\frac{1}{y}-\frac{1}{z+y}\right)=\sum\frac{z(x+y)}{y(z+y)}\overset{AM\geq GM}{\geq}$$

$$\geq 3\cdot\sqrt[3]{\frac{z(x+y)}{y(z+y)}\cdot\frac{x(y+z)}{z(x+z)}\cdot\frac{y(x+z)}{x(y+x)}}=3.$$ ***True.***

327. In $\Delta\ ABC$:

$$\sum\left(\frac{h_a}{w_a}\right)^k\geq 3\left(\prod\frac{h_a}{w_a}\right)^{\frac{k}{3}}\geq 3\left(\frac{2r}{R}\right)^{\frac{k}{3}}, k>0$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

Lemma. $w_aw_bw_c\leq rp^2.\quad h_ah_bh_c=\frac{2r^2p^2}{R}$

$$MA\geq MG\rightarrow\sum\left(\frac{h_a}{w_a}\right)^k\geq 3\left(\prod\frac{h_a}{w_a}\right)^{\frac{k}{3}}\geq 3\left(\frac{\frac{2r^2p^2}{R}}{rp^2}\right)^{\frac{k}{3}}=3\left(\frac{2r}{R}\right)^{\frac{k}{3}}$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

$\sum\left(\frac{h_a}{w_a}\right)^k\geq 3\left(\prod\frac{h_a}{w_a}\right)^{\frac{k}{3}}\geq 3\left(\frac{2r}{R}\right)^{\frac{k}{3}}$. ***Aplicando*** $MA\geq MG$

$$\sum\left(\frac{h_a}{w_a}\right)^k=\left(\frac{h_a}{w_a}\right)^k+\left(\frac{h_b}{w_b}\right)^k+\left(\frac{h_c}{w_c}\right)^k\geq 3\sqrt[3]{\left(\frac{h_a}{w_a}\cdot\frac{h_b}{w_b}\cdot\frac{h_c}{w_c}\right)^k}=3\left(\prod\frac{h_a}{w_a}\right)^{\frac{k}{3}}$$

Ahora bien $h_a=\frac{bc}{2R}, h_b=\frac{ca}{2R}, h_c=\frac{ab}{2R}, p=4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$$w_a=\frac{2bc}{b+c}\cos\frac{A}{2}, w_b=\frac{2ca}{c+a}\cos\frac{B}{2}, w_c=\frac{2ab}{a+b}\cos\frac{C}{2}$$

Por lo tanto, aplicando $MA\geq MG$

$$3\left(\prod\frac{h_a}{w_a}\right)^{\frac{k}{3}}=3\left(\frac{\prod(a+b)}{16R^2p}\right)^{\frac{k}{3}}\ge 3\left(\frac{8abc}{16R^2p}\right)^{\frac{k}{3}}=3\left(\frac{32pRr}{16R^2p}\right)^{\frac{k}{3}}=3\left(\frac{2r}{R}\right)^{\frac{k}{3}}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$RHS=\sum\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}=\sum\frac{\frac{p(p-a)}{bc}}{\frac{p(p-b)}{ac}}=\sum\frac{a(p-a)}{b(p-b)}=$$

$$=\sum\frac{(p-a)\cdot[p-b+p-c]\cdot S^2}{(p-b)\cdot[p-a+p-c]\cdot S^2}=\sum\frac{\frac{1}{r_a}\cdot\left[\frac{1}{r_b}+\frac{1}{r_c}\right]}{\frac{1}{r_b}\cdot\left[\frac{1}{r_a}+\frac{1}{r_c}\right]}=\sum\frac{r_b+r_c}{r_a+r_c}$$

$$\left.\begin{matrix}r_a=x\\ r_b=y\\ r_c=z\end{matrix}\right\}\Rightarrow\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\ge\frac{y+z}{x+z}+\frac{y+x}{z+x}+\frac{z+y}{x+y}$$

$$\sum\left(\frac{x}{y}-\frac{y+z}{x+z}\right)=\sum\frac{x^2+xz-y^2-yz}{y(x+z)}\overset{Cauchy}{\gtreqless}$$

$$\ge\frac{4}{(x+y+z)^2}\cdot\left[\sum(x^2+xz-y^2-yz)\right]=$$

$$=\frac{4}{(x+y+z)^2}\cdot\left((x^2+xz-y^2-yz)+(y^2+yx-z^2-zx)+(z^2+zy-x^2-xy)\right)=$$

$$=\frac{4}{(x+y+z)^2}\cdot 0=0\,;\sum\left(\frac{x}{y}-\frac{y+z}{x+z}\right)\ge 0$$

328. In $\Delta\ ABC$:

$$\frac{a^3}{b+c-a}+\frac{b^3}{c+a-b}+\frac{c^3}{a+b-c}\ge 4\sqrt{3}S$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

Lemma. $S\le\frac{p^2}{3\sqrt{3}}$.

Hölder's inequality $\rightarrow LHS \geq \frac{(a+b+c)^3}{3\left((b+c-a)+(a+c-b)+(b+a-c)\right)} = \frac{4p^2}{3} \geq$

$$\geq 4\sqrt{3}S \leftrightarrow S \leq \frac{p^2}{3\sqrt{3}}.$$

Solution 2 by Mehmet Șahin-Ankara-Turkey

$f(x) = \frac{x^3}{s-x}$ ***is concave,*** $(0 < x < s)$***. Using Jensen Inequality***

$$f\left(\frac{a+b+c}{3}\right) \leq \frac{1}{3}[f(a)+f(b)+f(c)]$$

$$\frac{1}{2}f\left(\frac{a+b+c}{3}\right) \leq \frac{1}{2}\cdot\frac{1}{3}\cdot[f(a)+f(b)+f(c)]$$

$$\frac{1}{2}[f(a)+f(b)-f(c)] \geq \frac{3}{2}f\left(\frac{a+b+c}{3}\right) \geq \frac{3}{2}\cdot\frac{\left(\frac{a+b+c}{3}\right)^3}{\frac{a+b+c}{2}-\frac{a+b+c}{3}}$$

$$\geq \frac{3}{2}\cdot\frac{(a+b+c)^3}{27}\cdot\frac{6}{a+b+c} \geq \frac{1}{3}(a+b+c)^2 = \frac{4}{3}s^2 \geq 4\sqrt{3}\cdot\Delta$$

Solution 3 by Nirapada Pal – Jhargram – India

$$\sum\frac{a^3}{b+c-a} = \sum\frac{a^4}{a(b+c-a)} \overset{Bergstrom}{\geq} \frac{(\sum a^2)^2}{\sum a\,(b+c-a)}$$

$= \frac{\left(\sum a^2\right)^2}{2\sum ab-\sum a^2} \geq \frac{(\sum ab)^2}{2\sum ab-\sum ab}$***. As*** $\sum a^2 \geq \sum ab; \sum ab \geq 4\sqrt{3}S$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\frac{a^3}{b+c-a}+\frac{b^3}{c+a-b}+\frac{c^3}{a+b-c} \overset{(1)}{\geq} 4\sqrt{3}S$$

$$\textbf{(1)} \quad \Leftrightarrow \frac{a^3}{s-a}+\frac{b^3}{s-b}+\frac{c^3}{s-c} \overset{(2)}{\geq} 8\sqrt{3}S$$

WLOG, we may assume $a \geq b \geq c$***. Then,*** $\frac{1}{s-a} \geq \frac{1}{s-b} \geq \frac{1}{s-c}$

$$\therefore \text{LHS of (2)} \geq \frac{1}{3}\left(\sum a^3\right)\left(\sum \frac{1}{s-a}\right) \quad \text{(Chebyshev)}$$

$$\overset{A-G}{\geq} \frac{1}{3} \cdot (3abc)\frac{\sum\{s^2 - s(b+c) + bc\}}{\prod(s-a)}$$

$$= 4Rrs\frac{(3s^2 - 4s^2 + s^2 + 4Rr + r^2)s}{r^2s^2}$$

$$= \frac{4Rr^2s^2}{r^2s^2}(4R + r) = 4R(4R + r)$$

$$\overset{Euler}{\geq} 8r(4R + r) \overset{Trucht}{\geq} 8r\left(s\sqrt{3}\right) = 8\sqrt{3}S$$

Solution 5 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\sum \frac{a^3}{b+c-a} \overset{Chebyshev}{\geq} \frac{1}{3}(a^3 + b^3 + c^3)\left(\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c}\right) \geq$$

$$\geq \frac{3(a^3 + b^3 + c^3)}{a+b+c} \overset{Chebyshev}{\geq} \frac{(a+b+c)(a^2 + b^2 + c^2)}{a+b+c} = a^2 + b^2 + c^2$$

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S \quad \text{(Ionescu-Weizenböck)}$$

Solution 6 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{a^3}{2(p-a)} = \sum \frac{a^3}{\left(\sqrt{2 \cdot (p-a)}\right)^2} \overset{Radon's}{\geq}$$

$$\geq \frac{(a+b+c)^3}{\left(\sum \sqrt{2 \cdot (p-a)}\right)^2} = \frac{(a+b+c)^3}{\left(\sqrt{2(p-a)} + \sqrt{2(p-b)} + \sqrt{2(p-c)}\right)^2} \overset{CBS}{\geq}$$

$$\geq \frac{(a+b+c)^3}{(1^2 + 1^2 + 1^2) \cdot \left(2(p-a) + 2(p-b) + 2(p-c)\right)} = \frac{(a+b+c)^3}{6p} =$$

$$= \frac{(a+b+c)^3}{3(a+b+c)} = \frac{(a+b+c)^2}{3} \geq ab + bc + ca \geq 4\sqrt{3}S$$

True

329. In $\Delta\ ABC$:

$$\sqrt[6]{\left(\sum a^2 r_a r_b\right)\left(\sum a^2 r_b r_c\right)\left(\sum a^2 r_c r_a\right)} \le \frac{9R^2}{2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC $\sqrt[6]{(\sum a^2 r_a r_b)(\sum a^2 r_b r_c)(\sum a^2 r_c r_a)} \le \frac{9R^2}{2}$

Teniendo en cuenta las siguientes desigualdades e identidades en un

ΔABC $r_a r_b + r_b r_c + r_c r_a = s^2$, $a^2 + b^2 + c^2 \le 9R^2$ ***(Leibniz),*** $s \le \frac{3\sqrt{3}R}{2}$

Aplicando MA ≥ MG

$$\sum a^2 r_a r_b + \sum a^2 r_b r_c + \sum a^2 r_c r_a \ge \sqrt[3]{(\sum a^2 r_a r_b)(\sum a^2 r_b r_c)(\sum a^2 r_c r_a)}$$

$$\sum a^2 s^2 \ge 3\sqrt[3]{\left(\sum a^2 r_a r_b\right)\left(\sum a^2 r_b r_c\right)\left(\sum a^2 r_c r_a\right)}$$

$$\Leftrightarrow \sqrt[6]{(\sum a^2 r_a r_b)(\sum a^2 r_b r_c)(\sum a^2 r_c r_a)} \le \frac{s\sqrt{a^2+b^2+c^2}}{\sqrt{3}} \le \frac{3\sqrt{3}R\cdot\sqrt{9R^2}}{2\sqrt{3}} = \frac{9R^2}{2}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt[6]{\left(\sum a^2 r_a r_b\right)\left(\sum a^2 r_b r_c\right)\left(\sum a^2 r_c r_a\right)} \overset{(1)}{\le} \frac{9R^2}{2}$$

$$\sum a^2 r_a r_b = a^2 r_a r_b + b^2 r_b r_c + c^2 r_c r_a$$

$$= \Delta^2\left[\frac{a^2}{(s-a)(s-b)} + \frac{b^2}{(s-b)(s-c)} + \frac{c^2}{(s-c)(s-a)}\right]$$

$$= \frac{\Delta^2\{a^2(s-c) + b^2(s-a) + c^2(s-b)\}}{(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta^2 s}{\Delta^2}\left(s\sum a^2 - \sum ab^2\right) \overset{(i)}{=} s^2\sum a^2 - s\sum ab^2$$

$$\sum a^2 r_b r_c = a^2 r_b r_c + b^2 r_c r_a + c^2 r_a r_b$$

$$= \Delta^2 \left[\frac{a^2}{(s-b)(s-c)} + \frac{b^2}{(s-c)(s-a)} + \frac{c^2}{(s-a)(s-b)}\right]$$

$$= \frac{\Delta^2\{a^2(s-a) + b^2(s-b) + c^2(s-c)\}}{(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta^2 s}{\Delta^2}\left(s\sum a^2 - \sum a^3\right) \overset{(ii)}{=} s^2\sum a^2 - s\sum a^3$$

$$\sum a^2 r_c r_a = a^2 r_c r_a + b^2 r_a r_b + c^2 r_b r_c$$

$$= \Delta^2 \left[\frac{a^2}{(s-c)(s-a)} + \frac{b^2}{(s-a)(s-b)} + \frac{c^2}{(s-b)(s-c)}\right]$$

$$= \frac{\Delta^2 s}{\Delta^2}\{a^2(s-b) + b^2(s-c) + c^2(s-a)\}$$

$$= s\left(s\sum a^2 - \sum a^2 b\right) \overset{(iii)}{=} s^2\sum a^2 - s\sum a^2 b$$

(1) $\Leftrightarrow \sqrt[3]{\left(\sum a^2 r_a r_b\right)\left(\sum a^2 r_b r_c\right)\left(\sum a^2 r_c r_a\right)} \overset{(2)}{\leq} \frac{81R^4}{4}$

Now, LHS of (2) $\overset{G\leq A}{\leq} \frac{\sum a^2 r_a r_b + \sum a^2 r_b r_c + \sum a^2 r_c r_a}{3}$

$$= \frac{\left(s^2\sum a^2 - s\sum ab^2\right) + \left(s^2\sum a^2 - s\sum a^3\right) + \left(s^2\sum a^2 - s\sum a^2 b\right)}{3}$$

(using (i), (ii), (iii))

$$= \frac{3s^2\sum a^2 - s\left(\sum a^3 + \sum a^2 b + \sum ab^2\right)}{3}$$

$$= \frac{3s^2\left(\sum a^2\right) - s\{3abc + 2s\left(\sum a^2 - \sum ab\right) + \sum ab(2s-c)\}}{3}$$

$$=\frac{3s^2\sum a^2-\{3abc+2s(\sum a^2)-2s(\sum ab)+2s(\sum ab)-3abc\}}{3}$$

$$=\frac{3s^2(\sum a^2)-2s^2(\sum a^2)}{3}=\frac{s^2(\sum a^2)}{3}$$

$$\overset{Mitrinovic}{\leq}\frac{27R^2}{4}\left(\frac{\sum a^2}{3}\right)\overset{Leibnitz}{\leq}\frac{27R^2}{4}\cdot\frac{9R^2}{3}=\frac{81R^4}{4}\Rightarrow \textit{(2) is true (Proved)}$$

330. In $\Delta\, ABC$:

$$\frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}+\frac{s^2}{s^2+r(R-2r)}\geq 2$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Soumava Chakraborty-Kolkata-India

$$m_am_b\leq\frac{2c^2+ab}{4}, \text{etc} \therefore \sum m_am_b\leq\frac{2\sum a^2+\sum ab}{4}$$

$$\therefore\frac{\sum m_a^2}{\sum m_am_b}\geq\frac{\frac{3}{4}(\sum a^2)\cdot 4}{2\sum a^2+\sum ab}=\frac{3\sum a^2}{2\sum a^2+\sum ab}$$

$$=\frac{6(s^2-4Rr-r^2)}{s^2+4Rr+r^2+2(s^2-4Rr-r^2)}=\frac{6s^2-24Rr-6r^2}{3s^2-4Rr-r^2}$$

$$\therefore\frac{\sum m_a^2}{\sum m_am_b}-2\overset{(1)}{\geq}\frac{6s^2-24Rr-6r^2}{3s^2-4Rr-r^2}-2$$

$$=\frac{6s^2-24Rr-6r^2-6s^2+8Rr+2r^2}{3s^2-4Rr-r^2}=\frac{-16Rr-4r^2}{3s^2-4Rr-r^2}$$

(1)⇒ it suffices to prove:

$$\frac{-16Rr-4r^2}{3s^2-4Rr-r^2}+\frac{s^2}{s^2+Rr-2r^2}\geq 0\Leftrightarrow\frac{s^2}{s^2+Rr-2r^2}\geq\frac{16Rr+4r^2}{3s^2-4Rr-r^2}$$

$$\Leftrightarrow s^2(3s-4Rr-r^2)\geq(16Rr+4r^2)(s^2+Rr-2r^2)\ \textit{(2)}$$

$$\text{Now, } 3s^2-4Rr-r^2 \overset{Gerretsen}{\geq} 3(16Rr-5r^2)-4Rr-r^2$$

$$= 44Rr-16r^2$$

$$\Rightarrow s^2(3s^2-4Rr-r^2) \geq s^2(44Rr-16r^2) \quad \textit{(3)}$$

(2),(3) ⇒ it suffices to prove:

$$s^2(44Rr-16r^2) \geq (16Rr+4r^2)(s^2+Rr-2r^2)$$

$$\Leftrightarrow s^2(11R-4r) \geq (4R+r)(s^2+Rr-2r^2)$$

$$\Leftrightarrow 11Rs^2-4rs^2 \geq 4Rs^2+4R^2r-8Rr^2+rs^2+Rr^2-2r^3$$

$$\Leftrightarrow s^2(7R-5r) \geq 4R^2r-7Rr^2-2r^3 \quad \textit{(4)}$$

$$\text{Gerretsen} \Rightarrow s^2(7R-5r) \geq (16Rr-5r^2)(7R-5r) \quad \textit{(5)}$$

(4),(5) ⇒ it suffices to prove:

$$(16Rr-5r^2)(7R-5r) \geq 4R^2r-7Rr^2-2r^3$$

$$\Leftrightarrow 112R^2-115Rr+25r^2 \geq 4R^2-7Rr-2r^2$$

$$\Leftrightarrow 108R^2-108Rr+27r^2 \geq 0$$

$$\Leftrightarrow 4R^2-4Rr+r^2 \geq 0 \Leftrightarrow (2R-r)^2 \geq 0 \rightarrow \textit{true (Proved)}$$

331. In $\Delta\, ABC$ the following relationship holds:

$$\frac{2m_a}{r_b+r_c} \geq \cos\frac{B-C}{2}$$

Proposed by Bogdan Fustei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c = \frac{\Delta}{s-b}+\frac{\Delta}{s-c} = \frac{\Delta(2s-b-c)}{(s-b)(s-c)} = \frac{a\Delta}{(s-b)(s-c)}$$

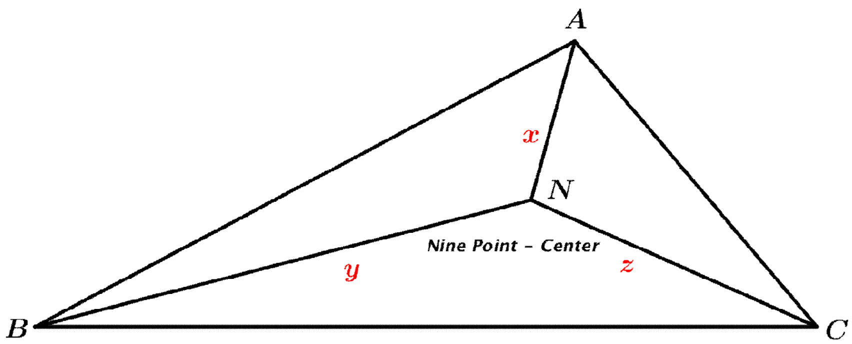
$$= \frac{a\sqrt{s(s-a)(s-b)(s-c)}}{(s-b)(s-c)} = \frac{a\sqrt{s(s-a)}}{\sqrt{(s-b)(s-c)}} \rightarrow \quad \textit{(1)}$$

By (1) and $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}, LHS \geq \left(\frac{b+c}{a}\right)\sqrt{\frac{s(s-a)(s-b)(s-c)}{bcs(s-a)}}$

$$= \frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{a}\sin\frac{A}{2} = \left(\frac{2R\sin A}{a}\right)\cos\frac{B-C}{2} = \cos\frac{B-C}{2} \quad \textit{(Done)}$$

332. In ΔABC :

$$x^2 + y^2 + z^2 \geq 12r^2$$



Proposed by Abdilkadir Altintas - Afyonkarashisar-Turkey

Solution by Soumava Chakraborty-Kolkata-India

In any $\Delta\ ABC$, $AN^2 + BN^2 + CN^2 \geq 12r^2$, Where $N \to$ nine – point center

For any point P in the plane of $\Delta\ ABC$, $\sum AP^2 = 3PG^2 + \sum(AG^2)$

$$\therefore \sum AN^2 = 3NG^2 + \sum(AG^2) \text{ (choosing } P \text{ as } N) = 3\left(\frac{1}{6}OH\right)^2 + \frac{4}{9}\cdot\frac{3}{4}\sum a^2$$

$$= \frac{OH^2}{12} + \frac{\sum a^2}{3} = \frac{9R^2 - \sum a^2}{12} + \frac{\sum a^2}{3}$$

$$= \frac{9R^2 + 3\sum a^2}{12} = \frac{3R^2 + \sum a^2}{4} = \frac{3R^2 + 2s^2 - 8Rr - 2r^2}{4}$$

$$\overset{Gerretsen}{\geq} \frac{3R^2 - 8Rr - 2r^2 + 32Rr - 10r^2}{4}$$

$$= \frac{3R^2+24Rr-12r^2}{4} \overset{Euler}{\geq} \frac{3(4r^2)+24r(2r)-12r^2}{4} = 12r^2 \quad \textit{(Proved)}$$

333. **Prove that in any triangle ABC the following relationship holds:**

$$\sum \frac{(a^2-ab+b^2)^2}{a^2+4ab+b^2} \geq \frac{2S}{\sqrt{3}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Probar en un triángulo $ABC \sum \frac{(a^2-ab+b^2)^2}{a^2+4ab+b^2} \geq \frac{2S}{\sqrt{3}}$.

Tener en cuenta las siguientes desigualdades

$$6(a^2-ab+b^2) \geq a^2+4ab+b^2 \Leftrightarrow 5(a-b)^2 \geq 0$$

$ab+bc+ca \geq 4S\sqrt{3}$. ***La desigualdad propuesta es equivalente***

$$\sum \frac{(a^2-ab+b^2)^2}{a^2+4ab+b^2} \geq \frac{a^2-ab+b^2}{6} + \frac{b^2-bc+c^2}{6} + \frac{c^2-ca+a^2}{6} =$$

$$= \frac{2\sum a^2 - \sum ab}{6} \geq \frac{\sum ab}{6} \geq \frac{4S\sqrt{3}}{6} = \frac{2S}{\sqrt{3}}.$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{(a^2-ab+b^2)^2}{a^2+4ab+b^2} \overset{Schwarz}{\geq} \frac{\big(2\cdot(a^2+b^2+c^2)-(ab+bc+ca)\big)^2}{2\cdot(a^2+b^2+c^2)+4(ab+bc+ca)}$$

$$\overset{\sum a^2 \geq \sum ab}{\geq} \frac{(\sum a^2 + \sum ab - \sum ab)^2}{2\cdot(a+b+c)^2} = \frac{(\sum a^2)^2}{2\cdot(\sum a)^2} \overset{Chebyshev}{\geq}$$

$$\geq \frac{\frac{1}{9}\cdot(\sum a)^4}{2\cdot(\sum a)^2} = \frac{1}{18}\left(\sum a\right)^2 = \frac{4}{18}\cdot p^2 = \frac{2}{9}p^2 \geq \frac{2}{9}\cdot p\cdot 3\sqrt{3}r = \frac{2S}{\sqrt{3}}.$$

334. **In $\Delta\, ABC$:**

$$\sqrt{\frac{m_a^2m_b^2+m_b^2m_c^2+m_c^2m_a^2}{9S^2}} \geq \frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}$$

Proposed by Adil Abdullayev – Baku – Azerbaidian

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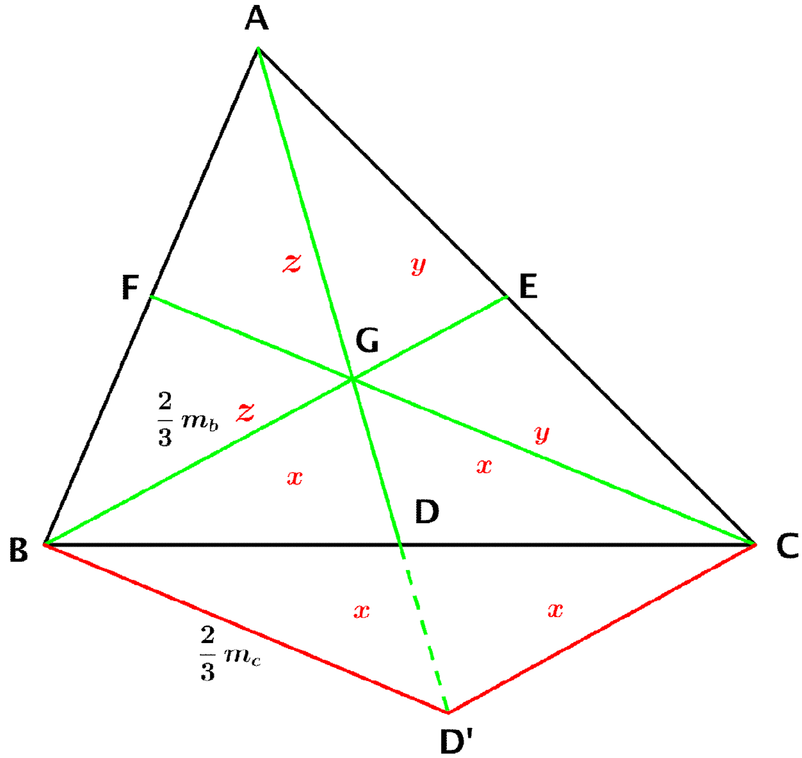
Solution by Soumava Chakraborty-Kolkata-India



In $\Delta\ BGD', GD' = \frac{2}{3}m_a, BG = \frac{2}{3}m_b$ ***and*** $BD' = \frac{2}{3}m_c$

$\because$ BG ***is a median in*** $\Delta\ ABD'$, $\therefore 2z = 2x \Rightarrow z = x$

$\because$ CG ***is a median in*** $\Delta\ ACD'$, $\therefore 2y = 2x \Rightarrow x = y$

$$\therefore x = y = z \therefore arc\ (\Delta\ BGD') = 2x = \frac{1}{3}(6x) = \frac{1}{3}arc\ (\Delta\ ABC) \overset{(1)}{=} \frac{S}{3}$$

For any $\Delta\ ABC$***, Hadwiger Finsler inequality*** $\Rightarrow 2\sum ab - \sum a^2 \geq 4\sqrt{3}S$

Applying this on $\Delta\ BGD'$***, we get,***

$$2\left(\frac{4}{9}\sum m_a m_b\right) - \frac{4}{9}\left(\sum m_a^2\right) \geq 4\sqrt{3}\left(\frac{S}{3}\right) \quad \textit{(from (1))}$$

$$\Rightarrow 2\sum m_a m_b - \sum m_a^2 \geq 3\sqrt{3}S$$

$$\Rightarrow 2\sum m_a m_b \geq \sum m_a^2 + 3\sqrt{3}S$$

$$\Rightarrow \frac{2\sum m_a m_b}{\sum m_a^2} \geq 1 + \frac{3\sqrt{3}S}{\sum m_a^2} = 1 + \frac{3\sqrt{3}S \cdot 4}{3\sum a^2} = 1 + \frac{4\sqrt{3}S}{\sum a^2}$$

$$\Rightarrow \frac{2\sum m_a m_b}{\sum m_a^2} \geq \frac{\sum a^2 + 4\sqrt{3}S}{\sum a^2} \Rightarrow \frac{\sum m_a m_b}{\sum m_a^2} \geq \frac{\sum a^2 + 4\sqrt{3}S}{2\sum a^2} \quad \textit{(2)}$$

Now, $\sum m_a^2 m_b^2 = \frac{\sum\{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)\}}{16} = \frac{9\sum a^2b^2}{16}$ ***(on simplification)***

$$\therefore \frac{\sum m_a^2 m_b^2}{9S^2} = \frac{9\sum a^2b^2}{16\cdot 9S^2} = \frac{\sum a^2b^2}{16S^2} \Rightarrow \sqrt{\frac{\sum m_a^2 m_b^2}{9S^2}} = \frac{\sqrt{\sum a^2b^2}}{4S} \overset{(3)}{\geq} \frac{\sqrt{\frac{1}{3}(\sum ab)^2}}{4S} \quad \textit{(Chebyshev)}$$

$$= \frac{\sum ab}{4\sqrt{3}S}; \textit{ (2), (3)} \Rightarrow \sqrt{\frac{\sum m_a^2 m_b^2}{9S^2}} \cdot \frac{\sum m_a m_b}{\sum m_a^2} \overset{(4)}{\geq} \left(\frac{\sum ab}{4\sqrt{3}S}\right)\left(\frac{\sum a^2 + 4\sqrt{3}S}{2\sum a^2}\right)$$

(4) $\Rightarrow$ ***it suffices to prove:*** $\frac{\sum ab}{4\sqrt{3}S} \cdot \frac{\sum a^2 + 4\sqrt{3}S}{2\sum a^2} \geq 1$

$$\Leftrightarrow \left(\sum ab\right)\left(\sum a^2\right) + 4\sqrt{3}S\left(\sum ab\right) \geq 8\sqrt{3}S\left(\sum a^2\right)$$

$$\Leftrightarrow 2\sum ab\left(\sum a^2 + 4\sqrt{3}S\right) \geq 16\sqrt{3}S\left(\sum a^2\right) \quad \textit{(5)}$$

Now, Hadwiger – Finsler inequality

$$\Rightarrow 2\sum ab\left(\sum a^2 + 4\sqrt{3}S\right) \geq \left(\sum a^2 + 4\sqrt{3}S\right)\left(\sum a^2 + 4\sqrt{3}S\right) =$$

$$= \left(\sum a^2 + 4\sqrt{3}S\right)^2 \overset{A-G}{\geq} 4\left(\sum a^2\right)\left(4\sqrt{3}S\right) = 16\sqrt{3}S\left(\sum a^2\right) \Rightarrow \textit{(5) is true}$$

335. If in $\Delta ABC, abc = 1$ then:

$$R^2\left(\frac{a^{r_a}}{r_a} + \frac{b^{r_b}}{r_b} + \frac{c^{r_c}}{r_c}\right)\left(\frac{a^{h_a}}{h_a} + \frac{b^{h_b}}{h_b} + \frac{c^{h_c}}{h_c}\right) \geq 4$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^{r_a}\left(\frac{1}{r_a}\right)+b^{r_b}\left(\frac{1}{r_b}\right)+c^{r_c}\left(\frac{1}{r_c}\right)}{\sum\frac{1}{r_a}}\overset{weighted\ A-G}{\geq}\sqrt[\left(\sum\frac{1}{r_a}\right)]{(a^{r_a})^{\left(\frac{1}{r_a}\right)}(b^{r_b})^{\left(\frac{1}{r_b}\right)}c^{r_c\left(\frac{1}{r_c}\right)}}=$$

$$=\sqrt[\left(\sum\frac{1}{r_a}\right)]{abc}=1\ (\because abc=1)\Rightarrow\sum\frac{a^{r_a}}{r_a}\overset{(1)}{\geq}\sum\frac{1}{r_a}=\frac{1}{r}$$

Again, $a^{h_a}\left(\frac{1}{h_a}\right)+b^{h_b}\left(\frac{1}{h_b}\right)+c^{h_c}\left(\frac{1}{h_c}\right)\overset{weighted\ A-G}{\geq}\sqrt[\left(\sum\frac{1}{r_a}\right)]{(a^{h_a})^{\left(\frac{1}{h_a}\right)}(b^{h_b})^{\left(\frac{1}{h_b}\right)}c^{h_c\left(\frac{1}{h_c}\right)}}$

$$=\sqrt[\left(\sum\frac{1}{r_a}\right)]{abc}=1(\because abc=1)\Rightarrow\sum\frac{a^{h_a}}{h_a}\overset{(2)}{\geq}\sum\frac{1}{h_a}=\frac{1}{r}$$

$$\text{(1), (2)}\Rightarrow LHS\geq\frac{R^2}{r^2}\overset{Euler}{\geq}4\ \text{(proved)}$$

336. If in $\Delta\ ABC, S=\frac{1}{2}$ then:

$$\min(a,b,c)\leq\frac{a^2+b^2+c^2}{abc(\sin A+\sin B+\sin C)}\leq\max(a,b,c)$$

Proposed by Dan Radu Seclaman – Romania

Solution by Adil Abdullayev-Baku-Azerbaidian

Let $a\leq b\leq c. abc=4RS=2R.\ a\leq\frac{a^2+b^2+c^2}{abc\cdot\frac{a+b+c}{2R}}\leq c\leftrightarrow a\leq\frac{a^2+b^2+c^2}{a+b+c}\leq c\leftrightarrow$

$$\leftrightarrow a^2+ab+ac\leq a^2+b^2+c^2\leq ac+bc+c^2$$

337. In acute $\Delta\ ABC$:

$$\frac{1}{\sqrt{\tan\frac{A}{2}}}+\frac{1}{\sqrt{\tan\frac{B}{2}}}+\frac{1}{\sqrt{\tan\frac{C}{2}}}\geq 3\sqrt[4]{4-\frac{2r}{R}}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

Lemma 1. $p^2 \geq r^2\left(4-\frac{2r}{R}\right)^3$ ***(Stronger inequality Mitrinovic)***

Lemma 2. $r_a r_b r_c = rp^2. \quad \tan\frac{A}{2} = \frac{r_a}{p}.$

$$AM \geq GM \to LHS = \sqrt{\frac{p}{r_a}} + \sqrt{\frac{p}{r_b}} + \sqrt{\frac{p}{r_c}} \geq 3 \cdot \sqrt[3]{\sqrt{\frac{p^3}{r_a r_b r_c}}} =$$

$$= 3 \cdot \sqrt[6]{\frac{p}{r}} \geq 3 \cdot \sqrt[6]{\sqrt{\left(4-\frac{2r}{R}\right)^3}} = RHS$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS \underbrace{\overbrace{\geq}^{A-G}}_{(1)} 3\sqrt[3]{\frac{1}{\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}}}$$

$$= 3\sqrt[3]{\frac{1}{\sqrt{\sqrt{\frac{(s-b)(s-c)(s-c)(s-a)(s-a)(s-b)}{s(s-a)\cdot s(s-b)\cdot s(s-c)}}}}}$$

$$= 3\sqrt[3]{\frac{1}{\sqrt{\frac{(s-a)(s-b)(s-c)}{s\sqrt{s(s-a)(s-b)(s-c)}}}}} = 3\sqrt[3]{\frac{1}{\sqrt{\frac{r^2s^2}{s^2rs}}}} = 3\sqrt[3]{\sqrt{\frac{s}{r}}}$$

(1) $\Rightarrow$ ***it suffices to prove:*** $\left(\frac{s}{r}\right)^{\frac{1}{6}} \geq \left(\frac{4R-2r}{R}\right)^{\frac{1}{4}}$

$$\Leftrightarrow \left(\frac{s}{r}\right)^2 \geq \left(\frac{4R-2r}{R}\right)^3 \Leftrightarrow s^2R^2 \geq r^2(4R-2r)^3 \quad \textbf{(2)}$$

$$\textit{LHS of (2)} \underbrace{\overbrace{\geq}^{Gerretsen}}_{(3)} (16Rr - 5r^2)R^3$$

(2), (3) $\Rightarrow$ ***it suffices to prove:*** $(16R - 5r)R^3 \geq 8r(2R - r)^3$

$$\Leftrightarrow 16t^4 - 69t^3 + 96t^2 - 48t + 8 \geq 0 \quad \left(\text{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)\{(t - 2)^2(16t + 27) + 66(t - 2) + 20\} \geq 0$$

$\rightarrow$ ***true*** $\because t = \frac{R}{r} \geq 2$ ***(Euler) (Proved)***

338. **Let ABC be a triangle and m_a, m_b, m_c are the medians.**

Prove that

$$m_a^2 + m_b^2 + m_c^2 \geq \frac{s^2 + 3r^2 + 12Rr}{2}$$

where s, r and R semiperimeter, inradius and circumradius of ABC respectively.

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Daniel Sitaru – Romania

$$\sum m_a^2 = \frac{3}{4}\sum a^2 = \frac{3}{4}\cdot 2(s^2 - r^2 - 4Rr) \geq \frac{s^2 + 3r^2 + 12Rr}{2} \quad \textit{(to prove)}$$

$$3s^2 - 3r^2 - 12Rr \geq s^2 + 3r^2 + 12Rr$$

$$s^2 \geq 3r^2 + 12Rr \quad \textit{(to prove)}$$

$$s^2 \overbrace{\geq}^{GERRETSEN} 16Rr - 5r^2 \geq 3r^2 + 12Rr \leftrightarrow$$

$$4Rr \geq 8r^2 \leftrightarrow R \geq 2r$$

339. In $\Delta\ ABC$:

$$\frac{2\begin{vmatrix} s+a^2 & ab & ac \\ ab & s+b^2 & bc \\ ac & bc & s+c^2 \end{vmatrix}}{s^2} \geq 8\sqrt{3}S + 3\sqrt[3]{4RS}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

To show:

$$\frac{2}{s^2}\begin{pmatrix}(s+a^2)\{(s+b^2)(s+c^2)-b^2c^2\}+ab\{abc^2-ab(s+c^2)\}+\\ +ac\{acb^2-ac(s+b^2)\}\end{pmatrix} \geq$$

$$\geq 8\sqrt{3}S + 3\sqrt[3]{4RS}$$

$$LHS = \frac{2}{s^2}\left((s+a^2)(s^2+sb^2+sc^2)+a^2b^2(-s)+a^2c^2(-s)\right)$$

$$= \frac{2}{s^2}\left(s^3+s^2(a^2+b^2+c^2)\right) = 2s+2(a^2+b^2+c^2)$$

$$= (a+b+c)+2(a^2+b^2+c^2)$$

Now, * $a+b+c \geq 3\cdot\sqrt[3]{abc}$ ***(AM ≥ GM)*** $= 3\sqrt[3]{4RS}$ ***(1)***

Now, $2(a^2+b^2+c^2) \geq 8\sqrt{3}S$

$$\Leftrightarrow (a^2+b^2+c^2)^2 \geq 48S^2 = 48s(s-a)(s-b)(s-c)$$

$$\Leftrightarrow (a^2+b^2+c^2)^2 \geq 48\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{c+a-b}{2}\right)\left(\frac{a+b-c}{2}\right)$$

$$\Leftrightarrow (a^2+b^2+c^2)^2 \geq 3((a+b)^2-c^2)(c^2-(a-b)^2)$$

$$\Leftrightarrow (a^2+b^2+c^2)^2 \geq 3(c^2(a+b)^2-(a^2-b^2)^2-c^4+c^2(a-b)^2)$$

$$\Leftrightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2 \geq 3(2c^2(a^2+b^2)+2a^2b^2-a^4-b^4-c^4)$$

$$\Leftrightarrow 4a^4+4b^4+4c^4-4a^2b^2-4b^2c^2-4c^2a^2 \geq 0$$

$$\Leftrightarrow 2a^4+2b^4+2c^4-2a^2b^2-2b^2c^2-2c^2a^2 \geq 0$$

$\Leftrightarrow (a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2 \geq 0$ ***which is true***

$$*\therefore 2(a^2+b^2+c^2)\ge 8\sqrt{3}S \quad \textit{(2)}$$

$$\textit{(1)} + \textit{(2)} \Rightarrow a+b+c+2(a^2+b^2+c^2)\ge 8\sqrt{3}S+3\sqrt[3]{S} \textit{ (QED)}$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo ABC. Probar que:

$$2\begin{bmatrix} s+a^2 & ab & ac \\ ab & s+b^2 & bc \\ ac & bc & s+c^2 \end{bmatrix} \ge 8\sqrt{3}Ss^2+3\sqrt[3]{4RS}s^2$$

$2s$ ***= perimetro,*** S ***= Área de región triangular***

$$2(s+a^2)\begin{bmatrix} s+b^2 & bc \\ bc & s+c^2 \end{bmatrix} - 2ab\begin{bmatrix} ab & bc \\ ac & s+c^2 \end{bmatrix} + 2ac\begin{bmatrix} ab & s+b^2 \\ ac & bc \end{bmatrix} \ge$$

$$\ge 8\sqrt{3}Ss^2+3\sqrt[3]{abc}s^2$$

$$2(s+a^2)[(s+b^2)(s+c^2)-b^2c^2]-2ab[(s+c^2)(ab)-c^2ab]+$$

$$+2ac[b^2ac-(s+b^2)(ac)]\ge 8\sqrt{3}Ss^2+3\sqrt[3]{abc}s^2$$

$$\Rightarrow 2(s+a^2)[s(s+(b^2+c^2)]-2ab[sab]+2ac[-sac]\ge$$

$\ge 8\sqrt{3}Ss^2+3\sqrt[3]{abc}s^2$***. Divididiendo*** $(\div s^2)$ ***a la desigualdad:***

$$\Rightarrow \frac{2s^2+2sa^2+2s(b^2+c^2)+2a^2b^2+2a^2c^2-2a^2b^2-2a^2c^2}{s}\ge 8\sqrt{3}S+3\sqrt[3]{abc}$$

$$\Rightarrow 2s+2(a^2+b^2+c^2)\ge 3\sqrt[3]{abc}+8\sqrt{3}S$$

$2s=a+b+c\ge 3\sqrt[3]{abc} \Leftrightarrow$ ***Válido por: (MA ≥ MG)***

Por lo cual falta demostrar que: $a^2+b^2+c^2\ge 4\sqrt{3}S$

$$\Rightarrow 4R^2(\sin^2 A+\sin^2 B+\sin^2 C)\ge 4\sqrt{3}(2R^2\sin A\sin B\sin C)\rightarrow$$

$$\rightarrow \frac{\sin(B+C)}{\sin B\sin C}+\frac{\sin(A+C)}{\sin A\sin C}+\frac{\sin(A+B)}{\sin A\sin B}\ge 2\sqrt{3}$$

$$\Rightarrow 2(\cot A+\cot B+\cot C)\ge 2\sqrt{3}\rightarrow \cot A+\cot B+\cot C\ge\sqrt{3}\rightarrow$$

$\rightarrow$ ***(Válido en un triángulo*** ABC***)***

340. In $\Delta\, ABC$:

$$\prod\left(\frac{m_a^8+m_b^8}{m_a^6+m_b^6}\cdot\frac{w_a^8+w_b^8}{w_a^6+w_b^6}\cdot\frac{h_a^8+h_b^8}{h_a^6+h_b^6}\right)\geq\left(\frac{2S^2}{R}\right)^6$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC\ \prod\left(\frac{m_a^8+m_b^8}{m_a^6+m_b^6}\cdot\frac{w_a^8+w_b^8}{w_a^6+w_b^6}\cdot\frac{h_a^8+h_b^8}{h_a^6+h_b^6}\right)\geq\left(\frac{2S^2}{R}\right)^6$

1) Siendo $x, y > 0$ ***se cumple la siguiente desigualdad***

$$x^8+y^8\geq xy(x^6+y^6)$$

Proof $\rightarrow 7x^8+y^8\geq 8\sqrt[8]{(x^8)^7y^8}=8x^7y,\ 7y^8+x^8\geq\sqrt[8]{(y^8)^7x^8}=8xy^7x$

$$\Leftrightarrow 8(x^8+y^8)\geq 8xy(x^6+y^6)\Leftrightarrow x^8+y^8\geq xy(x^6+y^6)$$

2) Recordar las siguientes desigualdades e identidad en un $\Delta\, ABC$

$$w_a\geq h_a, w_b\geq h_b, w_c\geq h_c, h_ah_bh_c=\frac{8S^3}{abc}=\frac{8S^3}{4RS}=\frac{2S^2}{R}$$

$$m_a\geq\sqrt{s(s-a)}, m_b\geq\sqrt{s(s-b)}, m_c\geq\sqrt{s(s-c)}, R\geq 2r$$

$$\Leftrightarrow m_am_bm_c\geq\sqrt{s^2\cdot s(s-a)(s-b)(s-c)}=S\cdot s=S\cdot\frac{S}{r}=\frac{S^2}{r}$$

Aplicando 1) $\wedge$ ***2) en la desigualdad propuesta***

$$\prod\left(\frac{m_a^8+m_b^8}{m_a^6+m_b^6}\cdot\frac{w_a^8+w_b^8}{w_a^6+w_b^6}\cdot\frac{h_a^8+h_b^8}{h_a^6+h_b^6}\right)\geq\prod(m_am_b\cdot w_aw_b\cdot h_ah_b)=$$

$$=(m_am_bm_c\cdot w_aw_bw_c\cdot h_ah_bh_c)^2\geq(m_am_bm_c)^2(h_ah_bh_c)^4$$

$$\Rightarrow(m_am_bm_c)^2(h_ah_bh_c)^4\geq\frac{S^4}{r^2}\cdot\frac{16S^8}{R^4}\geq\frac{64S^{12}}{4r^2\cdot R^4}\geq\frac{64S^{12}}{R^6}=\left(\frac{2S^2}{R}\right)^6$$

(LQQD)

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\Delta: x; y > 0 \Rightarrow x^8 + y^8 \geq \frac{1}{2} \cdot (x^6 + y^6) \cdot (x^2 + y^2) \quad \textbf{(Chebyshev)}$$

$$\prod f(m) \cdot f(w) \cdot f(h) \geq \prod \frac{1}{2^3} \cdot (m_a^2 + m_b^2) \cdot (w_a^2 + w_b^2)(h_a^2 + h_b^2) \overset{Cauchy}{\geq}$$

$$\geq \prod \frac{1}{2^3} \cdot 2^3 \cdot (m_a \cdot m_b) \cdot (w_a \cdot w_b) \cdot (h_a \cdot h_b) =$$

$$= (m_a \cdot m_b \cdot m_c)^2 \cdot (w_a \cdot w_b \cdot w_c)^2 \cdot (h_a \cdot h_b \cdot h_c)^2 \geq$$

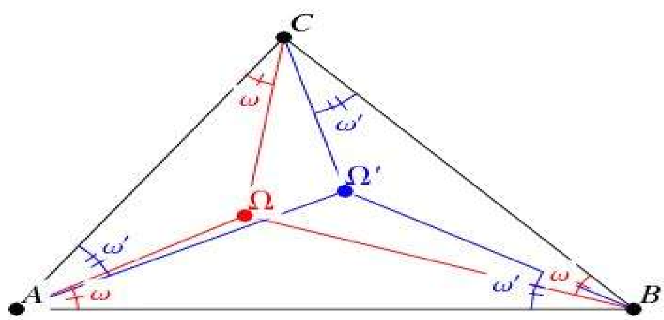
$$\overset{m_a \geq l_a \geq h_a}{\geq} (h_a \cdot h_b \cdot h_c)^6 = \left(\frac{8 \cdot S^3}{abc}\right)^6 = \left(\frac{8 \cdot S^3}{4RS}\right)^6 = \left(\frac{2S^2}{R}\right)^6$$

341. In $\Delta\, ABC, \Omega$ - first BROCARD point:

$$A\Omega^2 \cdot B\Omega^2 + B\Omega^2 \cdot C\Omega^2 + C\Omega^2 \cdot A\Omega^2 \leq 4R^2 S \cdot \tan \omega$$

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Daniel Sitaru – Romania



$$\frac{A\Omega}{\sin(B-\omega)} = \frac{2R\sin C}{\sin B} \to A\Omega = \frac{2R\sin(B-\omega)\sin C}{\sin B}$$

$$A\Omega \cdot B\Omega \cdot C\Omega = 8R^3 \sin(A-\omega)\sin(B-\omega)\sin(C-\omega) = 8R^3 \sin^3 \omega$$

$$\sum (A\Omega \cdot B\Omega)^2 \overset{AM-GM}{\geq} 3\sqrt[3]{(A\Omega \cdot B\Omega \cdot C\Omega)^4} = 48R^4 \sin^4 \omega \leq 4R^2 S \tan \omega \leftrightarrow$$

$$\leftrightarrow 12R^2 \cdot \sin^4 \omega \leq S \tan \omega$$

$$3R^2\frac{16S^4}{(\sum a^2b^2)^2}\le\frac{S^2}{\sum a^2};\ 3a^2b^2c^2\sum a^2\le\left(\sum a^2b^2\right)^2\leftrightarrow\sum(a^2-b^2)^2\ge 0$$

342. In $\Delta\ ABC$:

$$9r\le\sqrt{m_ar_a}+\sqrt{m_br_b}+\sqrt{m_cr_c}\le 4R+r$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Daniel Sitaru-Romania

$$\sum\sqrt{m_ar_a}\overset{AM-GM}{\ge}3\sqrt[6]{\prod m_a\cdot\prod r_a}\ge 3\sqrt[3]{\prod r_a}=3\sqrt[3]{rs^2}\ge$$

$$\overset{GERRETSEN}{\ge}3\sqrt[3]{r(16Rr-5r^2)}\ge 9r\leftrightarrow 16Rr-5r^2\ge 27r^2\leftrightarrow R\ge 2r$$

$$\sum\sqrt{m_ar_a}\overset{CBS}{\le}\sqrt{\sum m_a\cdot\sum r_a}\overset{TERESHIN}{\le}\sqrt{\sum\frac{b^2+c^2}{4R}\cdot(4R+r)}=$$

$$=\sqrt{\frac{4R+r}{4R}\cdot 2\sum a^2}=\sqrt{\frac{4R+r}{R}\cdot(s^2-r^2-4Rr)}\overset{GERRETSEN}{\le}$$

$$\le\sqrt{\frac{4R+r}{R}\cdot(4R^2+2r^2)}\le 4R+r\leftrightarrow\frac{4R^2+2r^2}{R}\le 4R+r\leftrightarrow R\ge 2r$$

343. In $\Delta\ ABC$:

$$\frac{a^3+s}{3\sqrt{3}r+b^2c}\cdot\frac{b^3+s}{3\sqrt{3}r+c^2a}\cdot\frac{c^3+s}{3\sqrt{3}r+a^2b}\ge 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$(a^3+s)(b^3+s)(c^3+s)\geq\left(3\sqrt{3}r+b^2c\right)\left(3\sqrt{3}r+c^2a\right)\left(3\sqrt{3}r+a^2b\right)$$

Tener en cuenta la siguiente desigualdad en un $\Delta\ ABC\ s\geq 3\sqrt{3}r$

Como $a,b,c>0$. ***Por la desigualdad de Holder***

$$(b^3+s)(b^3+s)(c^3+s)\geq(b^2c+s)^3\quad \textbf{(A)}$$

$$(c^3+s)(c^3+s)(a^3+s)\geq(c^2a+s)^3\quad \textbf{(B)}$$

$$(a^3+s)(a^3+s)(b^3+s)\geq(a^2b+s)^3\quad \textbf{(C)}$$

Multiplicando (A) · (B) · (C)

$$(a^3+s)(b^3+s)^3(c^3+s)^3\geq(b^2c+s)^3(c^2a+s)^3(a^2b+s)^3$$

$$\Leftrightarrow(a^3+s)(b^2+s)(c^3+s)\geq(b^2c+s)(c^2a+s)(a^2b+s)\geq$$

$$\geq\left(b^2c+3\sqrt{3}r\right)\left(c^2a+3\sqrt{3}r\right)\left(a^2b+3\sqrt{3}r\right)$$

(LQQD)

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$(a^3+s)(b^3+s)(c^3+s)\geq\left(3\sqrt{3}r+b^2c\right)\left(3\sqrt{3}r+c^2a\right)\left(3\sqrt{3}r+a^2b\right)$$

$$s\geq 3\sqrt{3}r=x$$

$$(a^3+x)(b^3+x)(c^3+x)\geq(a^2b+x)(b^2c+x)(c^2a+x)$$

$$(a^3b^3+a^3x+b^3x+x^2)(c^3+x)\geq(b^3a^2c+a^2bx+b^2cx+x^2)(c^2a+x)$$

$$a^3b^3c^3+a^3b^3x+a^3c^3x+a^3x^2+b^3c^3x+b^3x^2+c^3x^2+x^3$$

$$\geq a^3b^3c^3+b^3a^2cx+a^3c^2bx+a^2bx^2+c^3b^2ax+b^2cx^2+c^2ax^2+x^3$$

$$\textbf{(1)}\begin{cases}a^3c^3+a^3c^3+a^3b^3\geq 3a^3c^2b\\ b^3c^3+b^3c^3+a^3c^3\geq 3c^3b^2a\\ a^3b^3+a^3b^3+b^3c^3\geq 3b^3a^2c\end{cases}\Rightarrow\textbf{(2)}\begin{cases}a^3+a^3+b^3\geq 3a^2b\\ b^3+b^3+c^3\geq 3b^2c\\ c^3+c^3+a^3\geq 3c^2a\end{cases}$$

(1)+(2) → completely proved $x=3\sqrt{3}r$

344. If $0 \leq A \leq B \leq C < \frac{\pi}{2}$ then:

$$\left|\tan\frac{A-B}{2}\right| + \left|\tan\frac{B-C}{2}\right| + \left|\tan\frac{C-A}{2}\right| \geq C - A$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

For $0 \leq x \leq y < \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < x - y \leq 0 \Rightarrow 0 \leq y - x < \frac{\pi}{2}$

$$\therefore \left|\tan\left(\frac{x-y}{2}\right)\right| = -\tan\left(\frac{x-y}{2}\right) = \tan\left(\frac{y-x}{2}\right) \geq \frac{y-x}{2}$$

Thus, $\left|\tan\left(\frac{A-B}{2}\right)\right| \geq \frac{B-A}{2}$ *and* $\left|\tan\left(\frac{B-C}{2}\right)\right| \geq \frac{C-B}{2}$. *Also, as* $0 \leq C - A < \frac{\pi}{2}$

$\left|\tan\left(\frac{C-A}{2}\right)\right| = \tan\left(\frac{C-A}{2}\right) \geq \frac{C-A}{2}$. *Thus*

$$\left|\tan\left(\frac{A-B}{2}\right)\right| + \left|\tan\left(\frac{B-C}{2}\right)\right| + \left|\tan\left(\frac{C-A}{2}\right)\right| \geq \frac{B-A}{2} + \frac{C-B}{2} + \frac{C-A}{2} = C - A.$$

345. $f: \mathbb{R} \to \mathbb{R}, f(x) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & x \\ a^2 & b^2 & c^2 & x^2 \\ a^4 & b^4 & c^4 & x^4 \end{vmatrix}$, a, b, c sides in scalene ΔABC

Prove that: $\frac{f'(s)}{f(s)} > \frac{8}{s}$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$f(x) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & x \\ a^2 & b^2 & c^2 & x^2 \\ a^4 & b^4 & c^4 & x^4 \end{vmatrix}$$

As $f(x) = 0$ *when* $b = a, b = c, c = a, x = a, x = b, x = c$,

$(a-c), (a-b), (b-c), (x-c), (x-a), (x-b)$ *are factors of* $f(x)$

Also, $f(x)$ *is a homogenous expression of degree* 7 *in* a, b, c, x

$$\therefore f(x) \equiv k(b-c)(a-b)(a-c)(x-c)(x-a)(x-b)(x+a+b+c)$$

where k *is a constant. When* $x = 0$,

$$f(0) = -\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^4 & b^4 & c^4 \end{vmatrix} = -\begin{vmatrix} a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \\ a^4 & b^4-a^4 & c^4-a^4 \end{vmatrix}$$

$$= -(b-a)(c-a)\begin{vmatrix} a & 1 & 1 \\ a^2 & b+a & c+a \\ a^4 & b^3+b^2a+ba^2+a^3 & c^3+c^2a+ca^2+a^3 \end{vmatrix}$$

$$R_3 \to R_3 - a^2R_1, R_2 \to R_2 - aR_1$$

$$f(0) = -(b-a)(c-a)\begin{vmatrix} a & 1 & 1 \\ 0 & b & c \\ 0 & b^3+b^2a & c^3+c^2a \end{vmatrix}$$

$$= -a(b-a)(c-a)\begin{vmatrix} b & c \\ b^3+b^2a & c^3+c^2a \end{vmatrix}$$

$$= -abc(b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2+ab & c^2+ca \end{vmatrix}$$

$$= -abc(b-a)(c-a)(c-b)(a+b+c)$$

Also, $f(0) = k(b-c)(a-b)(a-c)(-1)abc(a+b+c)$

$= kabc(b-a)(c-a)(c-b)(a+b+c) \therefore k = -1$. *Thus,*

$$f(x) = (a-b)(b-c)(c-a)(x-a)(x-b)(x-c)(x+a+b+c)$$

$$\log|f(x)| = \log|(a-b)(b-c)(c-a)| + \log|x-a| + \log|x-b| +$$

$$+\log|x-c| + \log|x+a+b+c|$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} + \frac{1}{x+a+b+c}$$

$$\frac{f'(s)}{f(s)} = \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{1}{s+a+b+c}$$

$$\geq \frac{9}{(s-a)+(s-b)+(s-c)} + \frac{1}{3s} = \frac{9}{s} + \frac{1}{3s} = \frac{28}{3s} > \frac{8}{s}$$

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346. In $\Delta\, ABC$:

$$\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\le\frac{s}{r\sqrt{3}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Soumitra Mandal-Chandar Nagore-India

We have, $w_a\le\sqrt{p(p-a)}, w_b\le\sqrt{p(p-b)}$ ***and*** $w_c\le\sqrt{p(p-c)}$

$\therefore h_a=\frac{2\Delta}{a}, h_b=\frac{2\Delta}{b}$ ***and*** $h_c=\frac{2\Delta}{c}$ ***then,***

$$\sum_{cyc}\frac{w_a}{h_a}\le\frac{a\sqrt{p(p-a)}}{2\Delta}+\frac{b\sqrt{p(p-b)}}{2\Delta}+\frac{c\sqrt{p(p-c)}}{2\Delta}$$

$$=\frac{\sqrt{p}}{2\Delta}\left(a\sqrt{p-a}+b\sqrt{p-b}+c\sqrt{p-c}\right)$$

$$\overset{CHEBYSHEV'S}{\le}\frac{\sqrt{p}}{6\Delta}(a+b+c)\left(\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}\right)$$

$$\left[\begin{matrix}let\ a\ge b\ge c\ then\\ \sqrt{p-a}\le\sqrt{p-b}\le\sqrt{p-c}\end{matrix}\right]$$

$$\overset{JENSENS'S\ INEQUALITY}{\le}\frac{\sqrt{p}}{6\Delta}(a+b+c)\sqrt{3\sum_{cyc}(p-a)}=\frac{p}{r\sqrt{3}}$$

(Proved)

Solution 2 by Soumava Chakraborty-Kolkata-India

In $\Delta\ ABC, \frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\le\frac{s}{r\sqrt{3}}$. ***WLOG, we may assume*** $a\ge b\ge c$

Then, $w_a\le w_b\le w_c$ ***and*** $h_a\le h_b\le h_c\Rightarrow\frac{1}{h_a}\ge\frac{1}{h_b}\ge\frac{1}{h_c}$

$\therefore$ ***by Chebeyshev, LHS*** $\le\frac{1}{3}\left(\sum w_a\right)\left(\sum\frac{1}{h_a}\right)$

$$\overset{C-B-S}{\leq} \frac{1}{3}\sqrt{3}\sqrt{\sum w_a^2} \cdot 2R\frac{2s}{abc}$$

$$\overset{w_a \leq \sqrt{s(s-a)}}{\leq} \frac{\sqrt{3}}{3}\sqrt{\sum s(s-a)} \cdot \frac{4Rs}{4Rrs} = \frac{\sqrt{3}}{3} \cdot \frac{s}{r} = \frac{s}{r\sqrt{3}}$$ **(Proved)**

347. In $\Delta\ ABC$ the following relationship holds:

$$\prod\left(h_a + \sqrt[3]{w_a + \sqrt[4]{m_a}}\right) \geq \sqrt[16]{h_a h_b h_c w_a w_b w_c m_a m_b m_c}$$

Proposed by Bogdan Fustei-Romania

Solution by Daniel Sitaru-Romania

$$h_a + \sqrt[3]{w_a + \sqrt[4]{m_a}} \geq h_a \quad (1)$$

$$\left(h_a + \sqrt[3]{w_a + \sqrt[4]{m_a}}\right)^3 \geq m_a \quad (2)$$

$$\left(h_a + \sqrt[3]{w_a + \sqrt[4]{m_a}}\right)^{12} > w_a \quad (3)$$

By multiplying (1), (2), (3) $\rightarrow \left(h_a + \sqrt[3]{w_a + \sqrt[4]{m_a}}\right)^{16} > h_a m_a w_a \rightarrow$

$$h_a + \sqrt[3]{w_a + \sqrt[4]{m_a}} > \sqrt[16]{h_a m_a w_a}$$

$$\prod\left(h_a + \sqrt[3]{w_a + \sqrt[4]{m_a}}\right) > \sqrt[16]{h_a h_b h_c w_a w_b w_c m_a m_b m_c}$$

348. In $\Delta\ ABC$:

$$\sqrt[9]{(m_a + r_a)(m_b + r_b)(m_c + r_c)} \geq \frac{\sqrt[9]{m_a m_b m_c} + \sqrt[9]{sS}}{\sqrt[3]{4}}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un $\Delta\, ABC$ $\sqrt[9]{(m_a + r_a)(m_b + r_b)(m_c + r_c)} \geq \frac{\sqrt[9]{m_a m_b m_c} + \sqrt[9]{sS}}{\sqrt[3]{4}}$

Tener en cuenta la siguiente identidad $r_a r_b r_c = sS$

Aplicando la desigualdad de Holder

$$\sqrt[9]{\frac{(m_a+r_a)(m_b+r_b)(m_c+r_c)(1+1)(1+1)(1+1)(1+1)(1+1)(1+1)}{2^6}} \geq \frac{\sqrt[9]{m_a m_b m_c} + \sqrt[9]{r_a r_b r_c}}{\sqrt[3]{4}} =$$

$$= \frac{\sqrt[9]{m_a m_b m_c} + \sqrt[9]{sS}}{\sqrt[3]{4}}$$

349. In $\Delta\, ABC$:

$$3\sqrt{3}sr \leq \sqrt{m_a m_b r_a r_b} + \sqrt{m_b m_c r_b r_c} + \sqrt{m_c m_a r_c r_a} \leq \frac{3\sqrt{3}}{2} sR$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un $\Delta\, ABC$

$$3\sqrt{3}sr \leq \sqrt{m_a m_b r_a r_b} + \sqrt{m_b m_c r_b r_c} + \sqrt{m_c m_a r_c r_a} \leq \frac{3\sqrt{3}}{2} sR$$

Tener en cuenta las siguientes desigualdades e identidades en un $\Delta\, ABC$

$$m_a m_b + m_b m_c + m_c m_a \leq m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2) \leq$$

$$\leq \frac{3}{4} \cdot 9R^2 = \frac{27R^2}{4} \text{ (Leibniz)}$$

$$r_a r_b + r_b r_c + r_c r_a = s^2, r_a r_b r_c = Ss = S \cdot \frac{S}{r} = \frac{S^2}{r}, S = sr \geq 3\sqrt{3}r^2$$

$$m_a \geq \sqrt{s(s-a)} = \sqrt{r_b r_c}, m_b \geq \sqrt{r_a r_c}, m_c \geq \sqrt{r_a r_b} \Leftrightarrow$$

$\Leftrightarrow m_a m_b m_c \geq r_a r_b r_c$. ***En RHS. Aplicando la desigualdad de Cauchy***

$$\sqrt{m_a m_b r_a r_b} + \sqrt{m_b m_c r_b r_c} + \sqrt{m_c m_a r_c r_a} \leq$$

$$\leq\sqrt{(m_am_b+m_bm_c+m_cm_a)(r_ar_b+r_br_c+r_cr_a)}\leq$$

$$\leq\sqrt{\frac{27R^2}{4}\cdot s^2}=\frac{3\sqrt{3}sR}{2}$$. *En LHS. Aplicando MA $\geq$ MG*

$$\sqrt{m_am_br_ar_b}+\sqrt{m_bm_cr_br_c}+\sqrt{m_cm_ar_cr_a}\geq 3\sqrt[6]{(m_am_bm_c\cdot r_ar_br_c)^2}\geq$$

$$\geq 3\sqrt[3]{\frac{S}{r^2}\cdot S^3}\geq 3\sqrt[3]{3\sqrt{3}S^3}=3\sqrt{3}S=3\sqrt{3}sr$$

350. In $\Delta\, ABC$:

$$\frac{s}{R}\leq\frac{\sqrt{r_br_c}}{a}+\frac{\sqrt{r_cr_a}}{b}+\frac{\sqrt{r_cr_b}}{c}\leq\frac{s}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Daniel Sitaru – Romania

$$\sum\frac{\sqrt{r_br_c}}{a}\overset{CBS}{\leq}\sqrt{\sum r_br_c\cdot\sum\frac{1}{a^2}}\leq\sqrt{s^2\cdot\frac{1}{4r^2}}=\frac{s}{2r}$$

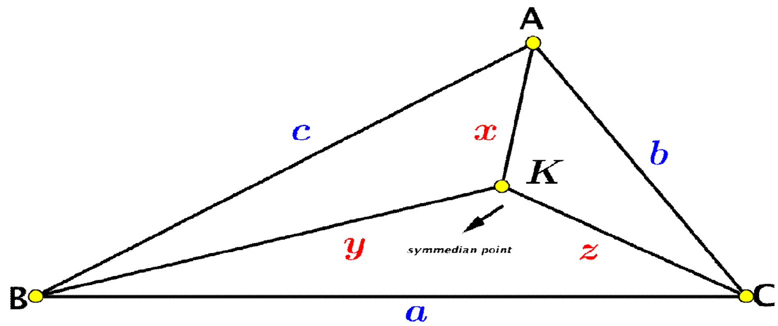
$$\sum\frac{\sqrt{r_br_c}}{a}=\sum\frac{\sqrt{s(s-a)}}{a}=\sum\frac{\sqrt{bc}\cdot\cos\frac{A}{2}}{a}=\frac{1}{2R}\sum\frac{\sqrt{bc}\cdot\cos\frac{A}{2}}{a}=$$

$$=\frac{1}{4R}\sum\frac{bc}{\sqrt{(s-a)(s-b)}}\overset{AM-GM}{\geq}\frac{1}{4R}\cdot 3\sqrt[3]{\frac{a^2b^2c^2s}{S^2}}=$$

$$=\frac{3}{4R}\sqrt[3]{\frac{16R^2S^2s}{S^2}}\geq\frac{s}{R}$$ *(to prove)* $\leftrightarrow$

$$\leftrightarrow\frac{27}{64R^3}\cdot 16R^2s\geq\frac{s^3}{R^3}\leftrightarrow s^2\leq\frac{27R^2}{4}\leftrightarrow s\leq\frac{3\sqrt{3}}{2}R$$ *(MITRINOVIC)*

351. $\left(\frac{x}{bc}\right)^2+\left(\frac{y}{ca}\right)^2+\left(\frac{z}{ab}\right)^2\le\frac{1}{12r^2}$



Proposed by Abdilkadir Altintas-Afyonkarashisar-Turkey

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c ***los lados de un*** $\Delta\ ABC$***. Probar que***

$$\left(\frac{x}{bc}\right)^2+\left(\frac{y}{ca}\right)^2+\left(\frac{z}{ab}\right)^2\le\frac{1}{12r^2}$$

Donde $x=KA,y=KB,z=KC\rightarrow K$ ***"Symmedian Point"***

Recordar la siguientes identidades siendo $\rightarrow K$ ***"Symmedian Point":***

$$KA=\frac{bc\sqrt{2b^2+2c^2-a^2}}{a^2+b^2+c^2},KB=\frac{ca\sqrt{2c^2+2a^2-b^2}}{a^2+b^2+c^2},KC=\frac{ab\sqrt{2a^2+2b^2-c^2}}{a^2+b^2+c^2}$$

La desigualdad propuesta es equivalente $\left(\frac{KA}{bc}\right)^2+\left(\frac{KB}{ca}\right)^2+\left(\frac{KC}{ab}\right)^2\le\frac{1}{12r^2}$

$$\frac{2b^2+2c^2-a^2}{(a^2+b^2+c^2)^2}+\frac{2c^2+2a^2-b^2}{(a^2+b^2+c^2)^2}+\frac{2a^2+2b^2-c^2}{(a^2+b^2+c^2)^2}=\frac{3}{a^2+b^2+c^2}\le\frac{1}{12r^2}$$

Es necesario demostrar $\frac{3}{a^2+b^2+c^2}\le\frac{1}{12r^2}\Leftrightarrow a^2+b^2+c^2\ge 36r^2$

Como: $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}\Leftrightarrow ab+bc+ca\ge 18Rr$ ***(Válido por desigualdad de Cauchy)***

$\Rightarrow$ ***Por lo tanto:*** $a^2+b^2+c^2\ge ab+bc+ca\ge 18Rr\ge 36r^2$ ***(LQQD)***

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352. In $\Delta\ ABC$:

$$\sqrt{\frac{4r_ar_br_c}{R^2}} \le \sqrt{\frac{h_ah_b}{r_c}} + \sqrt{\frac{h_bh_c}{r_a}} + \sqrt{\frac{h_ch_a}{r_b}} \le \sqrt{\frac{2r_ar_br_c}{rR}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC $\sqrt{\frac{4r_ar_br_c}{R^2}} \le \sqrt{\frac{h_ah_b}{r_c}} + \sqrt{\frac{h_bh_c}{r_a}} + \sqrt{\frac{h_ch_a}{r_b}} \le \sqrt{\frac{2r_ar_br_c}{rR}}$

Tener en cuenta las siguientes identidades y desigualdades en un $\Delta\ ABC$

$$h_a = \frac{2S}{a}, h_b = \frac{2S}{b}, h_c = \frac{2S}{c}, h_ah_bh_c = \frac{8S^3}{abc}, S = pr$$

$$r_a = \frac{S}{s-a}, r_b = \frac{S}{s-b}, r_c = \frac{S}{s-c}, r_ar_br_c = Sp = p^2r = \frac{S^2}{r}, \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}$$

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}, abc = 4Rpr, 2p \le 3\sqrt{3}R \Leftrightarrow 16p^4 \le 3^6R^4$$

En LHS, aplicando MA ≥ MG

$$\sum\sqrt{\frac{h_ah_b}{r_c}} \ge 3\sqrt[6]{\frac{(h_ah_bh_c)^2}{r_ar_br_c}} = 3\sqrt[6]{\frac{64S^6}{(abc)^2r_ar_br_c}} = \frac{6S}{\sqrt[6]{(16p^4)(R^2)(r^2)(r)}} \ge$$

$$\ge \frac{6S}{\sqrt[6]{(3R)^6 \cdot r^3}} = \frac{2S}{R\sqrt{3}} = \sqrt{\frac{4r_ar_br_c}{R^2}}$$

En RHS, por la desigualdad de Cauchy

$$\sqrt{\frac{h_ah_b}{r_c}} + \sqrt{\frac{h_bh_c}{r_a}} + \sqrt{\frac{h_ch_a}{r_b}} \le \sqrt{(h_ah_b + h_bh_c + h_ch_a)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right)} =$$

$$= \sqrt{\frac{4S^2}{2Rr} \cdot \frac{1}{r}} = \sqrt{\frac{4r_ar_br_c}{2Rr}} = \sqrt{\frac{2r_ar_br_c}{rR}}$$

353. Let $\Delta\ DEF$ be the Gergonne's triangle of $\Delta\ ABC$ and R_A, R_B, R_c circumradii of

$\Delta\ AFE, \Delta\ BDF$ respectively $\Delta\ CED$. Prove that:

$$2Rr - r^2 \le R_A^2 + R_B^2 + R_C^2 \le R^2 - Rr + r^2$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Rajsekhar Azaad-India

$$2R_a = \frac{s-a}{\cos\frac{A}{2}} \Rightarrow 4R_A^2 = \frac{(s-a)^2}{\cos^2\frac{A}{2}} = \frac{bc(s-a)}{s}$$

$$\therefore 4\sum R_a^2 = \frac{S\cdot\sum ab - 3abc}{S} = s^2 + r^2 + 4Rr - 12Rr$$

$$4\sum R_A^2 = s^2 + r^2 + 8Rr \quad (i)$$

Now, $16Rr - 5r^2 \le s^2 \le 4R^2 + 9Rr + 3r^2$ *(Gerretsen)*

$$16Rr - 5r^2 + r^2 - 8Rr \le s^2 + r^2 - 8Rr \le 4R^2 + 9Rr + 3r^2 + r^2 - 8Rr$$

$$\Rightarrow 8Rr - 4r^2 \le 4\sum R_A^2 \le 4R^2 - 9Rr + 9r^2 \quad \text{\{from (i)\}}$$

$$\Rightarrow 2Rr - r^2 \le \sum R_A^2 \le R^2 - Rr + r^2 \text{ (proved)}$$

354. In ΔABC:

$$8\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right) \le \prod\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$8\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right) \le \left(\cot\frac{A}{2} + \cot\frac{B}{2}\right)\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right)\left(\cot\frac{C}{2} + \cot\frac{A}{2}\right)$$

Como $\rightarrow \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$$

Siendo x, y, z números R^+ se cumple la siguiente desigualdad

$$(x+y)(y+z)(z+x)\geq 8xyz, \text{ donde}$$

$$x=\cot\frac{A}{2}>0, y=\cot\frac{B}{2}>0, z=\cot\frac{C}{2}>0$$

$$\Rightarrow\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)\geq 8\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

Solution 2 by Adil Abdullayev-Baku-Azerbaidian

Lemma 1. $\text{Cot}\frac{A}{2}=\frac{p}{r_a}$.

Lemma 2. $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$. $(r_a+r_b)(r_b+r_c)(r_c+r_a)=4Rp^2$. $r_ar_br_c=rp^2$.

$$LHS=8p\cdot\frac{1}{r}=\frac{8p}{r}.\ RHS=p^3\cdot\frac{(r_a+r_b)(r_b+r_c)(r_c+r_a)}{(r_ar_br_c)^2}=\frac{4Rp}{r^2}.$$

$$LHS\leq RHS\leftrightarrow\frac{8p}{r}\leq\frac{4Rp}{r^2}\leftrightarrow R\geq 2r\quad \textbf{(EULER).}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\cot\frac{A}{2}+\cot\frac{B}{2}\overset{A-G}{\geq}2\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}\quad (1)\ \cot\frac{B}{2}+\cot\frac{C}{2}\overset{A-G}{\geq}2\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}\quad (2)$$

$$\cot\frac{C}{2}+\cot\frac{A}{2}\overset{A-G}{\geq}2\sqrt{\cot\frac{C}{2}\cot\frac{A}{2}}\quad (3)$$

$$(1)\cdot(2)\cdot(3)\Rightarrow\prod\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\overset{(4)}{\geq}8\prod\cot\frac{A}{2}$$

$$=8\sqrt{\frac{(s-a)(s-b)(s-c)s^3}{(s-b)(s-c)(s-c)(s-a)(s-a)(s-b)}}=\frac{8s\cdot rs}{\prod(s-a)}=\frac{8s^3r}{r^2s^2}=\frac{8s}{r}$$

$\therefore$ ***(4) $\Rightarrow$ it suffices to prove:*** $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\leq\frac{s}{r}$ ***(5)***

$$\sum\cot\frac{A}{2}=\sqrt{s}\left(\sqrt{\frac{s-a}{(s-b)(s-c)}}+\sqrt{\frac{s-b}{(s-c)(s-a)}}+\sqrt{\frac{s-c}{(s-a)(s-b)}}\right)$$

$$\overset{CBS}{\leq}\sqrt{s}\sqrt{s-a+s-b+s-c}\sqrt{\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}+\frac{1}{(s-a)(s-b)}}$$

$$=s\sqrt{\frac{s}{(s-a)(s-b)(s-c)}}=s\sqrt{\frac{s^2}{r^2s^2}}=\frac{s}{r}\Rightarrow \textit{(5) is true (Proved)}$$

Solution 4 by Richdad Phuc-Vietnam

Denote $x=\cot\left(\frac{A}{2}\right), y=\cot\left(\frac{B}{2}\right), z=\cot\left(\frac{C}{2}\right)$

We have $x+y+z=xyz\Rightarrow xyz\geq 3\sqrt{3}\Rightarrow xy+yz+zx\geq 3(xyz)^{\frac{2}{3}}\geq 9$

Use $9(x+y)(y+z)(z+x)\geq 8(xy+yz+zx)(x+y+z)$ **and**

$xy+yz+zx\geq 9\Rightarrow (x+y)(y+z);\ (z+x)\geq 8(x+y+z)$

Solution 5 by Rozeta Atanasova-Skopje

$$RHS=\prod\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\overset{AM-GM}{\geq}\prod 2\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}$$

$$=8\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=8\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)=LHS$$

Solution 6 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\cot\frac{A}{2}=\sqrt{\frac{(x+y+z)\cdot x}{yz}}, \cot\frac{B}{2}=\sqrt{\frac{(x+y+z)\cdot y}{xz}}, \cot\frac{C}{2}=\sqrt{\frac{(x+y+z)\cdot z}{xy}}$$

$$LHS: 8\left(\sqrt{\frac{x+y+z}{xyz}}\cdot(x+y+z)\right); RHS: \prod\left(\sqrt{\frac{(x+y+z)x}{yz}}+\sqrt{\frac{(x+y+z)y}{xz}}\right)=$$

$$=\sqrt{\frac{x+y+z}{x}}\cdot\sqrt{\frac{x+y+z}{y}}\cdot\sqrt{\frac{x+y+z}{2}}\cdot\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\cdot\left(\sqrt{\frac{y}{z}}+\sqrt{\frac{z}{y}}\right)\cdot$$

$$\cdot\left(\sqrt{\frac{z}{x}}+\sqrt{\frac{x}{z}}\right)=\sqrt{\frac{x+y+z}{xyz}}\cdot(x+y+z)\prod\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)$$

$$LHS: 8\left(\sqrt{\frac{x+y+z}{xyz}}\cdot(x+y+z)\right)\le\sqrt{\frac{x+y+z}{xyz}}\cdot(x+y+z)\cdot$$

$$\cdot\prod\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)=RHS.,\ \ LHS:\ \underbrace{8\le\prod\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)}_{Cauchy}=RHS$$

Solution 7 by Soumitra Mandal-Chandar Nagore-India

$$\cot\frac{A}{2}=\frac{p(p-a)}{\Delta},\cot\frac{B}{2}=\frac{p(p-b)}{\Delta},\cot\frac{C}{2}=\frac{p(p-c)}{\Delta},\Delta=pr$$

and $ab+bc+ca=p^2+r^2+4Rr$. ***Now*** $\frac{A}{2},\frac{B}{2},\frac{C}{2}\ni\left(0,\frac{\pi}{2}\right)$

$\therefore\ \cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}>0$ ***for*** $x,y,z\ge 0$ ***we have,***

$$9\prod_{cyc}(x+y)\ge 8\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)$$

replacing x,y,z ***for*** $\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}$

$$9\prod_{cyc}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\ge 8\left(\sum_{cyc}\cot\frac{A}{2}\right)\left(\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}\right)=$$

$$=8\left(\sum_{cyc}\cot\frac{A}{2}\right)\frac{p^2}{\Delta^2}\left(\sum_{cyc}(p-a)(p-b)\right)$$

$$=8\left(\sum_{cyc}\cot\frac{A}{2}\right)\frac{p^2}{\Delta^2}(ab+bc+ca-p^2)=8\left(\sum_{cyc}\cot\frac{A}{2}\right)\frac{r+4R}{r}\ge 72\left(\sum_{cyc}\cot\frac{A}{2}\right)$$

$\because\ \frac{r+4R}{r}\ge 9$ ***and*** $R\ge 2r$. ***So we have,*** $\prod_{cyc}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\ge 8\left(\sum_{cyc}\cot\frac{A}{2}\right)$

Solution 8 by Uche Eliezer Okeke-Anambra-Nigeria

In ΔABC***, show***

$8\sum_{cyc}\cot\frac{A}{2}\le\prod_{cyc}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)^*$. ***We know:*** $\tan\frac{A}{2}=\frac{r}{\frac{1}{2}[b+c-a]}$

1) $\sum_{cyc}\cot\frac{A}{2}=\frac{1}{2r}\sum(b+c-a)=\frac{s}{r}$

2) $\prod_{cyc}\cot\frac{A}{2}=\frac{1}{(2r)^3}\prod_{cyc}(b+c-a)=\frac{1}{r^3}\prod_{cyc}(s-a)$

3) $\sum_{cyc}\cot\frac{A}{2}=\frac{s}{r}=\frac{s^2r^2}{r^3s}=\frac{1}{r^3}\cdot\frac{\Delta^2}{s}=\prod_{cyc}\cot\frac{A}{2}$

We proceed thus

$$RHS^*=\prod_{cyc}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\overset{AM-GM}{\geq}8\prod_{cyc}\cot\frac{A}{2}\overset{(3)}{=}8\sum_{cyc}\cot\frac{A}{2}$$

355. In $\Delta\ ABC$:

$$\frac{w_a^2+w_b^2+w_c^2}{w_aw_b+w_bw_c+w_cw_a}\leq\frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

$\frac{w_a^2+w_b^2+w_c^2}{w_aw_b+w_bw_c+w_cw_a}\leq\frac{R}{2r}$***. Teniendo en cuenta las siguientes desigualdades en un*** $\Delta\ ABC.\ w_a\leq\sqrt{s(s-a)},w_b\leq\sqrt{s(s-b)},w_c\leq\sqrt{s(s-c)}$

$$\Leftrightarrow w_a^2+w_b^2+w_c^2\leq s(s-a)+s(s-b)+s(s-c)=s^2$$

$$w_a\geq h_a,w_b\geq h_b,w_c\geq h_c$$

$$\Leftrightarrow w_aw_b+w_bw_c+w_cw_a\geq\sum h_ah_b=4S^2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)=\frac{4S^2}{2Rr}=$$

$=\frac{4s^2r^2}{2Rr}=\frac{2s^2r}{R}$***. Por lo tanto*** $\frac{w_a^2+w_b^2+w_c^2}{w_aw_b+w_bw_c+w_cw_a}\leq\frac{s^2}{\frac{2s^2r}{R}}=\frac{R}{2r}$

Solution 2 by Soumava Chakraborty-Kolkata-India

$w_a^2\leq s(s-a)$ ***etc.*** $\therefore\sum w_a^2\overset{(1)}{\leq}s(3s-\sum a)=s(3s-2s)=s^2$

Again, $\because w_a \geq h_a$ *etc*, $\therefore \sum w_a w_b \geq \sum h_a h_b \geq \frac{abc(2s)}{4R^2} = \frac{(14Rrs)(2s)}{4R^2} = \frac{2rs^2}{R}$

$$\Rightarrow \frac{1}{\sum w_a w_b} \leq \frac{R}{2rs^2} \quad (2)$$

$$(1) \times (2) \Rightarrow \frac{\sum w_a^2}{\sum w_a w_b} \leq \frac{R}{2rs^2} \cdot s^2 = \frac{R}{2r} \text{ (Proved)}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\left.\begin{aligned} \sin B + \sin C \leq 2 \cdot \cos\frac{A}{2} \\ \sin C + \sin A \leq 2 \cdot \cos\frac{B}{2} \\ \sin A + \sin B \leq 2 \cdot \cos\frac{C}{2} \end{aligned}\right\} \text{ ASSURE}$$

$$\sin B + \sin C = 2 \cdot \sin\frac{B+C}{2} \cdot \cos\frac{B-C}{2} \leq 2 \cdot \sin\frac{B+C}{2} = 2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right) = 2\cos\frac{A}{2}$$

$$\sin B + \sin C \leq 2 \cdot \cos\frac{A}{2} \Leftrightarrow \frac{b}{2R} + \frac{c}{2R} \leq 2\cos\frac{A}{2} \Leftrightarrow$$

$$\frac{1}{2R} \leq \frac{2 \cdot \cos\frac{A}{2}}{b+c} \Leftrightarrow \frac{bc}{2R} \leq \frac{2bc \cdot \cos\frac{A}{2}}{b+c} = \frac{2bc}{b+c} \cdot \sqrt{\frac{p(p-a)}{bc}} \Leftrightarrow \frac{bc}{2R} \leq \frac{2\sqrt{bc \cdot p \cdot (p-a)}}{b+c} = l_a.$$

$$\text{Similarly } \frac{ac}{2R} \leq l_b; \frac{ab}{2R} \leq l_c$$

$$\text{1) } \sum w_a \cdot w_b = \sum l_a \cdot l_b \sum \frac{abc^2}{4R^2} = \frac{1}{R} \cdot \frac{abc}{4R} \cdot (a+b+c) = \frac{S}{R} \cdot 2p = \frac{2p^2 \cdot r}{R}$$

$$\text{2) } l_a^2 + l_b^2 + l_c^2 = \sum l_a^2 = \sum \frac{2\sqrt{bc \cdot p \cdot (p-a)}}{b+c} \leq$$

$$\leq \sum \frac{2\sqrt{cb} \cdot \sqrt{p \cdot (p-a)}}{2\sqrt{bc}} = \sum p \cdot (p-a) = p^2$$

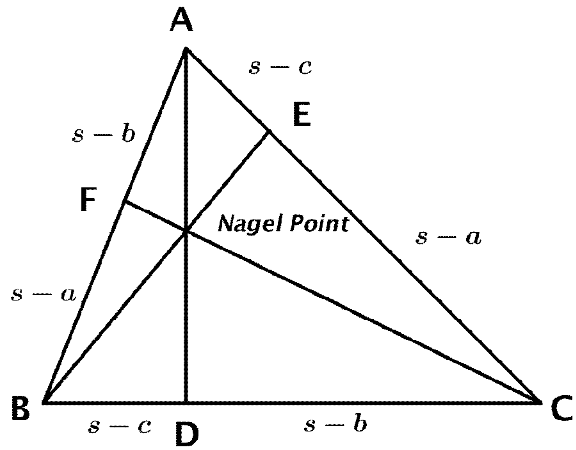
$$\text{3) } \frac{\sum l_a^2}{\sum l_a \cdot l_b} \leq \frac{p^2}{\frac{2p^2 \cdot r}{R}} = \frac{R}{2r}$$

356. In $\Delta\, ABC$, AD, BE, CF are NAGEL'S cevians. Prove that:

$$AD^2 + BE^2 + CF^2 \geq s^2$$

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Soumava Chakraborty-Kolkata-India



Using Stewart's theorem, $\frac{b^2(s-c)+c^2(s-b)}{a} = AD^2 + (s-b)(s-c)$

$$= AD^2 + s^2 - s(2s-a) + bc = AD^2 - s^2 + as + bc$$

$$\Rightarrow AD^2 \overset{(1)}{=} \frac{s(b^2+c^2)-(b^2c+bc^2)}{a} + s^2 - as - bc$$

Similarly, $BE^2 \overset{(2)}{=} \frac{s(c^2+a^2)-(c^2a+ca^2)}{b} + s^2 - bs - ca$, **and,**

$$CF^2 \overset{(3)}{=} \frac{s(a^2+b^2)-(a^2b+ab^2)}{c} + s^2 - cs - ab$$

(1)+(2)+(3) $\Rightarrow \sum AD^2 = 3s^2 - s(2s) - \sum ab +$

$$+\frac{s(b^3c+bc^3+c^3a+ca^3+a^3b+ab^3)-\{b^2c^2(b+c)+c^2a^2(c+a)+a^2b^2(a+b)\}}{abc}$$

$$= s^2 - \sum ab + \frac{s\{b^3(c+a)+c^3(a+b)+a^3(b+c)\}}{abc} -$$

$$-\frac{\{b^2c^2(2s-a)+c^2a^2(2s-b)+a^2b^2(2s-c)\}}{abc}$$

$$= s^2 - \sum ab + \frac{s\{b^3(2s-b)+c^3(2s-c)+a^3(2s-a)\}}{abc} -$$

$$-\frac{2s(\sum a^2b^2)-abc(\sum ab)}{abc}$$

$$=s^2-\sum ab+\sum ab+\frac{2s^2(\sum a^3)-s(\sum a^4)-2s(\sum a^2b^2)}{abc}$$

$$=s^2+\frac{2s^2(\sum a^3)-s(\sum a^4+2\sum a^2b^2)}{abc}=s^2+\frac{2s^2(\sum a^3)-s(\sum a^2)^2}{abc}$$

$$=s^2+\frac{s}{abc}\left\{\left(\sum a\right)\left(\sum a^3\right)-\left(\sum a^2\right)^2\right\}\overset{?}{\geq}s^2$$

$$\Leftrightarrow\left(\sum a\right)\left(\sum a^3\right)-\left(\sum a^2\right)^2\overset{?}{\geq}0$$

$$\Leftrightarrow\sum a^4+a^3(b+c)+b^3(c+a)+c^3(a+b)\overset{?}{\geq}\sum a^4+2\sum a^2b^2$$

$$\Leftrightarrow(a^3b+ab^3)+(b^3c+bc^3)+(c^3a+ca^3)\overset{?}{\geq}2\sum a^2b^2$$

$$\Leftrightarrow ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\overset{?}{\geq}2\sum a^2b^2\quad\textbf{(4)}$$

Indeed, $ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\geq$

$\overset{A-G}{\geq}ab(2ab)+bc(2bc)+ca(2ca)=2\sum a^2b^2\Rightarrow$ ***(4) is true (Proved)***

357. In $\Delta\ ABC$:

$$\frac{1}{\cos^4\frac{A}{2}}+\frac{1}{\cos^4\frac{B}{2}}+\frac{1}{\cos^4\frac{C}{2}}\geq\frac{36R^2}{s^2}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Probar en un triángulo ABC: $\sec^4\frac{A}{2}+\sec^4\frac{B}{2}+\sec^4\frac{C}{2}\geq\frac{36R^2}{s^2}$

Tener en cuenta la siguiente identidad en un triángulo ABC

$s = 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$. ***La desigualdad propuesta es equivalente***

$$\left(\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}\right)^2+\left(\frac{\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}\right)^2+\left(\frac{\cos\frac{A}{2}+\cos\frac{B}{2}}{\cos\frac{C}{2}}\right)^2\ge\frac{36}{16}=\frac{9}{4}$$

IRAN INEQUALITY. Siendo $x, y, z \ge 0$ ***se cumple la siguiente desigualdad***

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\ge\frac{9}{4}\quad \textit{(A)}$$

Siendo $x = \tan\frac{A}{2} > 0, y = \tan\frac{B}{2} > 0, z = \tan\frac{C}{2} > 0$

Se cumple $\rightarrow xy+yz+zx = 1, x+y = \frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}, y+z = \frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}},$

$z+x = \frac{\cos\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}$. ***Luego en (A)*** $\rightarrow \sum\left(\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}\right)^2 \ge \frac{9}{4}$. ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{\cos^4\frac{A}{2}}+\frac{1}{\cos^4\frac{B}{2}}+\frac{1}{\cos^4\frac{C}{2}}\overset{(1)}{\ge}\frac{36R^2}{s^2}\because \cos^4\frac{A}{2}=\frac{s^2(s-a)^2}{b^2c^2}\ \textit{etc}$$

$$\therefore \textit{(1)} \Leftrightarrow \frac{b^2c^2}{(s-a)^2}+\frac{c^2a^2}{(s-b)^2}+\frac{a^2b^2}{(s-c)^2}\ge 36R^2$$

$$\Leftrightarrow \frac{b^2c^2}{(s-a)^2}+\frac{c^2a^2}{(s-b)^2}+\frac{a^2b^2}{(s-c)^2}\overset{(2)}{\ge}\frac{36a^2b^2c^2}{16s(s-a)(s-b)(s-c)}$$

$\left(\because R = \frac{abc}{4\Delta}\right)$. ***Let*** $s-a = x, s-b = y, s-c = z$

Then, $s = x+y+z$ ***and*** $a = y = z, b = z+x, c = x+y$

$$\therefore \textit{(2)} \Leftrightarrow \frac{(z+x)^2(x+y)^2}{x^2}+\frac{(x+y)^2(y+z)^2}{y^2}+\frac{(y+z)^2(z+x)^2}{z^2}$$

$$\ge\frac{9}{4}\cdot\frac{(x+y)^2(y+z)^2(z+x)^2}{(x+y+z)xyz}$$

$$\Leftrightarrow 4\{y^2z^2(z+x)^2(x+y)^2+z^2x^2(x+y)^2(y+z)^2+x^2y^2(y+z)^2(z+x)^2\}\ge$$

$$\geq 9xyz(x+y)^2(y+z)^2(z+x)^2$$

$$\Leftrightarrow 4x^5(y^4+z^4)+4y^5(z^4+x^4)+4z^5(x^4+y^4)+2xyz\left(\sum x^3y^3\right)+6x^3y^3z^3$$

$$\stackrel{(3)}{\geq} x^5(y^3z+yz^3)+y^5(z^3x+zx^3)+z^5(x^3y+xy^3)+6\left(x^5y^2z^2+y^5z^2x^2+z^5x^2y^2\right)+$$

$$+2x^4(y^3z^2+y^2z^3)+2y^4(x^3z^2+x^2z^3)+2z^4(x^3y^2+x^2y^3)$$

Now, Schur $\Rightarrow \sum x^3y^3+3(xy)(yz)(zx)\geq$

$$\geq (xy)^2(yz)+(xy)(yz)^2+(yz)^2(zx)+(yz)(zx)^2+(zx)^2(xy)+(zx)(xy)^2$$

$$\Rightarrow 2xyz\left(\sum x^3y^3+3x^2y^2z^2\right)$$

$$\stackrel{(4)}{\geq} 2(x^3y^4z^2+z^3y^4x^2+y^3z^4x^2+x^3z^4y^2+z^3x^4y^2+y^3x^4z^2)$$

$$=2x^4(y^3z^2+y^2z^3)+2y^4(z^3x^2+z^2x^3)+2z^4(x^3y^2+x^2y^3)$$

Now, $\left.\begin{array}{l}2x^5(y^4+z^4)\geq 4x^5y^2z^2\\ 2y^5(z^4+x^4)\geq 4y^5z^2x^2\\ 2z^5(x^4+y^4)\geq 4z^5x^2y^2\end{array}\right\}$ ***Adding,***

$$2x^5(y^4+z^4)+2y^5(z^4+x^4)+2z^5(x^4+y^4)\geq 4x^5y^2z^2+4y^5z^2x^2+4z^5x^2y^2 \quad \textbf{(5)}$$

Again, $2x^5(y^4+z^4)\stackrel{Chebyshev}{\geq} 2\left(\frac{1}{2}\right)x^5(y+z)(y^3+z^3)=$

$$=x^5(y+z)(y^3+z^3)\stackrel{(i)}{\geq} x^5(y+z)\cdot yz(y+z)$$

$=x^5yz(y^2+z^2+2yz)=x^5(y^3z+yz^3)+2x^5y^2z^2$. ***Similarly,***

$2y^5(z^4+x^4)\stackrel{(ii)}{\geq} y^5(z^3x+zx^3)+2y^5z^2x^2$, ***and***

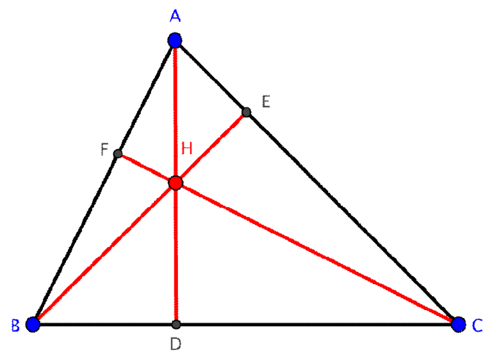
$$2z^5(x^4+y^4)\stackrel{(iii)}{\geq} z^5(x^3y+xy^3)+2z^5x^2y^2$$

(i)+(ii)+(iii) $\Rightarrow 2x^5(y^4+z^4)+2y^5(z^4+x^4)+2z^5(x^4+y^4)\geq$

$$\geq x^5(y^3z+yz^3)+y^5(z^3x+zx^3)+z^5(x^3y+xy^3)+2x^5y^2z^2+2y^5z^2x^2+2z^5x^2y^2 \quad \textbf{(6)}$$

$\therefore$ ***(4) + (5) + (6)*** $\Rightarrow$ ***(3) is true (Proved)***

358.

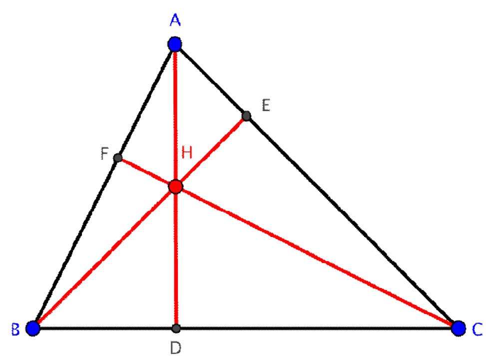


If in $\Delta\, ABC$, AD, BE, CF – altitudes, H – orthocenter then:

$$16r^2\left(\frac{h_a}{h_a-HD}+\frac{h_b}{h_b-HE}+\frac{h_c}{h_c-HF}\right)\le\frac{ab}{\cos C}+\frac{bc}{\cos A}+\frac{ca}{\cos B}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursarescu-Romania



$$\frac{ab}{\cos C}=\frac{ab\cdot\sin C}{\sin C\cos C}=\frac{2s}{\sin C\cos C}=\frac{4s}{\sin 2C}\Rightarrow\sum\frac{ab}{\cos C}=4s\cdot\sum\frac{1}{\sin 2A}\quad \textbf{(1)}$$

$$h_a-HD=AH=2R\cos A\Rightarrow\frac{h_a}{h_a\cdot HD}=\frac{\frac{2S}{a}}{2R\cos A}=$$

$$\frac{s}{Ra\cos A}=\frac{s}{2R^2\cdot\sin A\cos A}=\frac{s}{R^2\sin 2A}\Rightarrow\sum\frac{h_a}{h_a-HD}=\frac{s}{R^2}\sum\frac{1}{\sin 2A}\quad \textbf{(2)}$$

From (1)+(2) inequality becomes: $16r^2\cdot\frac{s}{R^2}\cdot\sum\frac{1}{\sin 2A}\le 4s\sum\frac{1}{\sin 2A}\Leftrightarrow$

$$\frac{4r^2}{R^2} \le 1 \Leftrightarrow 4r^2 \le R^2 \Leftrightarrow 2r \le R. \text{ True.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$16r^2\frac{h_a}{h_a - HD} \le \frac{bc}{\cos A} \Leftrightarrow \frac{16r^2(bc)}{2R\cdot AH} \le \frac{bc}{\cos A} \Leftrightarrow \frac{16r^2(bc)}{4R^2|\cos A|} \le \frac{bc}{\cos A} \Leftrightarrow 4R^2|\cos A|(bc) \overset{(1)}{\ge} 16r^2\cos A\,(bc)$$

If A is obtuse, $\cos A < 0 \Rightarrow$ RHS of (1) < 0 & LHS of (1) $> 0 \Rightarrow$ (1) is true.

If $A \le 90°$, LHS of (1) $= 4R^2\cos A\,(bc) \overset{Euler}{\ge} 16r^2\cos A\,(bc) =$ RHS of (1)

$\Rightarrow$ (1) is always true $\Rightarrow \frac{16r^2h_a}{h_a-HD} \overset{(a)}{\le} \frac{bc}{\cos A}$. Similarly, $\frac{16r^2h_b}{h_b-HE} \overset{(b)}{\le} \frac{ca}{\cos B}$ & $\frac{16r^2h_c}{h_c-HF} \overset{(c)}{\le} \frac{ab}{\cos C}$

(a)+(b)+(c) $\Rightarrow$ given inequality is true (Proved)

359. In $\Delta\ ABC$:

$$\frac{h_a^2 + h_b^2 + h_c^2}{h_ah_b + h_bh_c + h_ch_a} \le \frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{R}{2r} \ge \frac{h_a^2+h_b^2+h_c^2}{h_ah_b+h_bh_c+h_ch_a}$

Teniendo en cuenta las siguientes identidades y desigualdades en un $\Delta\ ABC$: $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \le \frac{1}{4r^2}, \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} = \frac{1}{2Rr}, h_a = \frac{2S}{a}, h_b = \frac{2S}{b}, h_c = \frac{2S}{c}$

El lado derecho es equivalente: $\frac{h_a^2+h_b^2+h_c^2}{h_ah_b+h_bh_c+h_ch_a} = \frac{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}} \le \frac{\frac{1}{4r^2}}{\frac{1}{2Rr}} = \frac{R}{2r}$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS = \frac{\frac{(\sum a^2b^2)}{4R^2}}{\frac{abc(a+b+c)}{4R^2}} = \frac{\sum a^2b^2}{4Rrs(2s)} \overset{Goldstone}{\le} \frac{4R^2s^2}{4Rrs(2s)} = \frac{R}{2r}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{\sum h_a^2}{\sum h_a\cdot h_b}=\frac{4\cdot S^2\cdot\sum\frac{1}{a^2}}{4S^2\cdot\sum\frac{1}{ab}}=\frac{\sum(ab)^2}{(abc)^2}\cdot\frac{abc}{a+b+c}=$$

$$=\frac{\sum(ab)^2}{abc\cdot(a+b+c)}=\frac{(\sum ab)^2-2abc\cdot(a+b+c)}{abc\cdot(a+b+c)}=\frac{(\sum ab)^2}{abc\cdot\sum a}\cdot 2\le\frac{R}{2r}$$

$$\frac{(\sum ab)^2}{abc\cdot\sum a}\le\frac{R+4r}{2r}\quad \textit{(ASSURE)}$$

$$\frac{(\sum ab)^2}{8Rr\cdot p^2}=\frac{(p^2+4Rr+r^2)^2}{8Rr\cdot p^2}\le\frac{R+4r}{2r}$$

$$\underbrace{p^4+2\cdot(4Rr+r^2)\cdot p^2+(4Rr+r^2)^2}_{LHS}\le\underbrace{p^2\cdot(4R^2+16Rr)}_{RHS}\quad (*)$$

$$RHS=p^2\cdot(4R^2+4Rr+3r^2)+(12Rr-3r^2)\cdot p^2\overset{GERRETSEN}{\geqq}$$

$$\ge p^4\cdot(12Rr-3r^2)\cdot p^2=p^4+(12Rr-3r^2)\cdot p^2$$

$$(*)\Rightarrow p^4+2\cdot(4Rr+r^2)\cdot p^2+(4Rr+r^2)^2\le p^4+(12Rr-3r^2)\cdot p^2$$

ASSURE

$$(4Rr-5r^2)\cdot p^2\ge(4Rr+r^2)^2$$

$$(4Rr-5r^2)\cdot p^2\overset{GERRETSEN}{\geqq}(5Rr-5r^2)(16Rr-5r^2)\overset{ASSURE}{\geqq}(4Rr+r^2)^2$$

$$(64R^2-100Rr+25r^2)\ge r^2(16R^2+8Rr+r^2)$$

$$48R^2-108Rr+24r^2\ge 0;\ 4R^2-9Rr+2r^2\ge 0|\cdot r^2$$

$$4t^2-9t+2\ge 0\Rightarrow 4\underbrace{(t-2)}_{\ge 0}\cdot\underbrace{\left(t-\frac{1}{4}\right)}_{>0}\ge 0$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

We have, $h_a=\frac{2\Delta}{a},h_b=\frac{2\Delta}{b},h_c=\frac{2\Delta}{c},a+b+c=2p$ ***and***

$$ab+bc+ca=p^2+r^2+4Rr;\ \frac{h_a^2+h_b^2+h_c^2}{h_ah_b+h_bh_c+h_ah_c}\le\frac{R}{2r}\Leftrightarrow\frac{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}\le\frac{R}{2r}$$

$$\Leftrightarrow\frac{a^2b^2+b^2c^2+a^2c^2}{abc(a+b+c)}\le\frac{R}{2r}\Leftrightarrow\frac{(p^2+r^2+4Rr)^2}{abc(a+b+c)}\le\frac{R+4r}{2r}$$

$$\Leftrightarrow\frac{p^4+r^4+16R^2r^2+2p^2r^2+8Rr^3+8Rp^2}{8Rrp^2}\le\frac{R+4r}{2r}$$

$$\Leftrightarrow p^4+r^4+16R^2r^2+2p^2r^2+8Rr^3+8Rrp^2\le 4R^2p^2+16Rrp^2$$

$$\Leftrightarrow p^4+r^4+16R^2r^2+2p^2r^2+8Rr^3\le 4R^2p^2+8Rrp^2$$

We know, $p^2\le 4R^2+4Rr+3r^2$***, then we need to prove,***

$$p^2(4R^2+4Rr+3r^2)+(r^2+4Rr)^2+2p^2r^2\le 4R^2p^2+8Rrp^2$$

$$\Leftrightarrow p^2(5r^2-4Rr)+(r^2+4Rr)^2\le 0\Leftrightarrow p^2\ge\frac{(r^2+4Rr)^2}{4Rr-5r^2}$$

Again, we know, $p^2\ge 16Rr-5r^2$***, we will show,*** $16Rr-5r^2\ge\frac{(r^2+4Rr)^2}{4Rr-5r^2}$

$\Leftrightarrow 4R^2-9Rr+2r^2\ge(R-2r)(4R-r)\ge 0$***, which is true.***

360. In $\Delta\,ABC, I-$ incentre, R_a, R_b, R_c- circumradii in $\Delta\,BIC, \Delta\,CIA, \Delta\,AIB$:

$$r_ar_br_c\le\frac{27}{8}R_aR_bR_c$$

Proposed by Mehmet Șahin – Ankara – Turkey

Solution by Daniel Sitaru – Romania

$$R_a=\frac{a}{2\sin\left(\pi-\frac{B+C}{2}\right)}=\frac{2R\sin A}{\cos\frac{A}{2}}=2R\sin\frac{A}{2}$$

$$\prod r_a\le\frac{28}{8}\prod R_a\leftrightarrow rs^2\le\frac{27}{8}\cdot 8R^3\prod\sin\frac{A}{2}\leftrightarrow$$

$$\leftrightarrow s^2\le\frac{27R^2}{4}\leftrightarrow s\le\frac{3\sqrt{3}R}{2}\quad\textit{(MITRINOVIC)}$$

361. In $\Delta\ ABC$:

$$\left(\sum\sqrt{\cos\frac{A}{2}}\right)\left(\sum\sqrt[4]{\cos\frac{A}{2}}\right)\left(\sum\sqrt[8]{\cos\frac{A}{2}}\right)\le 27\sqrt[16]{\left(\frac{3}{4}\right)^7}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\left(\sum\sqrt{\cos\frac{A}{2}}\right)\left(\sum\sqrt[4]{\cos\frac{A}{2}}\right)\left(\sum\sqrt[8]{\cos\frac{A}{2}}\right)\le$

$\le 27\sqrt[16]{\left(\frac{3}{4}\right)^7}$. ***Por la desigualdad de Cauchy***

$$\sum\sqrt{\cos\frac{A}{2}}\le\sqrt{3\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)}\le\sqrt{3\cdot\frac{3\sqrt{3}}{2}}=\frac{3\sqrt[4]{3}}{2}\quad (A)$$

$$\sum\sqrt[4]{\cos\frac{A}{2}}\le\sqrt{3\left(\sqrt{\cos\frac{A}{2}}+\sqrt{\cos\frac{B}{2}}+\sqrt{\cos\frac{C}{2}}\right)}\le\sqrt{3\cdot\frac{3\sqrt[4]{3}}{\sqrt{2}}}=\frac{3\sqrt[8]{3}}{\sqrt[4]{2}}\quad (B)$$

$$\sum\sqrt[8]{\cos\frac{A}{2}}\le\sqrt{3\left(\sqrt[4]{\cos\frac{A}{2}}+\sqrt[4]{\cos\frac{B}{2}}+\sqrt[4]{\cos\frac{C}{2}}\right)}\le\sqrt{3\cdot\frac{3\sqrt[8]{3}}{\sqrt[4]{2}}}=\frac{3\sqrt[16]{3}}{\sqrt[8]{2}}\quad (C)$$

Multiplicando (A), (B), (C)

$$\Rightarrow\left(\sum\sqrt{\cos\frac{A}{2}}\right)\left(\sum\sqrt[4]{\cos\frac{A}{2}}\right)\left(\sum\sqrt[8]{\cos\frac{A}{2}}\right)\le\frac{3\sqrt[4]{3}}{\sqrt{2}}\cdot\frac{3\sqrt[8]{3}}{\sqrt[4]{2}}\cdot\frac{3\sqrt[16]{3}}{\sqrt[8]{2}}=$$

$$=\frac{27\cdot\sqrt[16]{3^4}\cdot\sqrt[16]{3^2}\cdot\sqrt[16]{3}}{\sqrt[16]{4^4}\cdot\sqrt[16]{4^2}\cdot\sqrt[16]{4}}=27\sqrt[16]{\left(\frac{3}{4}\right)^7}\ \textbf{(LQQD)}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

In $\Delta\ ABC$: $\left(\sum\sqrt[2^{n-2}]{\cos\frac{A}{2}}\right)\cdot\left(\sum\sqrt[2^{n-1}]{\cos\frac{A}{2}}\right)\cdot\left(\sum\sqrt[2^n]{\cos\frac{A}{2}}\right)\le 27\cdot\sqrt[2^{n+1}]{\left(\frac{3}{4}\right)^7}$

$$\sum \cos\frac{A}{2} = k$$

$$I_n = \sum \sqrt[2^n]{\cos\frac{A}{2}} \le \sqrt{3\cdot I_{n-1}} \le \sqrt{3\sqrt{3\cdot I_{n-2}}} \le \cdots \le \sqrt[2^n]{3^{2^n-1}\cdot k} = \sqrt[2^{n+1}]{3^{2^{n+1}-2}\cdot k^2}$$

$$I_{n-1} = \sum \sqrt[2^{n-1}]{\cos\frac{A}{2}} \le \textit{Similarly} \le \sqrt[2^{n-1}]{3^{2^{n-1}}\cdot k} = \sqrt[2^{n+1}]{3^{2^{n+1}-4}\cdot k^4}$$

$$I_{n-2} = \sum \sqrt[2^{n-2}]{\cos\frac{A}{2}} \le \cdots \le \sqrt[2^{2^{n-2}}]{3^{2^{n-2}}\cdot k} = \sqrt[2^{n+1}]{3^{2^{n+1}-8}\cdot k^8}$$

$$I_n - I_{n-1}\cdot I_{n-2} \le \sqrt[2^{n+1}]{3^{2^{n+1}-2}\cdot k^2}\cdot \sqrt[2^{n+1}]{3^{2^{n+1}-4}\cdot k^4}\cdot \sqrt[2^{n+1}]{3^{2^{n+1}-8}\cdot k^8} =$$

$$= \sqrt[2^{n+1}]{(3^3)^{2^{n+1}}\cdot\left(\frac{k}{3}\right)^{14}} =$$

$$= 3^3\sqrt[2^{n+1}]{\left(\frac{k}{3}\right)^{14}} \le 27\cdot\sqrt[2^{n+1}]{\left(\frac{3\sqrt{3}}{2\cdot 3}\right)^{14}} = 27\cdot\sqrt[2^{n+1}]{\left(\frac{3}{4}\right)^{7}}.\ k = \sum\cos\frac{A}{2} \le \frac{3\sqrt{3}}{2}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

We know, $\sum_{cyc}\cos\frac{A}{2} \le \frac{3\sqrt{3}}{2}$ ***for all*** $A, B, C \in (0,\pi)$

$$\sum_{cyc}\sqrt{\cos\frac{A}{2}} \le 3\sqrt{\frac{1}{3}\left(\sum_{cyc}\cos\frac{A}{2}\right)} \le 3\sqrt{\frac{\sqrt{3}}{2}} = 3\sqrt[16]{\left(\frac{3}{4}\right)^4}$$

$$\sum_{cyc}\sqrt[4]{\cos\frac{A}{2}} \le 3\sqrt[4]{\frac{1}{3}\left(\sum_{cyc}\cos\frac{A}{2}\right)} \le 3\sqrt[4]{\frac{\sqrt{3}}{2}} = 3\sqrt[16]{\left(\frac{3}{4}\right)^2}$$

$$\sum_{cyc}\sqrt[8]{\cos\frac{A}{2}} \le 3\sqrt[8]{\frac{1}{3}\left(\sum_{cyc}\cos\frac{A}{2}\right)} \le 3\sqrt[8]{\frac{\sqrt{3}}{2}} = 3\sqrt[16]{\frac{3}{4}}$$

$$\left(\sum_{cyc}\sqrt{\cos\frac{A}{2}}\right)\left(\sum_{cyc}\sqrt[4]{\cos\frac{A}{2}}\right)\left(\sum_{cyc}\sqrt[8]{\cos\frac{A}{2}}\right)\le 27\sqrt[16]{\left(\frac{3}{4}\right)^7}$$

Solution 4 by Uche Eliezer Okeke-Anambra-Nigeria

First we need to study the function $f(x)=\sqrt[n]{\cos\left(\frac{x}{2}\right)}\equiv f_1(f_2)$

Where $f_1(x)=\sqrt[n]{x}, f_1''(x)=\frac{\frac{1}{n}\left(\frac{1-n}{n}\right)}{x^{2n-1}}<0\quad \forall n\ge 2, x>0$

Also $f_2(x)=\cos\left(\frac{x}{2}\right), f_2''(x)=-\left(\frac{1}{4}\cos\left(\frac{x}{2}\right)\right)<0\quad \forall x\in(0,\pi)$

Without doubts since ... $f(x)=f_1(f_2)\Rightarrow f''(x)=f_2''f_1'(f_2)+$

$+(f_2')^2f_1''(f_2)\Leftrightarrow f_2''<0$, ***and*** $f_1'(f_2)=\frac{1}{nx^{n-1}}>0$, ***and*** $f_1''(f_2)<0$

$$\Rightarrow\left[f''(x)<0\ldots so\ f(x)=\sqrt[n]{\cos\frac{x}{2}}\ is\ concave\ \forall\, n\ge 2, x\in(0,\pi)\right]$$

We now proceed with the Inequality...

$$\sum\left(\sqrt{\cos\frac{A}{2}}\right)\overset{Jensen}{\le}3\left(\frac{3}{4}\right)^{\frac{1}{2}}\quad (1)$$

$$\sum\left(\sqrt[4]{\cos\frac{A}{2}}\right)\overset{Jensen}{\le}3\left(\frac{3}{4}\right)^{\frac{1}{4}}\quad (2)$$

$$\sum\left(\sqrt[8]{\cos\frac{A}{2}}\right)\overset{Jensen}{\le}3\left(\frac{3}{4}\right)^{\frac{1}{8}}\quad (3)$$

We do... (1) × (2) × (3)

$$\Rightarrow\left(\sum\sqrt{\cos\frac{A}{2}}\right)\left(\sum\sqrt[4]{\cos\frac{A}{2}}\right)\sum\left(\sqrt[8]{\cos\frac{A}{2}}\right)\overset{(1)(2)(3)}{\le}27\left(\frac{3}{4}\right)^{\frac{7}{16}}\ \textit{(Proved)}$$

GENERALIZATION: In any triangle ABC***, prove:***

$$\prod_{j=1}^{n}\sum_{cyc}\left(\sin\frac{A}{2}\right)^{\frac{1}{(2)^j}} \leq 3^n \cdot 2^{-2\frac{\sum_{j=1}^{n} 2^{j-1}}{2^{n+1}}}$$

***GENERALIZATION:* In any triangle ABC, prove:**

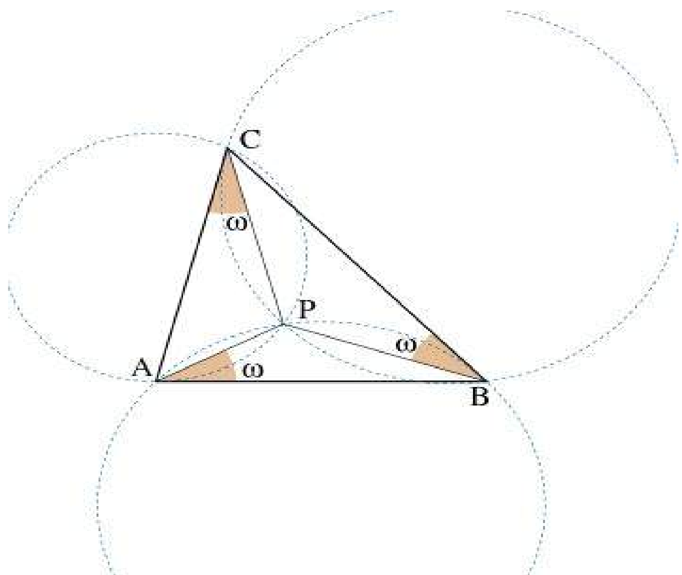
$$\prod_{j=1}^{n}\sum_{cyc}\left(\cos\frac{A}{2}\right)^{\frac{1}{2^j}} \leq 3^{\frac{\sum_{j=1}^{n} 2^{j-1}}{2^{n+1}}+n} \cdot 2^{-2\frac{\sum_{j=1}^{n} 2^{j-1}}{2^{n+1}}}$$

362. In $\Delta\, ABC, P -$ first Brocard point, R_a, R_b, R_c –circumradii in $\Delta\, BPC, \Delta CPA, \Delta APB$:

$$r_a r_b r_c \leq \frac{27}{8} R_a R_b R_c$$

Proposed by Mehmet Șahin – Ankara – Turkey

Solution by Daniel Sitaru – Romania



$$R_a = \frac{a}{2\sin\left(\pi - \omega - (B - \omega)\right)} = \frac{2R\sin A}{2\sin B}$$

$$\prod r_a \le \frac{27}{8}\prod R_a \leftrightarrow rs^2 \le \frac{27}{8}\prod \frac{R\sin A}{\sin B} \leftrightarrow$$

$$\leftrightarrow rs^2 \le \frac{27R^3}{8} \leftrightarrow s^2 \le \frac{27R^3}{8r} \quad \textit{(to prove)}$$

$$s^2 \overset{MITRINOVIC}{\overbrace{\le}} \frac{27R^2}{4} \le \frac{27R^3}{8r} \leftrightarrow 2r \le R \quad \textit{(EULER)}$$

363. In $\Delta\, ABC$

$$p(12r)^3 \le 6\sum \frac{a^4(b+c)}{r_a - r} \le p(6R)^3$$

Proposed by Marin Chirciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$s(12r)^3 \overset{(1)}{\overbrace{\le}} 6\sum \frac{a^4(b+c)}{r_a - r} \overset{(2)}{\overbrace{\le}} s(6R)^3$$

$$\frac{a^4(b+c)}{r_a - r} = \frac{a^4(b+c)}{\frac{\Delta}{s-c} - \frac{\Delta}{s}} = \frac{a^4(b+c)s(s-a)}{\Delta(s-s+a)}$$

$$= \frac{a^4(b+c)s(s-a)}{rs\cdot a} = \frac{a^3(b+c)(s-a)}{r}$$

Similarly, $\frac{b^4(c+a)}{r_b - r} = \frac{b^3(c+a)(s-b)}{r}$**, and,** $\frac{c^4(a+b)}{r_c - r} = \frac{c^3(a+b)(s-c)}{r}$

$\therefore$ **(1)** $\Leftrightarrow 6\{a^3(b+c)(s-a) + b^3(c+a)(s-b) + c^3(a+b)(s-c)\} \ge 1728sr^4$

$\Leftrightarrow a^3(b+c)(s-a) + b^3(c+a)(s-b) + c^3(a+b)(s-c) \ge 288sr^4$ **(3)**

LHS of (3) $\overset{A-G}{\ge} 3\sqrt[3]{a^3b^3c^3(b+c)(c+a)(a+b) + (s-a)(s-b)(s-c)}$

$$= 3abc\sqrt[3]{(a+b)(b+c)(c+a)\frac{r^2s^2}{s}}$$

$$\overset{A-G}{\ge} 12Rrs\sqrt[3]{8abc\, r^2s} = 24Rrs\sqrt[3]{4Rrs\cdot r^2s}$$

$$\overset{Euler}{\geq} 48r^2s\sqrt[3]{8r^4\cdot s^2} \overset{s\geq 3\sqrt{3}r}{\geq} 96r^2s\sqrt[3]{27r^6} = 288r^4s$$

$\Rightarrow$ ***(3) is true*** $\Rightarrow$ ***(1) is true***

(2) $\Leftrightarrow 6\{a^3(b+c)(s-a)+b^3(c+a)(s-b)+c^3(a+b)(s-c)\} \leq 216R^3sr$

$$\Leftrightarrow a^3(b+c)(s-a)+b^3(c+a)(s-b)+c^3(a+b)(s-c) \leq 36R^3sr$$

$$= 9R^2\cdot 4Rrs = 9R^2\cdot abc$$

$$\Leftrightarrow a^2(b+c)\frac{s(s-a)}{bc}+b^2(c+a)\frac{s(s-b)}{ca}+c^2(a+b)\frac{s(s-c)}{ab} \leq 9R^2s$$

$$\Leftrightarrow a^2(b+c)\cos^2\frac{A}{2}+b^2(c+a)\cos^2\frac{B}{2}+c^2(a+b)\cos^2\frac{C}{2} \leq 9R^2s \quad \textbf{(3)}$$

WLOG, we may assume $a\geq b\geq c \therefore \frac{A}{2}\geq\frac{B}{2}\geq\frac{C}{2}$ ***and*** $0<\frac{A}{2},\frac{B}{2},\frac{C}{2}<\frac{\pi}{2}$

$\therefore \cos^2\frac{A}{2}\leq\cos^2\frac{B}{2}\leq\cos^2\frac{C}{2}$***. Now,*** $a^2(b+c)\geq b^2(c+a)$

$$\Leftrightarrow ab(a-b)+c(a+b)(a-b)\geq 0$$

$\Leftrightarrow (a-b)(ab+bc+ca)\geq 0 \Leftrightarrow a\geq b \to$ ***true***

$\therefore a^2(b+c)\geq b^2(c+a)$***. Similarly,*** $b^2(c+a)\geq c^2(a+b)$

$$\therefore a^2(b+c)\geq b^2(c+a)\geq c^2(a+b)$$

$\therefore$ ***LHS of (3)*** $\overset{Chebyshev}{\leq} \frac{1}{3}\{a^2(b+c)+b^2(c+a)+c^2(a+b)\}$

$$\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)$$

$$=\frac{1}{3}\sum\{ab(a+b)\}\cdot\frac{1}{2}\left(2\cos^2\frac{A}{2}+2\cos^2\frac{B}{2}+2\cos^2\frac{C}{2}\right)$$

$$=\frac{1}{6}\sum\{ab(2s-c)\}\left(2+\sum\cos A\right)=\frac{1}{6}\left(2s\sum ab-12Rrs\right)\left(3+1+\frac{r}{R}\right)$$

$$=\frac{1}{6}\cdot 2s(s^2+r^2-2Rr)\left(\frac{4R+r}{R}\right)$$

$$\overset{Gerretsen}{\overbrace{\leq}} \frac{s(4R+r)}{3R}(4R^2+2Rr+4r^2) \overset{?}{\leq} 9R^2s$$

$$\Leftrightarrow (4R+r)(4R^2+2Rr+4r^2) \leq 27R^3$$

$$\Leftrightarrow 11R^3 - 12R^2r - 18Rr^2 - 4r^3 \geq 0$$

$$\Leftrightarrow 11t^3 - 12t^2 - 18t - 4 \geq 0 \quad \left(\text{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(11t^2+10t+2) \geq 0 \to \text{true} \because t = \frac{R}{r} \geq 2 \text{ (Euler)}$$

$$\Rightarrow \text{(3) is true} \Rightarrow \text{(2) is true (Proved)}$$

364. In $\Delta\, ABC$:

$$\left(\frac{r_a}{w_a}+\frac{r_b}{w_b}+\frac{r_c}{w_c}\right)^2 \geq 3\left(\frac{4R+r}{s}\right)^2$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

$$\text{We have: } r_a = \frac{2S}{b+c-a}; w_a \leq \sqrt{s(s-a)}; s = \frac{a+b+c}{2}; S = area$$

$$\Rightarrow \frac{r_a}{w_a} \geq \frac{2S}{(b+c-a)\sqrt{s(s-a)}}. \text{ Similarly } \frac{r_b}{w_b} \geq \frac{2S}{(c+a-b)\sqrt{s(s-b)}}; \frac{r_c}{w_c} \geq \frac{2S}{(a+b-c)\sqrt{s(s-c)}}$$

$$\Rightarrow \left(\frac{r_a}{w_a}+\frac{r_b}{w_b}+\frac{r_c}{w_c}\right)^2 \geq \frac{4S^2}{s}\cdot\left(\sum \frac{1}{\sqrt{\frac{b+c-a}{2}}\cdot(b+c-a)}\right)^2 \quad (1)$$

$$\text{Other: } 3\left(\frac{4R+r}{s}\right)^2 = 3\cdot\left[\frac{\frac{abc}{S}+\frac{2S}{a+b+c}}{s}\right]^2$$

$$= 3\left[\frac{abc(a+b+c)+2S^2}{S\cdot s(a+b+c)}\right]^2 = 3\left[\frac{8abc(a+b+c)+16S^2}{4S(a+b+c)^2}\right]^2$$

$$= 3\cdot\left[\frac{(a+b+c)^2\cdot[\sum(b+c-a)(c+a-b)]}{4S(a+b+c)^2}\right]^2 = \frac{3[\sum(b+c-a)(c+a-b)]^2}{16S^2} \quad (2)$$

(1), (2) $\Rightarrow \left(\sum \frac{r_a}{w_a}\right)^2 \ge 3\left(\frac{4R+r}{s}\right)^2$***. We will prove that:***

$$\frac{4S^2}{s}\cdot\left(\sum\frac{1}{\sqrt{\frac{(b+c-a)^3}{2}}}\right)^2 \ge \frac{3}{16S^2}\cdot\left[\sum(b+c-a)(c+a-b)\right]^2$$

$$\Leftrightarrow \frac{64S^4}{s}\cdot\left(\sum\frac{1}{\sqrt{\frac{(b+c-a)^3}{2}}}\right)^2 \ge 3\left[\sum(b+c-a)(c+a-b)\right]^2$$

$$\Leftrightarrow \frac{(a+b+c)^2\cdot\prod(b+c-a)^2}{6(a+b+c)}\cdot\left(\sum\frac{1}{\sqrt{\frac{(b+c-a)^3}{2}}}\right)^2 \ge$$

$$\ge \left[\sum(b+c-a)(c+a-b)\right]^2$$

$$\Leftrightarrow \left(\sum a\right)\prod(b+c-a)^2\cdot\left(\sum\frac{1}{\sqrt{\frac{(b+c-a)^3}{2}}}\right)^2 \ge$$

$$\ge 6\left[\sum(b+c-a)(c+a-b)\right]^2$$

$$\Leftrightarrow \left(\sum a\right)\prod(b+c-a)^2\cdot\left(\sum\frac{1}{\sqrt{(b+c-a)}}\right)^2 \ge$$

$$\ge 3\left[\sum(b+c-a)(c+a-b)\right] \quad \textbf{(3)}$$

Put: $\begin{cases} b+c-a=x>0 \\ c+a-b=y>0 \\ a+b-c=z>0 \end{cases}$***; (3)*** $\Leftrightarrow (\sum x)\prod x^2\cdot\left(\sum\frac{1}{\sqrt{x^3}}\right)^2 \ge 3(\sum xy)^2$

$$\Leftrightarrow (\sum x)\left(\sum\sqrt{x^3y^3}\right)^2 \ge 3xyz(\sum xy)^2 \quad \textbf{(6)}$$

Other, by Holder: $\left(\sum\sqrt{x^3y^3}\right)\left(\sum\sqrt{x^3y^3}\right)(1+1+1)\ge\left(\sum xy\right)^3$

$$\Rightarrow\left(\sum\sqrt{x^3y^3}\right)^2\ge\frac{(\sum xy)^3}{3}\quad \textit{(4)}$$

AM-GM: $(\sum x)(\sum xy)\ge 3\sqrt[3]{xyz}\cdot 3\sqrt[3]{x^2y^2z^2}=9xyz\quad$ ***(5)***

(4), (5) $\Rightarrow(\sum x)\left(\sum\sqrt{x^3y^3}\right)^2\ge 3xyz(\sum xy)^2\Rightarrow$ ***(6) True*** $\Rightarrow$ ***(3) True***

$$\Rightarrow\left(\sum\frac{r_a}{w_a}\right)^2\ge 3\left(\frac{4R+r}{s}\right)^2\Rightarrow \textit{Q.E.D.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{r_a}{w_a}+\frac{r_b}{w_b}+\frac{r_c}{w_c}=\frac{r_a^2}{r_aw_a}+\frac{r_b^2}{r_bw_b}+\frac{r_c^2}{r_cw_c}\overset{Bergstrom}{\underset{(1)}{\ge}}\frac{(r_a+r_b+r_c)^2}{\sum r_aw_a}$$

WLOG, we may assume $a\ge b\ge c$***. Then*** $r_a\ge r_b\ge r_c$ ***and*** $w_a\le w_b\le w_c$

$$\therefore\sum r_aw_a\overset{Chebyshev}{\le}\frac{1}{3}\left(\sum r_a\right)\left(\sum w_a\right)=\frac{1}{3}(4R+r)\left(\sum w_a\right)$$

$$\overset{C-B-S}{\le}\frac{4R+r}{3}\sqrt{3}\sqrt{\sum w_a^2}\underset{(2)}{\overset{w_a\le\sqrt{s(s-a)}}{\le}}\frac{\sqrt{3}(4R+r)}{3}\sqrt{\sum s(s-a)}$$

$$=\frac{\sqrt{3}(4R+r)}{3}\sqrt{3s^2-2s^2}=\frac{\sqrt{3}(4R+r)}{3}\cdot S=\frac{\sqrt{3}s(4R+r)}{3}$$

(1), (2) $\Rightarrow\sum\frac{r_a}{w_a}\ge\frac{3(4R+r)^2}{\sqrt{3}s(4R+r)}=\frac{3(4R+r)}{\sqrt{3}s}\Rightarrow\left(\sum\frac{r_a}{w_a}\right)^2\ge\frac{9(4R+r)^2}{3s^2}=3\left(\frac{4R+r}{s}\right)^2$

Solution 3 by Soumitra Mandal-Chandar Nagore

We know, $r_a=\frac{\Delta}{p-a},r_b=\frac{\Delta}{p-b},r_c=\frac{\Delta}{p-c},w_a\le\sqrt{p(p-a)}$

$w_b\le\sqrt{p(p-a)},w_c\le\sqrt{p(p-c)}$ ***and*** $\sum_{cyc}(p-a)(p-b)=r(r+4R)$

$$\therefore\left(\sum_{cyc}\frac{r_a}{w_a}\right)^2=\left(\sum_{cyc}\frac{\Delta}{p-a}\cdot\frac{1}{\sqrt{p(p-a)}}\right)^2\overset{Chebyshev}{\geq}$$

$$\geq\left(\frac{1}{3}\left(\sum_{cyc}\frac{\Delta}{p-a}\right)\left(\sum_{cyc}\frac{1}{\sqrt{p(p-a)}}\right)\right)^2$$

$$=\left(\frac{1}{3}\cdot\frac{\Delta}{(p-a)(p-b)(p-c)}\left(\sum_{cyc}(p-a)(p-b)\right)\cdot\left(\sum_{cyc}\frac{1}{\sqrt{p(p-a)}}\right)\right)^2$$

$$\overset{AM\geq GM}{\geq}\left(\frac{\Delta}{\frac{\Delta^2}{p}}\cdot r(r+4R)\cdot\frac{1}{\sqrt{p}}\cdot\frac{1}{\sqrt[6]{(p-a)(p-b)(p-c)}}\right)^2$$

$$\overset{Reverse\ AM\geq GM}{\geq}\left(\frac{pr}{\Delta}\cdot(r+4R)\cdot\frac{1}{\sqrt{p}}\cdot\frac{1}{\sqrt{\frac{(p-a)+(p-b)+(p-c)}{3}}}\right)^2=3\left(\frac{r+4R}{p}\right)^2$$

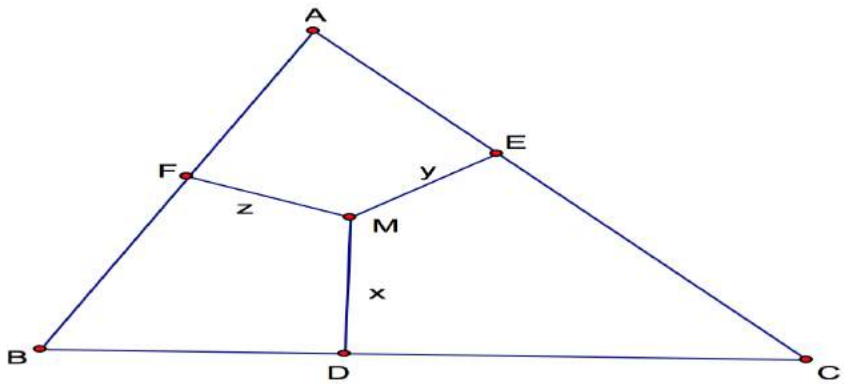
(Proved)

365. **Let $\Delta\ ABC$ and M is an arbitrary point in the domain of the triangle ABC. $D,E,F;D\in BC;E\in CA,F\in AB$ such that $MF\perp BC,ME\perp CA,MF\perp AB$. Let $MD=x,ME=y,MF=z$. Prove that**

$$(x^2+h_a^2)(y^2+h_b^2)(z^2+h_c^2)\geq\frac{1000}{729}(h_ah_bh_c)^2$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by proposer



We have $(x^2+h_a^2)(y^2+h_b^2)(z^2+h_c^2)\geq\frac{1000}{729}(h_ah_bh_c)^2 \Leftrightarrow$

$$\Leftrightarrow\left[\left(\frac{x}{h_a}\right)^2+1\right]\left[\left(\frac{y}{h_b}\right)^2+1\right]\left[\left(\frac{z}{h_c}\right)^2+1\right]\geq\frac{1000}{729}$$

Let $(X,Y,Z)=\left(\frac{x}{h_a},\frac{y}{h_b},\frac{z}{h_c}\right)\Rightarrow X,Y,Z>0$ ***and*** $X+Y+Z=1$ ***because***

$$X+Y+Z=\frac{x}{h_a}+\frac{y}{h_b}+\frac{z}{h_c}=\frac{\frac{1}{2}ax}{\frac{1}{2}ah_a}+\frac{\frac{1}{2}by}{\frac{1}{2}bh_b}+\frac{\frac{1}{2}cz}{\frac{1}{2}ch_c}=\frac{[MBC]}{[ABC]}+\frac{[MCA]}{[ABC]}+\frac{[MAB]}{[ABC]}=1$$

We need to prove $(X^2+1)(Y^2+1)(Z^2+1)\geq\frac{1000}{729}$ ***with*** $X,Y,Z>0$,

$X+Y+Z=1$. ***We have***

$$(X^2+1)(Y^2+1)(X^2+1)=(XY+YZ+ZX-1)^2+(1-XYZ)^2$$

$$\geq\left[1-\frac{(XY+YZ+ZX)^2}{3}\right](XY+YZ+ZX-1)^2$$

Let $t=XY+YZ+ZX\Rightarrow 0<t\leq\frac{1}{3}$ ***because***

$XY+YZ+ZX\leq\frac{1}{3}(X+Y+Z)^2=\frac{1}{3}$. ***Hence***

$$(X^2+1)(Y^2+1)(X^2+1)\geq f(t)=\left(1-\frac{t^2}{3}\right)(t-1)^2=\frac{t^4}{9}+\frac{t^2}{3}-2t+2,$$

$$0 < t \le \frac{1}{3}$$

$$f'(t) = \frac{4t^3}{9} + \frac{2t}{3} - 2 < 0, \forall t \in \left(0, \frac{1}{3}\right] \Rightarrow f(t) \ge f\left(\frac{1}{3}\right) = \frac{1000}{729}$$

Hence $(X^2+1)(Y^2+1)(Z^2+1) \ge \frac{1000}{729} \Rightarrow$

$$\Rightarrow (x^2+h_a^2)(y^2+h_b^2)(z^2+h_c^2) \ge \frac{1000}{729}(h_ah_bh_c)^2$$

366. In $\Delta\ ABC$:

$$\frac{r_ar_b}{\sin\frac{A}{2}\sin\frac{B}{2}} + \frac{r_br_c}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{r_cr_a}{\sin\frac{C}{2}\sin\frac{A}{2}} \ge \frac{h_ah_b}{\sin^2\frac{C}{2}} + \frac{h_bh_c}{\sin^2\frac{A}{2}} + \frac{h_ch_a}{\sin^2\frac{B}{2}}$$

Proposed by Hoang Tung Le Nhat-Hanoi-Vietnam

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{r_a \cdot r_b}{\sin\frac{A}{2}\sin\frac{B}{2}} \ge \sum \frac{h_a \cdot h_b}{\sin^2\frac{C}{2}}$$

1) $\sum \frac{r_a \cdot r_b}{\sin\frac{A}{2}\sin\frac{B}{2}} = \sum \frac{S^2}{(p-a)(p-b)\cdot\sqrt{\frac{(p-a)(p-c)(p-b)(p-a)}{acbc}}}$

$$\sum \frac{c\cdot\sqrt{ab}\cdot S^2}{\prod_\Delta(p-a)\cdot\sqrt{(p-a)(p-b)}} = S^2 \cdot \sum \frac{c}{\prod_\Delta(p-a)\cdot\sqrt{\frac{(p-a)(p-b)}{ab}}} =$$

$$= \frac{S^2}{\prod(p-a)} \cdot \sum \frac{c}{\sin\frac{C}{2}} = p \cdot \sum \frac{c}{\sin\frac{C}{2}} \quad (*)$$

2) $\sum \frac{h_a \cdot h_b}{\sin^2\frac{C}{2}} = 4S^2 \sum \frac{1}{ab\cdot\sin^2\frac{C}{2}} =$

$$= 4\cdot S^2 \cdot \sum \frac{1}{ab \cdot \frac{(p-a)(p-b)}{ab}} = 4\cdot S^2 \sum \frac{1}{(p-a)(p-b)} =$$

$$= 4\cdot S^2\cdot\frac{1}{\prod(p-a)}\cdot\sum(p-c) = 4p\cdot\sum(p-c) =$$

$$= 4p(3p-2p) = 4p^2 \quad (**)$$

$$(*), (**) \Rightarrow p\cdot\sum\frac{c}{\sin\frac{C}{2}} \ge 4p^2 \quad (ASSURE)$$

$$p\cdot\sum\frac{c}{\sin\frac{C}{2}} \overset{Chebyshev}{\ge} \frac{p}{3}\sum a\sum\frac{1}{\sin\frac{A}{2}} = \frac{2p^2}{3}\cdot\sum\frac{1}{\sin\frac{A}{2}} \ge \frac{2p^2}{3}\cdot\frac{3}{\sin\frac{\pi}{6}} = 4p^2;$$

$$f(x) = \frac{1}{\sin\frac{x}{2}} \Rightarrow f''(x) \ge 0$$

367. **If in $\Delta\, ABC, I -$ incentre, $R_a, R_b, R_c -$ circumradii of $\Delta\, BIC, \Delta\, CIA, \Delta\, AIB$**

$$3\sqrt{6}r \le \sqrt{m_aR_a} + \sqrt{m_bR_b} + \sqrt{m_cR_c} \le \frac{3\sqrt{6}R}{2}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo ABC, siendo, $I -$ incentro, R_a, R_b, R_c circunradios de los triángulos BIC, CIA, AIB. Probar que

$$3\sqrt{6}r \le \sqrt{m_aR_a} + \sqrt{m_bR_b} + \sqrt{m_cR_c} \le \frac{3\sqrt{6}}{2}R$$

Tener en cuenta las siguientes identidades y desigualdades

$$R_a = 2R\sin\frac{A}{2}, R_b = 2R\sin\frac{B}{2}, R_c = 2R\sin\frac{C}{2}, R \ge 2r$$

Lo cual implica $\rightarrow R_a + R_b + R_c = 2R\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \le 2R\cdot\frac{3}{2} = 3R$

$$R_aR_bR_c = 8R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 8R^2\cdot\frac{r}{4R} = 2R^2\cdot r \ge 8r^3$$

$$m_a + m_b + m_c \leq \sqrt{3(m_a^2 + m_b^2 + m_c^2)} = \sqrt{\frac{9}{4}(a^2 + b^2 + c^2)} \leq \sqrt{\frac{9}{4} \cdot 9R^2} = \frac{9R}{2}$$

$$m_a \geq \sqrt{s(s-a)}, m_b \geq \sqrt{s(s-b)}, m_c \geq \sqrt{s(s-c)} \Leftrightarrow$$

$\Leftrightarrow m_a m_b m_c \geq S \cdot s = s^2 r \geq 27r^3$. ***Aplicando MA ≥ MG***

$$\sqrt{m_a R_a} + \sqrt{m_b R_b} + \sqrt{m_c R_c} \geq 3\sqrt[6]{m_a m_b m_c \cdot R_a R_b R_c} \geq 3\sqrt[6]{27r^3 \cdot 8r^3} =$$

$= 3\sqrt[6]{6^3}r = 3\sqrt{6}r$ ***(LQQD). Por la desigualdad de Cauchy***

$$\sqrt{m_a R_a} + \sqrt{m_b R_b} + \sqrt{m_c R_c} \leq \sqrt{(m_a + m_b + m_c)(R_a + R_b + R_c)} \leq$$

$\leq \sqrt{\frac{9R}{2} \cdot 3R} = \frac{3\sqrt{6}}{2}R$ ***(LQQD)***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$6\sqrt{3}r \leq \sqrt{2 \cdot R_a \cdot m_a} + \sqrt{2 \cdot R_b \cdot m_b} + \sqrt{2 \cdot R_c \cdot m_c} \leq 3\sqrt{3}R.$$

i) $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \overset{f'' \leq 0}{\leq} 3 \cdot f\left(\frac{A+B+C}{6}\right) = \frac{3}{2}$

ii) $\sum m_a \leq \frac{9}{2}R$

iii) $\prod_A \sin\frac{A}{2} = \frac{r}{4R}$

iv) $m_a \geq \sqrt{p \cdot (p-a)} \quad \prod m_a \geq p \cdot S$

1) $RHS \Rightarrow \sum\sqrt{2 \cdot R_a \cdot m_a} = \sqrt{2} \cdot \sum\sqrt{R_a \cdot m_a} \leq$

$$\leq \sqrt{2} \cdot \sqrt{\sum R_a \cdot \sum m_a} = \sqrt{2} \cdot \sqrt{2R \cdot \sum \sin\frac{A}{2} \cdot \sum m_a} \overset{(i);(ii)}{\leq}$$

$$\leq 2 \cdot R \cdot \sqrt{\frac{3}{2} \cdot \frac{9}{2}} = \frac{2R}{2} \cdot 3\sqrt{3} = 3\sqrt{3}R$$

2) $LHS: \sum\sqrt{2 \cdot R_a \cdot m_a} \geq \sqrt{2} \cdot 3 \cdot \sqrt[3]{\sqrt{R_a \cdot R_b \cdot R_c \cdot m_a \cdot m_b \cdot m_c}}$

$$= 3\sqrt{2}\cdot\sqrt[6]{8R^3\cdot\prod_{\Delta}\sin\frac{A}{2}\cdot\prod_{\Delta}m_a}\overset{(iii);(iv)}{\geq}3\sqrt{2}\cdot\sqrt[6]{8R^3\cdot\frac{r}{4R}\cdot pS}=$$

$$=3\sqrt{2}\cdot\sqrt[6]{2R^2\cdot r^2\cdot p^2}\overset{\substack{p^2\geq 27r^2\\ R\geq 2r}}{\geq}3\sqrt{2}\cdot\sqrt[6]{8\cdot 27\cdot r^6}=$$

$$=3\sqrt{2}\cdot\sqrt{2\cdot 3}\cdot r=6\sqrt{3}r\ (LHS)$$

368. In $\Delta\ ABC$:

$$\left(\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}\right)^2+\frac{2r}{R}\geq 10$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triàngulo ABC: $\left(\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}\right)^2+\frac{2r}{R}\geq 10$

Tener en cuenta las siguientes identidades y desigualdades en un $\Delta\ ABC$

$$\frac{m_a}{s_a}=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right),\frac{m_b}{s_b}=\frac{1}{2}\left(\frac{c}{a}+\frac{a}{c}\right),\frac{m_c}{s_c}=\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)$$

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{2r}{R}\geq 10$***. Es suficiente demostrar***

$$\left(\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}\right)^2\geq(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$\left(\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}\right)\right)^2\geq 3+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}$***, donde***

$$x=\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}\geq 6$$

$\Leftrightarrow\frac{x^2}{4}\geq 3+x\Leftrightarrow x^2-4(3+x)=(x-6)(x+2)\geq 0$***, lo cual es cierto ya que $x\geq 6$. Por lo tanto***

$$\Rightarrow\left(\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}\right)^2+\frac{2r}{R}\geq(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{2r}{R}\geq 10$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\Delta: s_a=\frac{2bc\cdot m_a}{b^2+c^2}\qquad \frac{m_a}{s_a}=\frac{b^2+c^2}{2bc}$$

$$\left(\sum\frac{m_a}{s_a}\right)^2=\left(\sum\frac{b^2+c^2}{2bc}\right)^2=\frac{1}{4}\cdot\left(\sum\frac{a+b}{c}\right)^2=\frac{1}{4}\cdot\left(\frac{\sum(a^2b+ab^2)}{abc}\right)^2=$$

$$=\frac{1}{4}\cdot\left(\frac{p^2-2Rr+r^2}{2Rr}\right)^2\overset{GERRETSEN}{\geq}\frac{1}{4}\cdot\left(\frac{16Rr-5r^2-2Rr+r^2}{2Rr}\right)^2=$$

$$=\frac{1}{4}\cdot\left(\frac{14Rr-4r^2}{2Rr}\right)^2=\frac{1}{4}\cdot\left(7-\frac{2r}{R}\right)^2=LHS$$

$$\Rightarrow\frac{1}{4}\left(7-\frac{2r}{R}\right)^2+\frac{2r}{R}\geq 10\quad \textbf{(ASSURE)};\ \frac{2r}{R}=t\quad(1-t\geq 0)\ \textbf{Euler}$$

$$\frac{1}{4}\cdot(7-t)^2+t\geq 10;\ (7-t)^2+4t\geq 40\Leftrightarrow 49-14t+t^2+4t\geq 40$$

$$t^2-10t+9\geq 0;\ \underbrace{(t-1)}_{\leq 0}\cdot\underbrace{(t-9)}_{<0}\geq 0$$

369. In $\Delta\, ABC, O$ – circumcentre, R_a, R_b, R_c – circumradii in $\Delta\, BOC, \Delta\, COA, \Delta\, AOB$

$$r_ar_br_c\leq\frac{27}{8}R_aR_bR_c$$

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Daniel Sitaru – Romania

$$R_a=\frac{a}{2\sin 2A}=\frac{2R\sin A}{2\cdot 2\sin A\cos A}=\frac{R}{2\cos A}$$

$$\prod r_a \le \frac{27}{8}\prod R_a \leftrightarrow rs^2 \le \frac{27}{8}\cdot\frac{R^3}{8\cos A\cos B\cos C} \leftrightarrow$$

$$\leftrightarrow rs^2\cdot 8\cos A\cos B\cos C \le \frac{27R^3}{8}$$

$$rs^2\cdot 8\cos A\cos B\cos C \le rs^2\cdot 8\cdot\frac{1}{8} \le \frac{27R^3}{8} \quad \textit{(to prove)}$$

$$\leftrightarrow rs^2 \le \frac{27R^3}{8} \leftrightarrow s^2 \le \frac{27R^3}{8r} \textit{ (to prove)}$$

$$s^2 \overset{MITRINOVIC}{\le} \frac{27R^2}{4} \le \frac{27R^3}{8r} \leftrightarrow 2r \le R \textit{ (EULER)}$$

370. In $\Delta\, ABC$:

$$18p\cdot(2r)^3 \le \sum\frac{a^5}{r_a - r} \le 18p\cdot R^3$$

Proposed by Marin Chirciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$18p\cdot(2r)^3 \overset{(1)}{\le} \sum\frac{a^5}{r_a - r} \overset{(2)}{\le} 18p\cdot R^3;\ \frac{a^5}{r_a - r} = \frac{a^5}{\frac{\Delta}{s-a} - \frac{\Delta}{s}} = \frac{a^5\cdot s(s-a)}{\Delta(s-s+a)}$$

$$= \frac{a^4\cdot s(s-a)}{rs} \overset{(3)}{=} \frac{a^4(s-a)}{r} \therefore \textit{using (3), (1)} \Leftrightarrow \sum a^4(s-a) \ge 144sr^4 \quad \textit{(4)}$$

$$\text{Let } s-a = x, s-b = y, s-c = z$$

$$\therefore s = x+y+z \text{ and } a = y+z, b = z+x, c = x+y$$

$$\therefore (4) \Leftrightarrow (y+z)^4x + (z+x)^4y + (x+y)^4\cdot z \ge$$

$$\ge 144s\left(\frac{\Delta}{s}\right)^4 = \frac{144s\cdot\Delta^4}{s^4} = \frac{144s^2(s-a)^2(s-b)^2(s-c)^2}{s^3}$$

$$= \frac{144(s-a)^2(s-b)^2(s-c)^2}{s} = \frac{144x^2y^2z^2}{x+y+z}$$

$$\Leftrightarrow \{(y+z)^4x+(z+x)^4y+(x+y)^4z\}(x+y+z)\geq$$

$$\geq 144x^2y^2z^2 \quad \textbf{(4a)}$$

Now, $\{(y+z)^4x+(z+x)^4y+(x+y)^4z\}(x+y+z)\geq$

$$\overset{A-G}{\geq}(16y^2z^2x+16z^2x^2y+16x^2y^2z)(x+y+z)=$$

$$=16xyz\left(\sum xy\right)\left(\sum x\right)\overset{A-G}{\geq}16xyz\cdot 3\sqrt[3]{x^2y^2z^2}\cdot 3\sqrt[3]{xyz}$$

$=144x^2y^2z^2\Rightarrow$ ***(4a) is true*** $\Rightarrow$ ***(1) is true***

Using (3), (2) $\Leftrightarrow \sum a^4(s-a)\leq 18R^3sr$

$$\Leftrightarrow 2\sum a^4(s-a)\leq 36R^3sr=9R^2\cdot 4Rrs=9R^2abc$$

$$\Leftrightarrow 2\left\{\frac{a^4(s-a)}{abc}+\frac{b^4(s-b)}{abc}+\frac{c^4(s-c)}{abc}\right\}\leq 9R^2$$

$$\Leftrightarrow 2\left\{a^3\cdot\frac{s(s-a)}{bc}+b^3\cdot\frac{s(s-b)}{ca}+c^3\cdot\frac{s(s-c)}{ab}\right\}\leq 9R^2s$$

$$\Leftrightarrow 2\left(a^3\cos^2\frac{A}{2}+b^3\cos^2\frac{B}{2}+c^3\cos^2\frac{C}{2}\right)\leq 9R^2s$$

$$\Leftrightarrow a^3(1+\cos A)+b^3(1+\cos B)+c^3(1+\cos C)\leq 9R^2s$$

$$\Leftrightarrow \sum a^3+(a^3\cos A+b^3\cos B+c^3\cos C)\leq 9R^2s$$

$$\Leftrightarrow 3abc+2s\left(\sum a^2-\sum ab\right)+\sum a^3\cos A\leq 9R^2s$$

$$\Leftrightarrow 12Rrs+2s(s^2-12Rr-3r^2)+\sum a^3\cos A\leq 9R^2s$$

$$\Leftrightarrow 2s(s^2-6Rr-3r^2)+\sum(a^3\cos A)\leq 9R^2s \quad \textbf{(5)}$$

Now, $\sum a^3\cos A=a^3\cos A+b^3\cos B+c^3\cos C$

$$=\frac{a^3(b^2+c^2-a^2)}{2bc}+\frac{b^3(c^2+a^2-b^2)}{2ca}+\frac{c^3(a^2+b^2-c^2)}{2ab}$$

$$\overset{(6)}{=}\frac{a^4b^2+b^4c^2+c^4a^2+a^2b^4+b^2c^4+c^2a^4-\sum a^6}{2abc}$$

Schur $\Rightarrow \sum x^3+3xyz\ge x^2y+y^2z+z^2x+xy^2+yz^2+zx^2$

$$\Rightarrow \sum a^6+3a^2b^2c^2\ge a^4b^2+b^4c^2+c^4a^2+a^2b^4+b^2c^4+c^2a^4$$

(taking $x=a^2, y=b^2, z=c^2$***)***

$$\Rightarrow a^4b^2+b^4c^2+c^4a^2+a^2b^4+b^2c^4+c^2a^4-\sum a^6\le 3a^2b^2c^2$$

$$\Rightarrow \frac{a^4b^2+b^4c^2+c^4a^2+a^2b^4+b^2c^4+c^2a^4-\sum a^6}{2abc}\overset{(7)}{\le}\frac{3a^2b^2c^2}{2abc}$$

$= \frac{3}{2}(4Rrs)=6Rrs;$ ***(6), (7)*** $\Rightarrow \sum a^3\cos A\le 6Rrs$ ***(8)***

$$2s(s^2-6Rr-3r^2)\overset{Gerretsen}{\le}2s(4R^2-2Rr)\quad (9)$$

(8), (9) $\Rightarrow$ ***LHS of (5)*** $\le 2s(4R^2-2Rr)+6Rrs$

$$=s(8R^2+2Rr)\overset{Euler}{\le}s(8R^2+R^2)=9R^2s$$

= ***RHS of (5)*** $\Rightarrow$ ***(5) is true*** $\Rightarrow$ ***(2) is true (Proved)***

371. Prove that in ΔABC:

$$\frac{a^{11}}{b+c-a}+\frac{b^{11}}{c+a-b}+\frac{c^{11}}{a+b-c}\ge a^{10}+b^{10}+c^{10}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nguyen Ngoc Tu-Ha Giang-Vietnam

$$\sum\frac{a^{11}}{b+c-a}\ge\sum a^{10}\Leftrightarrow\sum a^{10}\cdot\left(\frac{a}{b+c-a}-1\right)\ge 0$$

$$\Leftrightarrow\sum a^{10}\cdot\frac{(a-b)+(a-c)}{b+c-a}\ge 0$$

$$\Leftrightarrow \sum\left(\frac{a^{10}}{b+c-a}-\frac{b^{10}}{c+a-b}\right)(a-b)\geq 0$$

$$\Leftrightarrow \sum \frac{1}{(b+c-a)(c+a-b)}\cdot(a-b)[(a-b)(a^{10}+a^9b+\dots+ab^9+b^{10})$$

$$+c(a^5+b^5)(a-b)(a^4+a^3b+\dots+ab^3+b^4)$$

$$-ab(a-b)(a^8+a^7b+\dots+ab^7+b^8)]\geq 0$$

$$\Leftrightarrow \sum\frac{1}{(b+c-a)(c+a-b)}\cdot[a^{10}+b^{10}\ +c(a^5+b^5)(a^4+a^3b+\dots+ab^3+b^4](a-b)^2\geq 0$$

It's true!

Solution 2 by Boris Colakovic-Belgrade-Serbia

$$LHS \overset{Chebishev}{\geq} \frac{1}{3}\left(\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\right)(a^{10}+b^{10}+c^{10})$$

Substitution $b+c-a=x; c+a-b=y; a+b-c=z$

$$a=\frac{y+z}{2}, b=\frac{z+x}{2}, c=\frac{x+y}{2}$$

$$\frac{1}{3}\left(\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\right)=\frac{1}{3}\left(\frac{y+z}{2x}+\frac{z+x}{2y}+\frac{x+y}{2z}\right)=$$

$$=\frac{1}{6}\left(\frac{x+y}{z}+\frac{y+z}{x}+\frac{z+x}{y}\right)=$$

$$=\frac{1}{6}\left(\frac{x}{z}+\frac{z}{x}+\frac{y}{z}+\frac{z}{y}+\frac{x}{y}+\frac{y}{x}\right)\overset{AM-GM}{\geq}\frac{1}{6}(2+2+2)\geq 1$$

$$LHS\geq a^{10}+b^{10}+c^{10}$$

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$a\geq b\geq c$$

$$\sum\frac{a^{11}}{b+c-a}\overset{Chebishev}{\geq}\frac{1}{3}\cdot\left(\sum a^{11}\right)\cdot\left(\sum\frac{1}{b+c-a}\right)\geq 3\left(\sum a^{11}\right)\left(\frac{1}{\sum a}\right)\geq$$

$$\overset{\sum a^{11}\geq\frac{1}{3}(\sum a^2)(\sum a)}{\geq}\sum a^{10}$$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{a^{11}}{2\cdot(p-a)} = \frac{1}{2}\cdot\sum\frac{a^{11}}{p-a} \overset{Chebishev}{\geq}$$

$$\geq \frac{1}{6}\cdot\sum a^{11}\cdot\sum\frac{1}{p-a} \geq \frac{1}{18}\cdot\left(\sum a\right)\left(\sum a^{10}\right)\cdot\sum\frac{1}{p-a} =$$

$$= \left(\frac{1}{18}\cdot\sum a\cdot\sum\frac{1}{p-a}\right)\cdot\sum a^{10} \overset{Bergstrom}{\geq}$$

$$\geq \frac{1}{8}\cdot\sum a\cdot\frac{9}{3p-2p}\cdot\sum a^{10} = \sum a^{10}$$

Solution 5 by Sanong Hauerai-Nakon Pathom-Thailand

$$\frac{a^{11}}{b+c-a} + \frac{b^{11}}{c+a-b} + \frac{c^{11}}{a+b-c} = \frac{a^{20}}{a^9b+a^9c-a^{10}} + \frac{b^{20}}{b^9c+b^9a-b^{10}} + \frac{c^{20}}{c^9a+c^9b-c^{10}}$$

$$\geq \frac{(a^{10}+b^{10}+c^{10})^2}{a^9b+b^9c+c^9a+b^9a+a^9c+c^9b-a^{10}+b^{10}+c^{10}}$$

$$\geq \frac{(a^{10}+b^{10}+c^{10})^2}{a^{10}+b^{10}+c^{10}+b^9a+a^9c+c^9b-(a^{10}+b^{10}+c^{10})}$$

$$= \frac{(a^{10}+b^{10}+c^{10})^2}{b^9a+a^9c+c^9b} \geq \frac{(a^{10}+b^{10}+c^{10})^2}{a^{10}+b^{10}+c^{10}}$$

$$= a^{10}+b^{10}+c^{10}$$

Because $a+b>c, b+c>a, c+a>b$

Solution 6 by Soumitra Mandal-Chandar Nagore-India

Let $a \geq b \geq c$ **then** $\frac{1}{b+c-a} \geq \frac{1}{c+a-b} \geq \frac{1}{a+b-c}$

$$\therefore \sum_{cyc}\frac{a^{11}}{b+c-a} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum_{cyc} a^{11}\right)\left(\sum_{cyc}\frac{1}{b+c-a}\right)$$

$$\geq \left(\sum_{cyc} a^{11}\right)\left(\frac{3}{a+b+c}\right) \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum_{cyc} a^{10}\right)(a+b+c)\left(\frac{3}{a+b+c}\right)$$

$$= a^{10} + b^{10} + c^{10}$$

372. In acute $\Delta\ ABC$:

$$8 + \prod \sec A \geq 4 \sum \csc^2 A$$

Proposed by Kevin Soto Palacios – Huarmey – Peru

Solution by Mehmet Sahin-Ankara-Turkey

$$8 + \prod_{cyclic} \sec A \geq 4 \cdot \sum_{cyclic} \csc^2 A$$

$$8 + \frac{1}{\cos A \cdot \cos B \cos C} \geq 4\left(\frac{1}{\sin^2 A} + \frac{1}{\sin^2 B} + \frac{1}{\sin^2 C}\right)$$

$f(x) = \frac{1}{\sin^2 x}$ *is convex function in* $\left(0, \frac{\pi}{2}\right)$

$$f\left(\frac{A+B+C}{3}\right) \leq \frac{1}{3}[f(A) + f(B) + f(C)]$$

$$\Leftrightarrow f(A) + f(B) + f(C) \geq 3 \cdot f\left(\frac{A+B+C}{3}\right)$$

$$\geq 3 \cdot \frac{1}{\left[\sin\left(\frac{A+B+C}{3}\right)\right]^2} \geq 3 \cdot \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} = 4$$

$$\Leftrightarrow 8 + \frac{1}{\cos A \cdot \cos B \cdot \cos C} \geq 4 \cdot 4 \Leftrightarrow \frac{1}{\cos A \cdot \cos B \cdot \cos C} \geq 8$$

$$\Leftrightarrow \cos A \cdot \cos B \cdot \cos C \leq \frac{1}{8}$$

$$\cos A + \cos B + \cos C \geq 3\sqrt[3]{\cos A \cdot \cos B \cdot \cos C}$$

$$\frac{3}{2} \geq 1 + \frac{r}{R} \geq 3\sqrt[3]{\cos A \cdot \cos B \cdot \cos C}$$

$$\Leftrightarrow \cos A \cdot \cos B \cdot \cos C \leq \frac{1}{8} \textit{ as desired } \therefore$$

373. In $\Delta\, ABC$:

$$\max(A, B, C) = 120° \Leftrightarrow s\sqrt{3} = r + 3R$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\max(A; B; C) = A = 120°$$

$$\textit{1)}\ \tan\frac{A}{2} = \frac{r}{p-a}$$

$$\textit{2)}\ \sin A = \frac{a}{2R} \Rightarrow \sin 120° = \frac{a}{2R} \Rightarrow a = \sqrt{3}R \quad (*)$$

$$\textit{1)} \Rightarrow \tan\frac{120°}{2} = \frac{r}{p-a} \Leftrightarrow \sqrt{3} = \frac{r}{p-a} \Rightarrow$$

$$\Rightarrow 3 \cdot (p - a) = \sqrt{3}r \Rightarrow 3 \cdot \left(\frac{b + c - a}{2}\right) = \sqrt{3}r \Rightarrow$$

$$\Rightarrow \frac{3 \cdot (b + c)}{2} - \frac{3a}{2} = \sqrt{3}r \Rightarrow \frac{3a}{2} - 3a + \frac{3(b + c)}{2} = \sqrt{3}r$$

$$3 \cdot \left(\frac{a+b+c}{2}\right) = \sqrt{3}r + 3a;\ 3p = \sqrt{3}r + 3a| \cdot \sqrt{3};\ 3 \cdot \sqrt{3}p = 3 \cdot r + 3\sqrt{3}a$$

$$\sqrt{3}p = r + \sqrt{3}a \overset{(*)}{\Rightarrow} \sqrt{3}p = r + 3R$$

374. If in $\Delta\, ABC, O$ – circumcentre,

$$d_a = d(O, BC), d_b = d(O, AC), d_c = d(O, AB)$$

$$\frac{d_a}{r_a} + \frac{d_b}{r_b} + \frac{d_c}{r_c} \geq 1$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

***En un triángulo** $ABC, O -$ **circuncentro,** $d_a = d(O, BC), d_b = d(O, AC)$,*

$d_c = d(O, AB)$. ***Probar que*** $\frac{d_a}{r_a} + \frac{d_b}{r_b} + \frac{d_c}{r_c} \geq 1$

Tener en cuenta las siugientes desigualdad e identidades en un ABC

$d_a = R\cos A, d_b = R\cos B, d_c = R\cos C, R \geq 2r$ ***(Inequality Euler)***

$$r_a = \frac{S}{s-a}, r_b = \frac{S}{s-b}, r_c = \frac{S}{s-c}, \cos A + \cos B + \cos C = 1 + \frac{r}{R} = \frac{R+r}{R}$$

$$\frac{S}{2R^2} = \sin A + \sin B + \sin C,$$

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C = \frac{2S}{R^2}$$

La desigualdad propuesta s equivalente

$$\frac{d_a}{r_a} + \frac{d_b}{r_b} + \frac{d_c}{r_c} = \frac{R}{S}\big((s-a)\cos A + (s-b)\cos B + (s-c)\cos C\big) \geq 1$$

$$\Leftrightarrow \frac{R}{s}(s(\cos A + \cos B + \cos C) - a\cos A - b\cos B - c\cos C) \geq 1$$

$$\Leftrightarrow \frac{R}{S} \cdot s\left(\frac{R+r}{R}\right) - \frac{R}{S} \cdot R(\sin 2A + \sin 2B + \sin 2C) \geq 1$$

$$\Leftrightarrow \frac{R}{sr} \cdot \left(\frac{R+r}{R}\right) - \frac{R}{S} \cdot \frac{2S}{R} = \frac{R+r}{r} - 2 = \frac{R-r}{r} \geq \frac{2r-r}{r} = 1 \textit{ (LQQD)}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{a \cdot d_a}{2} + \frac{b \cdot d_b}{2} + \frac{c \cdot d_c}{2} = S \Leftrightarrow \sum a \cdot d_a = 2S \quad \textit{(1)}$$

$$\sum \frac{d_a}{r_a} = \sum \frac{(p-a) \cdot d_a}{S} = \frac{p}{s} \cdot \sum d_a - \frac{1}{S} \cdot \sum a \cdot d_a =$$

$$\overset{(1)}{=} \frac{1}{r} \cdot \sum d_a - \frac{1}{S} \cdot 2S = \frac{1}{r} \cdot \sum d_a - 2 \geq 1 \quad \textit{(ASSURE)}$$

$$\sum d_a \geq 3r \Rightarrow \sum d_a = \sum R \cdot \cos A = R \cdot \left(1 + \frac{r}{R}\right) = R + r \overset{Euler}{\geq} 3r$$

375. Prove that in an acute-angled triangle having the perimeter equal with 1 we have: $\cos^3(aA+bB+cC)\geq 54Rr\cos A\cos B\cos C$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\{\cos x\}''=-\cos x\leq 0 \text{ for all } x\in\left(0,\frac{\pi}{2}\right)$$

hence, $\cos x$ *is a concave function and* $a+b+c=1$ *then*

$$\cos(aA+bB+cC)\geq a\cos A+b\cos B+c\cos C$$

$$\cos^3(aA+bB+cC)\geq(a\cos A+b\cos B+c\cos C)^3$$

$$=27\,abc\cos A\cos B\cos C=27\cdot 4Rrp\cdot\prod_{cyc}\cos A$$

$$=27\cdot 2Rr\cdot 2p\cdot\prod_{cyc}\cos A=54Rr\prod_{cyc}\cos A$$

(Proved) $\left[\begin{matrix}\because a+b+c=1\\ 2p=1\end{matrix}\right]$

376. In $\Delta\,ABC$:

$$(m_a+r_a)(m_b+r_b)(m_c+r_c)\geq 8w_aw_bw_c$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo ABC*. Probar que:*

$$(m_a+r_a)(m_b+r_b)(m_c+r_c)\geq 8w_aw_bw_c$$

Considerar lo siguiente en un triángulo ABC*:*

$$m_a=\frac{\sqrt{2b^2+2c^2-a^2}}{2},\ m_b=\frac{\sqrt{2a^2+2c^2-b^2}}{2},\ m_c=\frac{\sqrt{2a^2+2b^2-c^2}}{2}$$

$$r_ar_br_c = Sp, S = \sqrt{p(p-a)(p-b)(p-c)}$$

$\Leftrightarrow S$ = ***Área de región trianglular,*** p = ***semiperímetro***

$$w_a = \frac{2\sqrt{bc}\sqrt{p(p-a)}}{b+c}, w_b = \frac{2\sqrt{ac}\sqrt{p(p-b)}}{a+c}, w_c = \frac{2\sqrt{bc}\sqrt{p(p-c)}}{a+b}$$

Por: MA ≥ MG: $(m_a + r_a)(m_b + r_b)(m_c + r_c) \geq 8\sqrt{m_am_bm_cr_ar_br_c}$

Ahora demostraremos que: $m_am_bm_c \geq r_ar_br_c$. ***Por:***

$$MP \geq MA \to 2(b^2 + c^2) \geq (b+c)^2 \to 2(b^2 + c^2) - a^2 \geq (b+c)^2 - a^2$$

$$\to \frac{\sqrt{2(b^2+c^2)-a^2}}{2} \geq \sqrt{(b+c+a)(b+c-a)}(0,5) \to m_a \geq \sqrt{p(p-a)} \quad \textbf{(A)}$$

Por lo tanto: $m_b \geq \sqrt{p(p-b)}$ ***(B),*** $m_c \geq \sqrt{p(p-c)}$ ***(C)***

Multiplicando: (A)(B)(C) $\to m_am_bm_c \geq r_ar_br_c$. ***Por lo tanto:***

$$(m_a + r_a)(m_b + r_b)(m_c + r_c) \geq 8\sqrt{m_am_bm_cr_ar_br_c} \geq$$

$$\geq 8r_ar_br_c = 8Sp\frac{(a+b)(b+c)(a+c)}{(a+b)(b+c)(a+c)} \geq$$

$$\geq \frac{64abcp\sqrt{p(p-a)(p-b)(p-c)}}{(a+b)(b+c)(a+c)} = 8w_aw_bw_c \quad \textbf{(LQQD)}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

In ΔABC: $\prod(m_a + r_a) \geq 8 \cdot \prod w_a$

1) $m_a \geq \sqrt{p\cdot(p-a)}$; $r_a = \frac{S}{p-a}$

$$\prod(m_a + r_a) \overset{AM\geq GM}{\geq} 8 \cdot \sqrt{m_am_bm_c} \cdot \sqrt{r_ar_br_c} =$$

$$= 8\sqrt{\prod m_a} \cdot \sqrt{\prod r_a} \geq 8 \cdot \sqrt{\sqrt{\prod p\cdot(p-a)}} \cdot \sqrt{\frac{S^3}{\prod(p-a)}} =$$

$$= 8 \cdot \sqrt{\frac{p\cdot S^4\cdot p}{\prod(p-a)\cdot p}} = 8\sqrt{\frac{p^2\cdot S^4}{S^2}} = 8p\cdot S; \prod(m_a + r_a) \geq 8p\cdot S$$

$$\textit{2)}\ w_a \le \sqrt{p\cdot(p-a)} \Rightarrow 8\prod w_a \le 8\sqrt{p(p-a)p(p-b)p(p-c)} =$$
$$= 8\cdot p\cdot S$$

377. **Siendo a, b, c los lados de un triángulo ABC. Probar que**

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \ge 4r(4R+r)$$

Proposed by Kevin Soto Palacios – Huarmey – Peru

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \overset{AM-GM}{\ge} 3\sqrt[3]{(abc)^2} =$$
$$= 3\sqrt[3]{16p^2\cdot R^2\cdot r^2} \ge 4r\cdot(4R+r)\quad \textit{(ASSURE)}$$
$$27\cdot 16p^2\cdot R^2\cdot r^2 \ge 64r^3(4R+r)^3 \Leftrightarrow \frac{27}{4}\cdot p^2\cdot R^2 \ge r(4R+r)^3$$
$$\Rightarrow \textit{(Gerretsen)}\ \frac{27}{4}\cdot(16Rr-5r^2)\cdot R^2 \ge r\cdot(4R+r)^3 \Rightarrow$$
$$\left(\frac{R}{2}=t\right) \Rightarrow \frac{27}{4}\cdot(16t-5)t^2 \ge (4t+1)^3 \Leftrightarrow 176t^3-327t^2-48t-4 \ge 0$$
$$\underbrace{(t-2)}_{\substack{\Downarrow\\ t\ge 2}}\cdot\underbrace{(176t^2+25t+2)}_{\substack{\Downarrow\\ >0}} \ge 0.\ \textit{TRUE.}$$

378. **In $\Delta\ ABC$:**

$$\lambda\cdot\frac{R}{r} + \frac{p^2}{(4R+r)^2} \ge 2\lambda + \frac{1}{3}, \lambda \ge \frac{2}{15}$$

Proposed by Marin Chirciu – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un $\Delta\ ABC: n\cdot\frac{R}{r} + \frac{p^2}{(4R+r)^2} \ge 2n + \frac{1}{3}$***, donde*** $n \ge \frac{2}{15}$

$\Leftrightarrow n\left(\frac{R-2r}{r}\right)+\frac{p^2}{(4R+r)^2}\geq\frac{1}{3}$. ***Como*** $n\geq\frac{2}{15}$, $R-2r\geq 0$ ***(Inequality Euler),***

$p^2\geq 16Rr-5r^2$ ***(Inequality Gerretsen). Por lo tanto***

$$n\left(\frac{R-2r}{r}\right)+\frac{p^2}{(4R+r)^2}\geq\frac{2}{15}\left(\frac{R-2r}{r}\right)+\frac{16Rr-5r^2}{(4R+r)^2}$$

Es suficiente probar: $\frac{2}{15}\left(\frac{R-2r}{r}\right)+\frac{16Rr-5r^2}{(4R+r)^2}\geq\frac{1}{3}\Leftrightarrow\frac{2R}{15r}+\frac{16Rr-5r^2}{(4R+r)^2}\geq\frac{3}{5}$

$$\Leftrightarrow 2R(4R+r)^2+15r(16Rr-5r^2)\geq 9r(4R+r)^2$$

$$\Leftrightarrow 32R^3+16R^2r+242Rr^2-75r^3\geq 144R^2r+72r^2R+9r^3$$

$$\Leftrightarrow 32R^3-128R^2r+170Rr^2-84r^3=2(R-2r)(16R^2-32Rr+21r^2)=$$

$$=2(R-2r)(16(R-2r)+21r^2)\geq 0\quad \textbf{(LQQD)}$$

379. In $\Delta\ ABC$:

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$

La desigualdad propuesta es equivalente: $\frac{s-b}{s-a}+\frac{s-c}{s-b}+\frac{s-a}{s-c}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$

Como a,b,c ***son la dos de un triángulo*** ABC***, realizamos los siguientes cambios***

$$x=s-a>0, y=s-b>0, z=s-c>0\Leftrightarrow x+y=c, y+z=a, z+x=b$$

$\Leftrightarrow \frac{y}{x}+\frac{z}{y}+\frac{x}{z} \geq \frac{z+y}{z+x}+\frac{x+z}{x+y}+\frac{y+x}{y+z}$. ***Realizamos los siguienes cambios de variables*** $P=\frac{1+\frac{y}{x}}{2}>0, Q=\frac{1+\frac{z}{y}}{2}>0, R=\frac{1+\frac{x}{z}}{2}>0$. ***Aplicando la desigualdad de Holder*** $8PQR=\left(1+\frac{y}{x}\right)\left(1+\frac{z}{y}\right)\left(1+\frac{x}{z}\right) \geq (1+1)^3=8 \Leftrightarrow PQR \geq 1$

Ahora bien en (A)

$$\left(\frac{y}{x}+\frac{z+y}{z+x}\right)+\left(\frac{z}{y}-\frac{x+z}{x+y}\right)+\left(\frac{x}{z}-\frac{y+x}{y+z}\right)=\frac{z(y-x)}{x(z+x)}+\frac{x(z-y)}{y(x+y)}+\frac{y(x-z)}{z(y+z)} \geq 0$$

$$\Leftrightarrow \frac{\frac{y}{x}-1}{1+\frac{x}{z}}+\frac{\frac{z}{y}-1}{1+\frac{y}{x}}+\frac{\frac{x}{z}-1}{1+\frac{z}{y}}=\frac{2(P-1)}{2R}+\frac{2(Q-1)}{2P}+\frac{2(R-1)}{2Q} \geq 0$$

$\Leftrightarrow \frac{P}{R}+\frac{Q}{P}+\frac{R}{Q} \geq \frac{1}{R}+\frac{1}{P}+\frac{1}{Q}$. ***Como*** $PQR \geq 1$, ***donde*** $P, Q, R>0$

Aplicando MA $\geq$ ***MG:*** $\frac{P}{R}+\frac{P}{R}+\frac{Q}{P} \geq 3\sqrt[3]{\frac{PQ}{R^2}}=3\sqrt[3]{\frac{PQR}{R^3}} \geq 3\sqrt[3]{\frac{1}{R^3}}=\frac{3}{R}$ ***(B)***

Análogamente para los siguientes términos $\frac{Q}{P}+\frac{Q}{P}+\frac{R}{Q} \geq \frac{3}{P}$ ***(C)***

$\frac{R}{Q}+\frac{R}{Q}+\frac{P}{R} \geq \frac{3}{Q}$ ***(D). Sumando (B) + (C) + (D)***

$\Rightarrow 3\left(\frac{P}{R}+\frac{Q}{P}+\frac{R}{Q}\right) \geq \frac{3}{R}+\frac{3}{P}+\frac{3}{Q} \Leftrightarrow \frac{P}{R}+\frac{Q}{P}+\frac{R}{Q} \geq \frac{1}{R}+\frac{1}{P}+\frac{1}{Q}$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

In any $\Delta ABC, \frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a} \overset{(1)}{\geq} \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$

(1) $\Leftrightarrow \frac{s-b}{s-a}+\frac{s-c}{s-b}+\frac{s-a}{s-c} \overset{(2)}{\geq} \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$

$\left(\because r_a=\frac{\Delta}{s-a}, etc\right)$. ***Let*** $s-a=x, s-b=y, s-c=z$

Then, $a=y+z, b=z+x, c=x+y$

$\therefore$ ***(2)*** $\Leftrightarrow \frac{y}{x}+\frac{z}{y}+\frac{x}{z} \geq \frac{y+z}{z+x}+\frac{z+x}{x+y}+\frac{x+y}{y+z}$

$$\Leftrightarrow \frac{y^2z+z^2x+x^2y}{xyz} \geq \frac{(y+z)^2(x+y)+(z+x)^2(y+z)+(x+y)^2(z+x)}{(x+y)(y+z)(z+x)}$$

$$\Leftrightarrow (x+y)(y+z)(z+x)(x^2y+y^2z+z^2x) -$$

$$-xyz\{(x+y)^2(z+x)+(y+z)^2(x+y)+(z+x)^2(y+z)\} \geq 0$$

$$\Leftrightarrow (x^4y^2+y^4z^2+z^4x^2)+(x^3y^3+y^3z^3+z^3x^3) \geq 3x^2y^2z^2+x^3y^2z+y^3z^2x+z^3x^2y \quad (3)$$

Let $u, v, w > 0$. ***We have,*** $u^3+v^3+v^3 \overset{A-G}{\geq} 3uv^2$,

$$v^3+w^3+w^3 \overset{A-G}{\geq} 3vw^2, w^3+u^3+u^3 \overset{A-G}{\geq} 3wu^2$$

Adding the above 3, $3\sum u^3 \geq 3\sum uv^2 \Rightarrow \sum u^3 \geq \sum uv^2$

Putting $u = xy, v = yz, w = zx$, ***we get,***

$$\sum x^3y^3 \overset{(4)}{\geq} xy(yz)^2+(yz)(zx)^2+(zx)(xy)^2$$

$= x^3y^2z+y^3z^2x+z^3x^2y$. ***Again,*** $x^4y^2+y^4z^2+z^4x^2 \overset{A-G}{\geq} 3x^2y^2z^2$ ***(5)***

(4) + (5) ⇒ (3) is true (Proved)

380. In ΔABC:

$$\frac{1}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{2n+2}}+\frac{1}{\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)^{2n+2}}+\frac{1}{\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)^{2n+2}} \geq \frac{3^{n+2}}{4^{n+1}}, n \geq 0$$

Proposed by D.M. Bătinețu – Giurgiu and Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un ΔABC: $\frac{1}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{2n+2}}+\frac{1}{\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)^{2n+2}}+\frac{1}{\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)^{2n+2}} \geq \frac{3^{n+2}}{4^{n+1}}, n \geq 0$

IRAN INEQUALITY. Siendo $x, y, z \geq 0$ ***se cumple la siguiente desigualdad***

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right) \geq \frac{9}{4} \quad (A)$$

Realizamos los siguientes cambios de variables

$$x = \tan\frac{A}{2} > 0,\ y = \tan\frac{B}{2} > 0,\ z = \tan\frac{C}{2} > 0,\ xy + yz + zx = 1$$

Por lo tanto tenemos en (A) $\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \geq \frac{9}{4}$

De la desigualdad ponderada de Cauchy

$\left(\frac{1}{a^{n+1}} + \frac{1}{b^{n+1}} + \frac{1}{c^{n+1}}\right) \cdot 3^n \geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{n+1}$***, donde*** $a, b, c > 0, n \geq 0$

Siendo $a = (x+y)^2,\ b = (y+z)^2,\ c = (z+x)^2 \Leftrightarrow x, y, z > 0$

$$\Rightarrow \frac{1}{(x+y)^{2n+2}} + \frac{1}{(y+z)^{2n+2}} + \frac{1}{(z+x)^{2n+2}} \geq \frac{1}{3^n}\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right)^{n+1} \geq$$

$$\geq \frac{1}{3^n} \cdot \left(\frac{9}{4}\right)^{n+1} = \frac{3^{n+2}}{4^{n+1}} \textit{ (LQQD)}$$

Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam

$$\sum \frac{1}{\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)^{2n+2}} \geq \frac{3^{n+2}}{4^{n+1}};\forall n \geq 0$$

We have: $\tan\frac{A}{2} = x; \tan\frac{B}{2} = y; \tan\frac{C}{2} = z \Rightarrow xy + yz + zx = 1$

Chebyshev: $\sum \frac{1}{(x+y)^{2n+2}} \geq \frac{\left(\sum\frac{1}{(x+y)^{2n}}\right)\left(\sum\frac{1}{(x+y)^2}\right)}{3}$ ***(1)***

AM-GM: n ***positive real numbers:***

$$\frac{1}{(x+y)^{2n}} + \frac{1}{\left(\frac{2}{\sqrt{3}}\right)^{2n}} + \cdots + \frac{1}{\left(\frac{2}{\sqrt{3}}\right)^{2n}} \geq n \cdot \sqrt[n]{\frac{1}{(x+y)^{2n} \cdot \left(\frac{2}{\sqrt{3}}\right)^{2n(n-1)}}}$$

$$\Rightarrow \sum \frac{1}{(x+y)^{2n}} + \sum \frac{n-2}{\left(\frac{2}{\sqrt{3}}\right)^{2n}} \geq \frac{n}{(x+y)^2 \cdot \left(\frac{4}{3}\right)^{n-1}}$$

$$\Rightarrow \sum \frac{1}{(x+y)^{2n}} \geq \left(\frac{3}{4}\right)^{n-1} \cdot n \cdot \sum \frac{1}{(x+y)^2} - \frac{3(n-1)}{\left(\frac{2}{\sqrt{3}}\right)^{2n}} \quad \textit{(2)}$$

Because, inequality: $\sum\frac{1}{(x+y)^2}\ge\frac{9}{4\sum xy}=\frac{9}{4}$ ***(2)***

$$\Rightarrow\sum\frac{1}{(x+y)^{2n}}\ge n\cdot\left(\frac{3}{4}\right)^{n-1}\cdot\frac{9}{4}-\frac{3(n-1)}{\left(\frac{2}{\sqrt{3}}\right)^{2n}}\quad (3)$$

$$(1),(3)\Rightarrow\sum\frac{1}{(x+y)^{2n+2}}\ge\sum\frac{1}{(x+y)^{2n+2}}\ge\frac{\left[\frac{n\cdot 3^{n+1}}{4n}-\frac{3^{n+1}}{4^n}(n-1)\right]\cdot\frac{9}{4}}{3}\Rightarrow\sum\frac{1}{(x+y)^{2n+2}}\ge\frac{3^{n+2}}{4^{n+1}}$$

Solution 3 by Sanong Hauerai-Nakon Pathom-Thailand

Because $A+B+C'=\pi, 0<A,B,C<\pi$***, we get***

$$\sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2};\ \cos A+\cos B+\cos C\le\frac{3}{2}$$

$$\tan\frac{A}{2}=\frac{\sin A}{1+\cos A},\tan\frac{B}{2}=\frac{\sin B}{1+\cos B},\tan\frac{C}{2}=\frac{\sin C}{1+\cos C}$$

In $x=\tan\frac{A}{2},y=\tan\frac{B}{2},z=\tan\frac{C}{2}$***, we obtain***

$$(x+y)+(y+z)+(z+x)\le 2\sqrt{3};\ \frac{1}{(x+y)}+\frac{1}{(y+z)}+\frac{1}{(z+x)}\ge\frac{3\sqrt{3}}{2}$$

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\ge\frac{\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)^2}{3}\ge\frac{\frac{2y}{4}}{3}=\frac{9}{4}$$

$$\frac{1}{((x+y)^2)^{(n+1)}}+\frac{1}{((y+z)^2)^{(n+1)}}+\frac{1}{((z+x)^2)^{(n+1)}}\ge\frac{\left(\frac{1}{(x+1)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^{(n+1)}}{3^n}$$

$$\ge\frac{\left(\frac{9}{4}\right)^{(n+1)}}{3^n}=\frac{\frac{9^{(n+1)}}{4^{(n+1)}}}{3^n}=\frac{3^{(2n+2)}\cdot 3^{(-n)}}{4^{(n+1)}}=\frac{3^{n+2}}{4^{(n+1)}}$$

Solution 4 by SK Rejuan-West Bengal-India

$\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^2\le 2\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}\right)$ ***[by Cauchy Inequality]***

$$=2\left\{\frac{(s-b)(s-c)}{s(s-a)}+\frac{(s-a)(s-c)}{s(s-b)}\right\}=2\cdot\frac{2(s-a)(s-b)(s-c)}{s(s-a)(s-b)}$$

$$= 4\left(\frac{s-c}{s}\right) \Rightarrow \left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^2 \le 4\left(\frac{s-c}{s}\right)$$

$$\Rightarrow \left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{2n+2} \le 4^{n+1}\left(\frac{s-c}{s}\right)^{n+1} \Rightarrow \sum\frac{1}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{2n+2}} \ge$$

$$\frac{1}{4^{n+1}}\sum\left(\frac{s}{s-c}\right)^{n+1}$$ ***(1). Let,*** $P = \sum\left(\frac{s}{s-c}\right)^{n+1}$ ***(2)***

$\ge \frac{3}{3^{n+1}}\left(\sum\frac{s}{s-c}\right)^{n+1}$ ***[by m^th power theorem]*** $\ge \frac{3}{3^{n+1}}\left\{\frac{9}{\sum\frac{s-c}{s}}\right\}^{n+1}$ ***[by A.M ≥ H.M]***

$$\Rightarrow P \ge \frac{3}{3^{n+1}}\cdot 3^{2n+2} \quad [\because \sum\frac{s-c}{s} = 1]$$

$$\Rightarrow P \ge 3^{2n+2-n-1+1} = 3^{n+2} \Rightarrow \sum\left(\frac{s}{s-c}\right)^{n+1} \ge 3^{n+2} \quad (3)$$

[From (2) the value of P***]*** $\Rightarrow \frac{1}{4^{n+1}}\sum\left(\frac{s}{s-c}\right)^{n+1} \ge \frac{3^{n+2}}{4^{n+1}}$ ***(4)***

Combining (1) & (4) we get, $\sum\frac{1}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{2n+2}} \ge \frac{3^{n+2}}{4^{n+1}}$ ***[Proved]***

381. In $\Delta\, ABC, G -$ **centroid**

$$\left(\sum\cot^2(\measuredangle GAB)\right)\left(\sum\cot^4(\measuredangle GBC)\right)\left(\sum\cot^8(\measuredangle GCA)\right) \ge 3^{10}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$\left(\sum\cot^2(\measuredangle GAB)\right)\left(\sum\cot^4(\measuredangle GBC)\right)\left(\sum\cot^8(\measuredangle GCA)\right) \ge 3^{10}$$

Siendo $G -$ ***centroide, tener en cuenta las siguientes identidades***

conocidas $\cot(\measuredangle GAB) = \frac{3c^2+b^2-a^2}{4S}, \cot(\measuredangle GBC) = \frac{3a^2+c^2-b^2}{4S},$

$\cot(\sphericalangle GCA) = \frac{3b^2+a^2-c^2}{4S}$*. Lo cual implica*

$$\cot(\sphericalangle GAB) + \cot(\sphericalangle GBC) + \cot(\sphericalangle CGA) = \frac{3(a^2+b^2+c^2)}{4S} \geq \frac{3\cdot 4S\sqrt{3}}{4S} = 3\sqrt{3}$$

(Válido por Weitzenbock inequality)

Aplicando la desigualdad de Cauchy se obtiene que

$$\sum \cot^2(\sphericalangle GAB) \geq \frac{1}{3}\left(\sum \cot(\sphericalangle GAB)\right)^2 \geq \frac{1}{3}\cdot 27 = 3^2 \quad \textit{(A)}$$

$$\sum \cot^4(\sphericalangle GBC) \geq \frac{1}{3}\left(\sum \cot^2(\sphericalangle GAB)\right)^2 \geq \frac{1}{3}\cdot 3^4 = 3^3 \quad \textit{(B)}$$

$$\sum \cot^4(\sphericalangle GCA) \geq \frac{1}{3}\left(\sum \cot^4(\sphericalangle GBC)\right)^2 \geq \frac{1}{3}\cdot 3^6 = 3^5 \quad \textit{(C)}$$

$$\Rightarrow \left(\sum \cot^2(\sphericalangle GAB)\right)\left(\sum \cot^4(\sphericalangle GBC)\right)\left(\sum \cot^8(\sphericalangle GCA)\right) \geq 3^2\cdot 3^3\cdot 3^5 = 3^{10}$$

382. **In $\Delta\, ABC, I -$ incentre, $R_a, R_b, R_c -$ circumradii of $\Delta\, BIC, \Delta\, CIA, \Delta\, AIB$**

$$\frac{1}{R} \leq \frac{R_a}{a^2} + \frac{R_b}{b^2} + \frac{R_c}{c^2} \leq \frac{R}{4r^2}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Daniel Sitaru – Romania

$$R_a = 2R\sin\frac{A}{2}, \frac{1}{R} \leq \sum \frac{R_a}{a^2} \leq \frac{R}{4r^2} \leftrightarrow \frac{1}{2R^2} \leq \sum \frac{\sin\frac{A}{2}}{a^2} \leq \frac{1}{8r^2}$$

$$\sum \frac{\sin\frac{A}{2}}{a^2} \overset{CHEBYSHEV}{\leq} \frac{1}{3}\sum \sin\frac{A}{2}\sum \frac{1}{a^2} \leq \frac{1}{3}\cdot\frac{3}{2}\cdot\frac{1}{4r^2} = \frac{1}{8r^2}$$

$$\sum \frac{\sin\frac{A}{2}}{a^2} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{1}{a^2b^2c^2}\prod \sin\frac{A}{2}} = 3\sqrt[3]{\frac{1}{16R^2S^2}\cdot\frac{r}{4R}} = \frac{3}{4R}\sqrt[3]{\frac{r}{r^2s^2}} \geq \frac{1}{2R^2}$$

$$\leftrightarrow 27R^3 \geq 8rs^2 \textit{ (to prove)}$$

$$8rs^2 \overset{MITRINOVIC}{\leq} 8r\cdot\frac{27R^2}{4} \overset{EULER}{\leq} 4R\cdot\frac{27R^2}{4} = 27R^3$$

383. In $\Delta\ ABC, K$ – Lemoine's point

$$\frac{AK}{b^2+c^2}+\frac{BK}{c^2+a^2}+\frac{CK}{a^2+b^2}\le\frac{m_a+m_b+m_c}{a^2+b^2+c^2}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un $\Delta\ ABC$: $\frac{KA}{b^2+c^2}+\frac{KB}{c^2+a^2}+\frac{KC}{a^2+b^2}\le\frac{m_a+m_b+m_c}{a^2+b^2+c^2}$

Recordar la siguientes identidades en: K – Lemoine's Point

$$KA=\frac{bc\sqrt{2b^2+2c^2-a^2}}{a^2+b^2+c^2}, KB=\frac{ca\sqrt{2c^2+2a^2-b^2}}{a^2+b^2+c^2}, KC=\frac{ab\sqrt{2a^2+2b^2-c^2}}{a^2+b^2+c^2}$$

$$\Leftrightarrow KA=\frac{2m_abc}{a^2+b^2+c^2}, KB=\frac{2m_bca}{a^2+b^2+c^2}, KC=\frac{2m_cab}{a^2+b^2+c^2}$$

La desigualdad propuesta es equivalente

$$\frac{KA}{b^2+c^2}+\frac{KB}{c^2+a^2}+\frac{KC}{a^2+b^2}\le\frac{m_a+m_b+m_c}{a^2+b^2+c^2}$$

$$\frac{2m_abc}{(a^2+b^2+c^2)(b^2+c^2)}+\frac{2m_bca}{(a^2+b^2+c^2)(c^2+a^2)}+\frac{2m_cab}{(a^2+b^2+c^2)(a^2+b^2)}\le$$

$\le\frac{m_a+m_b+m_c}{a^2+b^2+c^2}$***. La cual es evidente ya que*** $\frac{2bc}{b^2+c^2}\le 1, \frac{2ca}{c^2+a^2}\le 1, \frac{2ab}{a^2+b^2}\le 1$

384. In ΔABC:

$$\frac{R}{r}\ge 1+\frac{r_a^2+r_b^2+r_c^2}{r_ar_b+r_br_c+r_cr_a}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{R}{r}\ge 1+\frac{r_a^2+r_b^2+r_c^2}{r_ar_b+r_br_c+r_cr_a}$

Realizamos los siguientes cambios de variables

$x = r_a > 0, y = r_b > 0, z = r_c > 0$. ***Se verifica lo siguiente***

$$(x+y)(y+z)(z+x) = 4p^2R, xyz = Sp = p^2r$$

La desigualdad propuesta es equivalente $\frac{(x+y)(y+z)(z+x)}{4xyz} \geq 1 + \frac{x^2+y^2+z^2}{xy+yz+zx}$

Ahora bien $\frac{(x+y)(y+z)(z+x)}{4xyz} = \frac{x+y}{4z} + \frac{y+z}{4x} + \frac{z+x}{4y} + \frac{1}{2} =$

$= \frac{1}{4}\left(\frac{x}{y}+\frac{y}{x}\right) + \frac{1}{4}\left(\frac{y}{z}+\frac{z}{y}\right) + \frac{1}{4}\left(\frac{z}{x}+\frac{x}{z}\right) + \frac{1}{2}$. ***Aplicando la desigualdad de Cauchy*** $\frac{(x+y)(y+z)(z+x)}{4xyz} = \frac{x^2+y^2}{4xy} + \frac{y^2+z^2}{4yz} + \frac{z^2+x^2}{4zx} + \frac{1}{2} \geq$

$$\geq \frac{\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2}{4(xy+yz+zx)} + \frac{1}{2} =$$

$$= \frac{2(x^2+y^2+z^2) + 2\sum\left(\sqrt{x^2+y^2}\right)\left(\sqrt{x^2+z^2}\right)}{4(xy+yz+zx)} + \frac{1}{2} \geq$$

$$\geq \frac{2\sum x^2 + 2\sum(x^2+yz)}{4(xy+yz+zx)} + \frac{1}{2} = \frac{4\sum x^2 + 2\sum xy}{4\sum xy} + \frac{1}{2} = 1 + \frac{x^2+y^2+z^2}{xy+yz+zx} \quad \textit{(LQQD)}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$r_a = \frac{\Delta}{p-a}, r_b = \frac{\Delta}{p-b}, r_c = \frac{\Delta}{p-c}$ ***and*** $\sum_{cyc}(p-a)(p-b) = r(r+4R)$

$$\frac{R}{r} \geq 1 + \frac{r_a^2+r_b^2+r_c^2}{r_ar_b+r_br_c+r_cr_a} \Leftrightarrow \frac{R}{r} + 1 \geq \frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}$$

$$\Leftrightarrow \frac{R}{r} + 1 \geq \frac{\left(\sum_{cyc}\frac{\Delta}{p-a}\right)^2}{\sum_{cyc}\frac{\Delta^2}{(p-a)(p-b)}} \Leftrightarrow \frac{r+R}{r} \geq \frac{\left(\sum_{cyc}(p-a)(p-b)\right)^2}{p(p-a)(p-b)(p-c)}$$

$\Leftrightarrow \frac{R+r}{r} \geq \frac{r^2(r+4R)^2}{\Delta^2} \Leftrightarrow p^2 \geq \frac{r(r+4R)^2}{r+r}$***; we know,*** $p^2 \geq 16Rr - 5r^2$***, we need to prove,*** $16Rr - 5r^2 \geq \frac{r(r+4R)^2}{r+R} \Leftrightarrow (16R-5r)(r+R) \geq (r+4R)^2$

$\Leftrightarrow 3r(R-2r) \geq 0$***, which is true, Hence proved***

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385. In $\Delta\ ABC$:

$$\left(\sum a^2\right)\left(\sum a^4\right)\left(\sum a^8\right) \ge \frac{4^7}{81}(R+r)^{14}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un Δ acutángulo ABC: $(\sum a^2)(\sum a^4)(\sum a^8) \ge \frac{4^7}{81}(R+r)^2$

LEMMA → WALKER INEQUALITY

En un triángulo acutángulo ABC se cumple la siguiente desigualdad

$$a^2 + b^2 + c^2 \ge 4(R+r)^2$$

SOLUCIÓN

$$\Leftrightarrow \sin^2 A + \sin^2 B + \sin^2 C \ge \left(1 + \frac{r}{R}\right)^2 = (\cos A + \cos B + \cos C)^2$$

Recordar las siguientes identidades en un ΔABC

$$\sin^2 A + \sin^2 B + \sin^2 C = 2\sin B \sin C \cos A + 2\sin C \sin A \cos B + 2\sin A \sin B \cos C$$

$$\cot A \cot B + \cot C \cot A + \cot A \cot B = 1$$

$$\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = -\frac{\cos(B+C)}{\sin B \sin C} - \frac{\cos(C+A)}{\sin B \sin C} - \frac{\cos(A+B)}{\sin A \sin B}$$

$$\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = -\cot B \cot C - \cot C \cot A - \cot A \cot B + 3 = -1 + 3 = 2$$

Aplicando la desigualdad de Cauchy

$$(2\sin B \sin C \cos A + 2\sin C \sin A \cos B + 2\sin A \sin B \cos C)\left(\frac{\cos A}{2\sin B \sin C} + \frac{\cos B}{2\sin C \sin A} + \frac{\cos C}{2\sin A \sin B}\right) \ge$$
$$\ge (\cos A + \cos B + \cos C)^2$$

$$\Leftrightarrow (\sin^2 A + \sin^2 B + \sin^2 C) \cdot 1 \ge (\cos A + \cos B + \cos C)^2$$

$$\Leftrightarrow \sin^2 A + \sin^2 B + \sin^2 C \ge (\cos A + \cos B + \cos C)^2 \ \textbf{(LQQD)}$$

En la desigualdad propuesta. Aplicamos la desigualdad de Cauchy

$$\sum a^2 \ge 4(R+r)^2 \quad \textbf{(A)}; \sum a^4 \ge \frac{\left(\sum a^2\right)^2}{3} \ge \frac{4^2(R+r)^4}{3} \quad \textbf{(B)}$$

$$\sum a^8 \geq \frac{\left(\sum a^4\right)^2}{3} \geq \frac{\frac{4^4(R+r)^8}{3^2}}{3} = \frac{4^4(R+r)^8}{3^3}$$ ***(C). Multiplicando (A) · (B) · (C)***

$$\left(\sum a^2\right)\left(\sum a^4\right)\left(\sum a^8\right) \geq 4(R+r)^2 \cdot \frac{4^2(R+r)^4}{3} \cdot \frac{4^4(R+r)^8}{3^3} = \frac{4^7}{81}(R+r)^2$$ ***(LQQD)***

386. In $\Delta\ ABC$:

$$\prod (h_a + h_b)^4 \leq 512 \prod \left(m_a^4 + m_b^4\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

Lemma 1: $m_a \geq w_a \geq h_a$

Lemma 2: $2^{n-1}(a^n + b^n) \geq (a+b)^n$

Then: $2^3\left(m_a^4 + m_b^4\right) \geq (m_a + m_b)^4;\ 2^9 \prod\left(m_a^4 + m_b^4\right) \geq \prod (m_a + m_b)^4$

completely proved

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$\Delta = \frac{a \cdot h_a}{2} = \frac{b \cdot h_b}{2} = \frac{c \cdot h_c}{2}$; $m_a \geq \frac{b^2+c^2}{2R}, m_b \geq \frac{c^2+a^2}{2R}, m_c \geq \frac{a^2+b^2}{2R}$ **and** $abc = 4R\Delta$

$$512 \prod_{cyc}\left(m_a^4 + m_b^4\right) = \prod_{cyc}\left\{8\left(m_a^4 + m_b^4\right)\right\} \geq \prod_{cyc}(m_a + m_b)^4$$

$\left[\because \frac{x^4+y^4}{2} \geq \left(\frac{(x+y)}{2}\right)^4\right]$***. We need to prove,***

$$\prod_{cyc}(m_a + m_b) \geq \prod_{cyc}(h_a + h_b)$$

$$\therefore \prod_{cyc}(m_a + m_b) \geq \frac{1}{R^3} \prod_{cyc}\left(\frac{(b^2 + c^2) + (c^2 + a^2)}{2}\right)$$

$$\overset{AM\geq GM}{\geq}\frac{1}{R^3}\prod_{cyc}(a^2+b^2)\geq\frac{1}{8R^3}\prod_{cyc}(a+b)^2\geq\frac{abc}{R^3}\prod_{cyc}(a+b)$$

$$=\frac{64\Delta^3}{(abc)^2}\prod_{cyc}(a+b)=8\prod_{cyc}\left(\frac{2\Delta}{a}+\frac{2\Delta}{b}\right)>\prod_{cyc}(h_a+h_b)$$

$$\therefore 512\prod_{cyc}(m_a^4+m_b^4)\geq\prod_{cyc}(h_a+h_b)^4$$

387. In $\Delta\ ABC$:

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3\geq\frac{27(a+b)(b+c)(c+a)}{8abc}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3\geq\frac{27(a+b)(b+c)(c+a)}{8abc}$

Tener en cuenta las siguientes identidades y desigualades previas en un $\Delta\ ABC$ $h_a=\frac{bc}{2R},\ h_b=\frac{ca}{2R},\ h_c=\frac{ab}{2R}$; $m_a\geq\frac{b^2+c^2}{4R},\ m_b\geq\frac{c^2+a^2}{4R},\ m_c\geq\frac{a^2+b^2}{4R}$.

Aplicando en la desigualdad propuesta

$$\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3\geq\left(\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)+\frac{1}{2}\left(\frac{c}{a}+\frac{a}{c}\right)+\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)\right)^3=$$

$$=\frac{1}{8}\left(\frac{a}{b}\left(1+\frac{b}{c}\right)+\frac{b}{c}\left(1+\frac{c}{a}\right)+\frac{c}{a}\left(1+\frac{a}{b}\right)\right)^3$$ ***. Como*** $a,b,c>0$

Utilizando MA ≥ MG

$$\frac{1}{8}\left(\frac{a}{b}\left(1+\frac{b}{c}\right)+\frac{b}{c}\left(1+\frac{c}{a}\right)+\frac{c}{a}\left(1+\frac{a}{b}\right)\right)^3\geq\frac{27}{8}\cdot\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\left(1+\frac{a}{b}\right)=$$

$= \frac{27(a+b)(b+c)(c+a)}{8abc}$. ***Pro transitividad:*** $\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)^3 \geq \frac{27(a+b)(b+c)(c+a)}{8abc}$

(LQQD)

388. In $\Delta\, ABC$:

$$\left(\frac{h_a}{h_b}+\frac{h_b}{h_c}+\frac{h_c}{h_a}\right)^2+\frac{2r}{R}\geq 10$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\left(\frac{h_a}{h_b}+\frac{h_b}{h_c}+\frac{h_c}{h_a}\right)^2+\frac{2r}{R}\geq 10$

LEMMA

1) Siendo $x, y, z > 0$ ***se cumple***

$$\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^2 \geq (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

2) Siendo a, b, c ***los lados de un triángulo se cumple***

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{2r}{R}\geq 10$. ***La desigualdad propuesta es equivalente*** $\left(\frac{b}{c}+\frac{c}{b}+\frac{a}{c}\right)^2+\frac{2r}{R}\geq 10$. ***Aplicando*** $1 \wedge 2$ ***se obtiene***

$$\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)^2+\frac{2r}{R}\geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{2r}{R}\geq 10 \quad \textit{(LQQD)}$$

389. In $\Delta\, ABC$:

$$1 \leq \frac{\cot A+\cot B+\cot C}{\sqrt{4-\frac{2r}{R}}} \leq \frac{R^2}{4r^2}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$\sqrt{4-\frac{2r}{R}} \le \cot A + \cot B + \cot C \le \frac{R^2}{4r^2}\cdot\sqrt{4-\frac{2r}{R}}$$

Recordar las siguientes identidades y desigualdades conocidas en un

$$\Delta\, ABC.\ \cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4S}$$

$$S = pr,\quad p \ge 3\sqrt{3}r,\ \ R \ge 2r,\quad a^2+b^2+c^2 \le 9R^2$$

$$\tan\frac{A}{2} = \frac{(s-b)(s-c)}{S} = \frac{\big(a-(b-c)\big)\big(a+(b-c)\big)}{4S} = \frac{a^2-(b-c)^2}{4S} \le \frac{a^2}{4S}$$

$$\tan\frac{B}{2} \le \frac{b^2}{4S},\ \tan\frac{C}{2} \le \frac{c^2}{4S}$$

$$\Rightarrow \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \le \frac{a^2+b^2+c^2}{4S} = \cot A + \cot B + \cot C,$$

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \ge \sqrt{4-\frac{2r}{R}}$$

En el LHS: $\cot A + \cot B + \cot C \ge \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \ge \sqrt{4-\frac{2r}{R}}$

En el RHS: $\cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4S} \le \frac{9R^2}{4r\cdot 3\sqrt{3}r} = \frac{R^2}{4r^2}\cdot\sqrt{3} \le \frac{R^2}{4r^2}\sqrt{4-\frac{2r}{R}}$

390. In ΔABC:

$$\frac{R}{r} \ge 1 + \frac{r_a^2+r_b^2+r_c^2}{r_ar_b+r_br_c+r_cr_a}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{R}{r} \geq 1 + \frac{r_a^2+r_b^2+r_c^2}{r_ar_b+r_br_c+r_cr_a}$

Realizamos los siguientes cambios de variables

$$x = r_a > 0, \quad y = r_b > 0, \quad z = r_c > 0$$

Se verifica lo siguiente $(x+y)(y+z)(z+x) = 4p^2R$, $xyz = Sp = p^2r$

La desigualdad propuesta es equivalente: $\frac{(x+y)(y+z)(z+x)}{4xyz} \geq 1 + \frac{x^2+y^2+z^2}{xy+yz+zx}$

Ahora bien: $\frac{(x+y)(y+z)(z+x)}{4xyz} = \frac{x+y}{4z} + \frac{y+z}{4x} + \frac{z+x}{4y} + \frac{1}{2} =$

$= \frac{1}{4}\left(\frac{x}{y}+\frac{y}{x}\right) + \frac{1}{4}\left(\frac{y}{z}+\frac{z}{y}\right) + \frac{1}{4}\left(\frac{z}{x}+\frac{x}{z}\right) + \frac{1}{2}$. ***Aplicando la desigualdad de Cauchy*** $\frac{(x+y)(y+z)(z+x)}{4xyz} = \frac{x^2+y^2}{4xy} + \frac{y^2+z^2}{4yz} + \frac{z^2+x^2}{4zx} + \frac{1}{2} \geq$

$$\geq \frac{\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2}{4(xy+yz+zx)} + \frac{1}{2} =$$

$$= \frac{2(x^2+y^2+z^2) + 2\sum\left(\sqrt{x^2+y^2}\right)\left(\sqrt{x^2+y^2}\right)}{4(xy+yz+zx)} + \frac{1}{2} \geq$$

$$\geq \frac{2\sum x^2 + 2\sum(x^2+yz)}{4(xy+yz+zx)} + \frac{1}{2} = \frac{4\sum x^2 + 2\sum xy}{4\sum xy} + \frac{1}{2} = 1 + \frac{x^2+y^2+z^2}{xy+yz+zx}$$

Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam

$$1 + \frac{r_a^2+r_b^2+r_c^2}{\sum r_a \cdot r_b} = 1 + \frac{(\sum r_a)^2 - 2\cdot\sum r_a\cdot r_b}{\sum r_a \cdot r_b} =$$

$$= 1 + \frac{(\sum r_a)^2}{\sum r_a \cdot r_b} - 2 = \frac{(\sum r_a)^2}{\sum r_a \cdot r_b} - 1$$

1) $r_a + r_b + r_c = 4R + r$

2) $\sum r_a \cdot r_b = p^2$

$$\frac{R}{r}+1\geq\frac{(\sum r_a)^2}{\sum r_a\cdot r_b}\Leftrightarrow\frac{R}{r}+1\geq\frac{(4R+r)^2}{p^2}\Leftrightarrow$$

$$\Leftrightarrow p^2\cdot\frac{R+r}{2}\geq(4R+r)^2\Rightarrow p^2\geq 16Rr-5r^2$$ ***(GERRETSEN)***

$$(16Rr-5r^2)\cdot\frac{R+r}{2}\geq(4R+r)^2;\ (16R-5r)\cdot(R+r)\geq(4R+r)^2$$

$$16R^2-5Rr+16Rr-5r^2\geq 16R^2+8Rr+r^2;\ 3Rr\geq 6r^2;\ R\geq 2r$$

Solution 3 by Mehmet Șahin – Ankara – Turkey

$r_a=\frac{\Delta}{s-a}$, $\frac{r_a^2}{r_a\cdot r_b+r_br_c+r_cr_a}=\frac{1}{(s-a)^2}\cdot r^2$. ***Let us define the following function.*** $f(x)=\frac{1}{(s-x)^2}$ ***is convex in*** $(0,s)$. ***Using the Jensen inequality then*** $f\left(\frac{a+b+c}{3}\right)\geq\frac{1}{3}[f(a)+f(b)+f(c)]$

$$\Rightarrow f(a)+f(b)+f(c)\leq 3\cdot f\left(\frac{a+b+c}{3}\right)$$

$$\frac{r_a^2+r_b^2+r_c^2}{r_ar_b+r_br_c+r_cr_a}\leq 3r^2\cdot\frac{1}{\left(\frac{a+b+c}{2}-\frac{a+b+c}{3}\right)^2}\leq\frac{27r^2}{s^2}$$

$$\frac{r_a^2+r_b^2+r_c^2}{r_a\cdot r_b+r_br_c+r_cr_a}+1\leq\frac{27r^2+s^2}{s^2}\leq\frac{R}{r}\Leftrightarrow 27r^3+s^2\cdot r\leq s^2\cdot R$$

$$\Leftrightarrow 27r^3+27r^3\leq\frac{27}{4}R^2\cdot R\Leftrightarrow 54\cdot r^3\leq\frac{27}{4}\cdot R^3\Leftrightarrow R-2r\geq 0$$ ***(Euler)***

391. In $\Delta\ ABC$:

$$\sum\frac{(a\sqrt{a}+b\sqrt{b})\sin C}{\sqrt{a}+\sqrt{b}}\leq\sum\frac{(a^3\sqrt{a}+b^3\sqrt{b})\sin C}{a^2\sqrt{a}+b^2\sqrt{b}}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

En un triángulo ABC. Probar que:

$$\sum\frac{(a\sqrt{a}+b\sqrt{b})\sin C}{\sqrt{a}+\sqrt{b}}\le\sum\frac{(a^3\sqrt{a}+b^3\sqrt{b})\sin C}{a^2\sqrt{a}+b^2\sqrt{b}}\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{(a^3\sqrt{a}+b^3\sqrt{b})\sin C}{a^2\sqrt{a}+b^2\sqrt{b}}-\sum\frac{(a\sqrt{a}+b\sqrt{b})\sin C}{\sqrt{a}+\sqrt{b}}\ge 0$$

Realizamos los siguientes cambios de variables:

$\sqrt{a}=x^2,\sqrt{b}=y^2,\sqrt{c}=z^2\Leftrightarrow a,b,c>0$***, por la tanto*** $x,y,z>0$

$$\Rightarrow\frac{(a\sqrt{a}+b\sqrt{b})\sin C}{\sqrt{a}+\sqrt{b}}\le\frac{(a^3\sqrt{a}+b^3\sqrt{b})\sin C}{a^2\sqrt{a}+b^2\sqrt{b}}$$

$$\Rightarrow\sin C\left(\frac{x^7+y^7}{x^5+y^5}-\frac{x^3+y^3}{x+y}\right)\ge 0\to\sin C\left(\frac{(x^7+y^7)(x+y)-(x^3+y^3)(x^5+y^5)}{(x+y)(x^5+y^5)}\right)\ge 0$$

$$\Rightarrow\sin C\frac{xy(x^6+y^6)-x^3y^3(x^2+y^2)}{(x+y)(x^5+y^5)}\ge 0\to\sin C\frac{xy[x^6+y^6-x^2y^2(x^2+y^2)]}{(x+y)(x^5+y^5)}\ge 0$$

$$\Rightarrow\sin C\frac{xy\left[(x^2+y^2)^3-4x^2y^2(x^2+y^2)\right]}{(x+y)(x^5+y^5)}\to\sin C\frac{xy(x^2+y^2)\left[(x^2+y^2)^2-4x^2y^2\right]}{(x+y)(x^5+y^5)}\ge 0$$

$\Rightarrow\sin C\dfrac{xy(x^2-y^2)^2}{(x+y)(x^5+y^5)}\ge 0$ ***(A), Por la tanto:*** $\sin A\dfrac{yz(y^2-z^2)^2}{(y+z)(y^5+z^5)}\ge 0$ ***(B)***

$\Rightarrow\sin B\dfrac{xz(z^2-x^2)^2}{(x+z)(x^5+z^5)}\ge 0$ ***(C)*** $\to$ ***Sumando: (A) + (B) + (C):***

$$\sum\frac{(a^3\sqrt{a}+b^3\sqrt{b})\sin C}{a^2\sqrt{a}+b^2\sqrt{b}}-\sum\frac{(a\sqrt{a}+b\sqrt{b})\sin C}{\sqrt{a}+\sqrt{b}}\ge 0$$

392. In $\Delta\ ABC$:

$$\sum(b+c-a)^2\cdot\sum(b+c-a)^3\ge 2592\sqrt{3}r^5$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

Lemma 1: $x^2 + y^2 + z^2 \geq \frac{(x+y+z)^2}{3}$

Lemma 2: $x^3 + y^3 + z^3 \geq \frac{(x+y+z)^3}{9}$

$$\sum(2S-2a)^2 \cdot \sum(2S-2a)^3 \geq 2592\sqrt{3}r^5$$

$$LHS \geq \frac{3S^2}{3} \cdot \frac{8S^3}{9} = \frac{4^5S^5}{3^3} \overset{S\geq 3\sqrt{3}r}{\geq} \frac{2^5 \cdot 3^7\sqrt{3}r^5}{3^3} = 2592\sqrt{3}r^5$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

We know, $\frac{1}{3}\sum_{cyc} x^2 \geq \left(\frac{x+y+z}{3}\right)^2$ **and** $\frac{1}{3}\sum_{cyc} x^3 \geq \left(\frac{x+y+z}{3}\right)^3$**. So,**

$$\sum_{cyc}(b+c-a)^2 \cdot \sum_{cyc}(b+c-a)^3$$

$$\geq \frac{1}{3}(a+b+c)^2 \cdot \frac{1}{9}(a+b+c)^3 = \frac{(a+b+c)^5}{27} \geq \frac{\left(6\sqrt{3}r\right)^5}{27} = 2592\sqrt{3}r^6$$

393. In $\Delta\, ABC$:

$$\frac{3\left(r_a^2 + r_b^2 + r_c^2\right)}{(r_a + r_b + r_c)^2} \geq 1 + \frac{R-2r}{2R-r}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

In $\Delta\, ABC$. Prove that: $\frac{3\left(r_a^2+r_b^2+r_c^2\right)}{(r_a+r_b+r_c)^2} \geq 1 + \frac{R-2r}{2R-r}$

$$\Leftrightarrow \frac{3\sum\left(\frac{2S}{b+c-a}\right)^2}{\left(\sum\frac{2S}{b+c-a}\right)^2} \geq \frac{3(R-r)}{2R-r} \Leftrightarrow 3 \cdot \frac{\sum\left(\frac{1}{b+c-a}\right)^2}{\left(\sum\frac{1}{b+c-a}\right)^2} \geq \frac{3 \cdot \left(\frac{abc}{4S} - \frac{2S}{2a}\right)}{2 \cdot \frac{abc}{4S} - 2\frac{S}{\sum a}}$$

$$\Leftrightarrow 3\frac{\sum\left(\frac{1}{b+c-a}\right)^2}{\left(\sum\frac{1}{b+c-a}\right)^2} \geq \frac{3}{2}\cdot\left[\frac{abc(\sum a)-8S^2}{abc(\sum a)-4S^2}\right]$$

$$\Leftrightarrow 3\cdot\frac{\sum\left(\frac{1}{b+c-a}\right)^2}{\left(\sum\frac{1}{b+c-a}\right)^2} \geq \frac{3(2abc-\prod(b+c-a))}{4abc-\prod(b+c-a)} \quad \textbf{(1)}; \begin{cases} b+c-a=2x \\ c+a-b=2y \\ a+b-c=2z \end{cases} \Leftrightarrow \begin{cases} a=y+z \\ b=z+x \\ c=x+y \end{cases}$$

$$\textbf{(1)} \Leftrightarrow 3\frac{\sum\left(\frac{1}{x}\right)^2}{\left(\sum\frac{1}{x}\right)^2} \geq \frac{3[2\prod(x+y)-8xyz]}{4\prod(x+y)-8xyz} \Leftrightarrow \frac{3\sum x^2y^2}{(\sum xy)^2} \geq \frac{3}{2}\cdot\left[\frac{\prod(x+y)-4xyz}{\prod(x+y)-2xyz}\right]$$

$$\Leftrightarrow 3\frac{\sum x^2y^2}{(\sum xy)^2}-1 \geq \frac{3}{2}\cdot\left[\frac{\prod(x+y)-4xyz}{\prod(x+y)-2xyz}-\frac{2}{3}\right]$$

$$\Leftrightarrow \frac{\sum x^2(y-z)^2}{(\sum xy)^2} \geq \frac{\sum x(y-z)^2}{2[\sum xy(x+y)]} \Leftrightarrow \sum(x-y)^2\cdot\left[\frac{2z^2}{(\sum xy)^2}-\frac{z}{\sum xy(x+y)}\right] \geq 0$$

$$\Leftrightarrow \sum(x+y)^2\left(y^2z^2+x^2z^2+2z^3(x+y)-2xyz^2-x^2y^2\right) \geq 0$$

$$\begin{cases} S_a = x^2y^2+x^2z^2+2x^3(y+z)-2x^2yz-y^2z^2 \\ S_b = x^2y^2+y^2z^2+2y^3(x+z)-2y^2xz-x^2z^2 \\ S_c = x^2z^2+y^2z^2+2z^3(x+y)-2xyz^2-x^2y^2 \end{cases}$$

Suppose: $x \geq y \geq z \Rightarrow S_a > 0$

$$S_b = 2y^2x(y-z)+x^2(y^2-z^2)+y^2z^2+2y^3z > 0 \ (x \geq y \geq z)$$

$$\Rightarrow S_a+S_b > 0$$

$$S_b+S_c = 2y^2z^2+2y^3(x+z)+2z^3(x+y)-2y^2xz-2xyz^2$$

$$= 2y^2z^2+2x(y^3+z^3)+2yz(y^2+z^2)-2xyz(y+z)$$

$$\geq 2y^2z^2+2xyz(y+z)+2yz(y^2+z^2)-2xyz(y+z)$$

$$= 2y^2z^2+2yz(y^2+z^2) > 0 \Rightarrow S_b+S_c > 0$$

We have $S_b > 0; S_b+S_a > 0; S_b+S_c > 0 \Rightarrow$ ***By*** $SOS \Rightarrow Q.E.D.$

Solution 2 by Mehmet Sahin-Ankara-Turkey

$$r_a + r_b + r_c = r + 4R$$

$$r_a^2 + r_b^2 + r_c^2 = (r + 4R)^2 - 2s^2$$

$$\frac{3 \cdot (r_a^2 + r_b^2 + r_c^2)}{(r_a + r_b + r_c)} = \frac{3 \cdot [(r + 4R)^2 - 2s^2]}{(r + 4R)^2} \geq 1 + \frac{R - 2r}{2R - r}$$

$$\Leftrightarrow 3\left[1 - \frac{2 \cdot s^2}{(r + 4R)^2}\right] \geq \frac{3(R - r)}{2R - r}$$

$$\Leftrightarrow 1 - \frac{2s^2}{(r + 4R)^2} \geq \frac{R - r}{2R - r} \Leftrightarrow 2s^2(2R - 1) \leq R(r + 4R)^2$$

$$\Leftrightarrow s^2 \leq \frac{R(r+4R)^2}{(4R-2r)} \quad \dots \textit{(1)}$$

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2 \quad \textit{(Gerretsen Inequality)} \quad \textit{(2)}$$

From (1) and (2): $16Rr - 5r^2 \leq s^2 \leq \frac{R(r+4R)^2}{4R-2r}$

$$\Leftrightarrow (16Rr - 5r^2) \cdot (4R \cdot 2r) \leq R(r + 4R)^2$$

$$\Leftrightarrow 16R^3 - 56R^2r + 53Rr^2 - 10r^3 \geq 0 \Leftrightarrow (R - 2r)(4R - r)(4R - 5r) \geq 0$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{3(r_a^2 + r_b^2 + r_c^2)}{(r_a + r_b + r_c)^2} \overset{(1)}{\geq} 1 + \frac{R - 2r}{2R - r}$$

$$\textit{(1)} \Leftrightarrow \frac{3\{(4R+r)^2 - 2s^2\}}{(4R+r)^2} \geq \frac{3(R-r)}{2R-r}$$

$$\Leftrightarrow (2R - r)(4R + r)^2 - 2(2R - r)s^2 \geq (R - r)(4R + r)^2$$

$$\Leftrightarrow R(4R + r)^2 \geq 2(2R - r)s^2 \Leftrightarrow 2(2R - r)s^2 \leq R(4R + r)^2$$

Rouche $\Rightarrow (4R - 2r)s^2 \leq$

$$(2R^2 + 10Rr - r^2)(4R - 2r) + 4(R - 2r)(2R - r) \cdot \sqrt{R^2 - 2Rr} \leq$$

$$\overset{?}{\leq} R(4R + r)^2$$

$$\Leftrightarrow (R-2r)(8R^2-12Rr+r^2) \overset{?}{\geq} 4(R-2r)(2R-r)\cdot\sqrt{R^2-2Rr}$$

$$\Leftrightarrow 8R^2-12Rr+r^2 \overset{?}{\geq} 4(2R-r)\sqrt{R^2-2Rr}\ (\because R-2r\geq 0\ (Euler))$$

$$\Leftrightarrow (8R^2-12Rr+r^2)^2 \overset{?}{\geq} 16(2R-r)^2(R^2-2Rr)$$

$$\Leftrightarrow 16R^2r^2+8Rr^3+r^4 \overset{?}{\geq} 0 \rightarrow \textit{ true (Proved)}$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

LEMMA: In any triangle ABC,

$$\frac{4r(12R^2-11Rr+r^2)}{3R-2r} \leq p^2 \leq \frac{R(r+4r)^2}{2(2R-r)} \leq 4R^2+4Rr+3r^2$$

… …

now, $r_a = \frac{\Delta}{p-a}, r_b = \frac{\Delta}{p-b}$ ***and*** $r_c = \frac{\Delta}{p-c}$***, then***

$$\sum_{cyc}(p-a)^2(p-b)^2 = \left(\sum_{cyc}(p-a)(p-b)\right)^2 - 2\prod_{cyc}(p-a)\left(\sum_{cyc}(p-a)\right)$$

$$= r^2(r+4R)^2 - 2p^2r^2\left[\sum_{cyc}(p-a)(p-b) = r(r+4R)\ and\ \prod_{cyc}(p-a) = pr^2\right]$$

$$\therefore \frac{3\left(\sum_{cyc} r_a^2\right)}{\left(\sum_{cyc} r_a\right)^2} \geq 1 + \frac{R-2r}{2R-r} = \frac{3(R-r)}{2R-r}$$

$$\Leftrightarrow \frac{\Delta^2 \sum_{cyc}\frac{1}{(p-a)^2}}{\Delta^2\left(\sum_{cyc}\frac{1}{p-a}\right)^2} \geq \frac{R-r}{2R-r} \Leftrightarrow \frac{\sum_{cyc}(p-a)^2(p-b)^2}{\left(\sum_{cyc}(p-a)(p-b)\right)^2} \geq \frac{R-r}{2R-r}$$

$$\Leftrightarrow \frac{r^2(r+4R)^2-2p^2r^2}{r^2(r+4R)^2} \geq \frac{R-r}{2R-r} \Leftrightarrow \frac{R}{2R-r} \geq \frac{2p^2}{(r+4R)^2}$$

$\Leftrightarrow \frac{R(r+4R)^2}{2(2R-r)} \geq p^2$***, which is true. Hence proved***

394. In $\Delta\, ABC$:

$$\sqrt[3]{\left(\frac{a^3+b^3+c^3}{3}\right)^2}\cdot\sqrt[5]{\left(\frac{a^5+b^5+c^5}{3}\right)^2}\cdot\sqrt[7]{\left(\frac{a^7+b^7+c^7}{3}\right)^2}\ge\left(\frac{4\sqrt{3}S}{3}\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nirapada Pal-Jhargram-India

$$\sqrt[3]{\left(\frac{\sum a^3}{3}\right)^2}\cdot\sqrt[5]{\left(\frac{\sum a^5}{3}\right)^2}\cdot\sqrt[7]{\left(\frac{\sum a^7}{3}\right)^2}\ge$$

$$\ge\sqrt[3]{\left(\frac{\sum a}{3}\right)^6}\cdot\sqrt[5]{\left(\frac{\sum a}{7}\right)^{10}}\cdot\sqrt[7]{\left(\frac{\sum a}{3}\right)^{14}}\ge\left(\frac{\sum a}{3}\right)^6$$

$$=\frac{(\sum a^2+2\sum ab)^3}{3^6}\ge\left(\frac{\sum ab}{3}\right)^3\ge\left(\frac{4\sqrt{3}S}{3}\right)^3$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\prod\left\{\begin{array}{l}\sqrt[3]{\left(\frac{a^3+b^3+c^3}{3}\right)^2}\overset{AM\ge GM}{\ge}(abc)^{\frac{2}{3}}\\ \sqrt[5]{\left(\frac{a^5+b^5+c^5}{3}\right)^2}\ge(abc)^{\frac{2}{3}}\\ \sqrt[7]{\left(\frac{a^7+b^7+c^7}{3}\right)^2}\ge(abc)^{\frac{2}{3}}\end{array}\right\}\prod$$

$$\underbrace{\qquad}_{\prod\sqrt[3]{}\cdot\prod\sqrt[5]{}\cdot\prod\sqrt[7]{}\ge(abc)^2\quad(*)}$$

$$\sin A\cdot\sin B\cdot\sin C\le\frac{3\sqrt{3}}{8}\quad \textit{(True)}$$

$$\frac{abc}{8R^3}\le\frac{3\sqrt{3}}{8}\Bigg|\cdot\frac{(abc)^2}{8}\Leftrightarrow\frac{(abc)^3}{64R^3}\le\frac{3\sqrt{3}}{64}\cdot(abc)^2\Leftrightarrow\left(\frac{4\sqrt{3}}{3}S\right)^3\overset{(*)}{\le}(abc)^2$$

Solution 3 by Boris Colakovic-Belgrade-Serbia

By Holder's inequality $\Rightarrow$

$$\Rightarrow \frac{a^n}{x}+\frac{b^n}{y}+\frac{c^n}{z}\ge\frac{(a+b+c)^n}{3^{n-2}(x+y+z)};\ a,b,c,x,y,z\in(0,\infty);\ n\in N;n\ge 2$$

$$\sqrt[3]{\left(\frac{a^3+b^3+c^3}{3}\right)^2}\ge\frac{(a+b+c)^2}{3^2}\qquad (1)$$

$$\sqrt[5]{\left(\frac{a^5+b^5+c^5}{3}\right)^2}\ge\frac{(a+b+c)^2}{3^2}\qquad (2)$$

$$\sqrt[7]{\left(\frac{a^7+b^7+c^7}{3}\right)^2}\ge\frac{(a+b+c)^2}{3^2}\qquad (3)$$

$$(1)*(2)*(3)\Rightarrow LHS\ge\left(\frac{a+b+c}{3}\right)^6\qquad a+b+c=2p\ (p\text{ - perimeter})$$

$$S=r\cdot p=\frac{abc}{4R}\quad R\ge 2r\ \text{ Euler's inequality}$$

$$a+b+c\overset{AM-GM}{\ge}3\sqrt[3]{abc}\Rightarrow(a+b+c)^3\ge 27abc=27\cdot 4R\cdot S\ge 27\cdot 8r\cdot S\Rightarrow$$

$$\Rightarrow\frac{(a+b+c)^4}{2}\ge 27\cdot 8\cdot S^2\Rightarrow a+b+c\ge 2\cdot 3^{\frac{3}{4}}\cdot\sqrt{S}\Rightarrow\frac{a+b+c}{3}\ge\frac{2\sqrt{S}}{\sqrt[4]{3}}$$

$$LHS\ge\left(\frac{2\sqrt{S}}{\sqrt[4]{3}}\right)^6=\frac{(2^2S)^3}{\left(\sqrt{3}\right)^3}=\left(\frac{4S}{\sqrt{3}}\right)^3=\left(\frac{4\sqrt{3}S}{3}\right)^3$$

395. In $\Delta\ ABC, d_a=d(O,BC), d_b=d(O,CA), d_c=d(O,AB)$,

O – circumcentre

1. $\frac{a^2}{d_a^2}+\frac{b^2}{d_b^2}+\frac{c^2}{d_c^2}\ge 36$
2. $\frac{d_a^2+d_b^2+d_c^2}{a^2+b^2+c^2}\ge\frac{1}{12}$

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Kevin Soto Palacios – Huarmey – Peru

En un triángulo ABC, $O -$ ***circuncentro,***

$$d_a = d(O, BC), d_b = d(O, AC), d_c = d(0, AB)$$

Probar que

$$\textbf{1. } \frac{a^2}{d_a^2} + \frac{b^2}{d_b^2} + \frac{c^2}{d_c^2} \geq 36$$

$$\textbf{2. } \frac{d_a^2 + d_b^2 + d_c^2}{a^2 + b^2 + c^2} \geq \frac{1}{12}$$

Tener en cuenta las siguientes identidades y desigualdad en un $\Delta\, ABC$

$$d_a = R\cos A,\ d_b = R\cos B,\ d_c = R\cos C,\ a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$$

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C \geq 1 - \frac{1}{4} = \frac{3}{4}$$

Ahora bien: $\frac{a^2}{d_a^2} + \frac{b^2}{d_b^2} + \frac{c^2}{d_c^2} \geq 36 \Leftrightarrow \tan^2 A + \tan^2 B + \tan^2 C \geq 9$

En un triángulo acutángulo ABC: $\tan A + \tan B + \tan C \geq 3\sqrt{3} \Leftrightarrow$

$$\Leftrightarrow \tan^2 A + \tan^2 B + \tan^2 C \geq \frac{1}{3}(\tan A + \tan B + \tan C)^2 \geq \frac{1}{3}\cdot 27 = 9$$

(LQQD). Por último: $\frac{d_a^2 + d_b^2 + d_c^2}{a^2 + b^2 + c^2} \geq \frac{1}{12} \Leftrightarrow$

$$\Leftrightarrow 3(\cos^2 A + \cos^2 B + \cos^2 C) \geq \sin^2 A + \sin^2 B + \sin^2 C$$

$$\Leftrightarrow 3(\cos^2 A + \cos^2 B + \cos^2 C) \geq 3 - (\cos^2 A + \cos^2 B + \cos^2 C) \Leftrightarrow$$

$$\Leftrightarrow \cos^2 A + \cos^2 B + \cos^2 C \geq \frac{3}{4} \textit{ (LQQD)}$$

396. In $\Delta\, ABC, I -$ incentre, $R_a, R_b, R_c -$ circumradii in $\Delta\, BIC, \Delta\, CIA, \Delta\, AIB$

$$\sqrt{6} \leq \sqrt{\frac{R_a}{h_a}} + \sqrt{\frac{R_b}{h_b}} + \sqrt{\frac{R_c}{h_c}} \leq \sqrt{\frac{3R}{r}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $I-$ Incentro, R_a, R_b, R_c circunradio en los triángulos BIC, CIA, AIB. Probar en un triángulo ABC

$\sqrt{6}\le\sqrt{\frac{R_a}{h_a}}+\sqrt{\frac{R_b}{h_b}}+\sqrt{\frac{R_c}{h_c}}\le\sqrt{\frac{3R}{r}}$***. Tener en cuenta las siguientes identidades***

$$R_a=2R\sin\frac{A}{2}, R_b=2R\sin\frac{B}{2}, R_c=2R\sin\frac{C}{2}, R_aR_bR_c=8R^3\cdot\frac{r}{4R}=2R^2r$$

$$\Leftrightarrow R_a+R_b+R_c=2R\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)\le 3R, 2p\le 3\sqrt{3}R, R\ge 2r$$

$$h_a=\frac{2S}{a}, h_b=\frac{2S}{b}, h_c=\frac{2S}{c}, \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$$

$\Leftrightarrow h_ah_bh_c=\frac{8S^3}{abc}=\frac{8S^3}{4RS}=\frac{2S^2}{R}=\frac{2p^2r^2}{R}\le\frac{27r^2}{2}$***. Aplicando MA ≥ MG***

$$\sqrt{\frac{R_a}{h_a}}+\sqrt{\frac{R_b}{h_b}}+\sqrt{\frac{R_c}{h_c}}\ge 3\sqrt[6]{\frac{R_aR_bR_c}{h_ah_bh_c}}=3\sqrt[6]{\frac{2R^2r}{\frac{2p^2r^2}{R}}}\ge 3\sqrt[6]{\frac{2R^2r}{\frac{27r^2}{2}}}=$$

$=3\sqrt[6]{\frac{4R}{27r}}\ge 3\sqrt[6]{\frac{8}{27}}=3\sqrt{\frac{2}{3}}=\sqrt{6}$***. Por la desigualdad de Cauchy***

$\sqrt{\frac{R_a}{h_a}}+\sqrt{\frac{R_b}{h_b}}+\sqrt{\frac{R_c}{h_c}}\le\sqrt{(R_a+R_b+R_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)}\le\sqrt{\frac{3R}{r}}$ ***(LQQD)***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$R_a=2R\sin\frac{A}{2}, R_b=2R\sin\frac{B}{2}, R_c=2R\sin\frac{C}{2}; \Delta=\frac{a\cdot h_a}{2}=\frac{b\cdot h_b}{2}=\frac{c\cdot h_c}{2}$***. Now,***

$$\sum_{cyc}\sqrt{\frac{R_a}{h_a}}=\sum_{cyc}\sqrt{\frac{a\,R\sin\frac{A}{2}}{\Delta}}=\sum_{cyc}\sqrt{\frac{2a\cdot 2R\sin\frac{A}{2}\cos\frac{A}{2}}{4\Delta\cos\frac{A}{2}}}=\sum_{cyc}\sqrt{\frac{a\cdot 2R\sin A}{4\Delta\cos\frac{A}{2}}}$$

$$=\frac{1}{\sqrt{4\Delta}}\sum_{cyc}\frac{a}{\sqrt{\cos\frac{A}{2}}}\overset{CHEBYSHEV}{\underset{\smile}{\geq}}\frac{1}{2\sqrt{\Delta}}\cdot\frac{1}{3}\cdot(a+b+c)\cdot\left(\sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}}\right)$$

$$=\sqrt{\frac{p}{r}}\cdot\frac{1}{3}\left(\sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}}\right)\geq\sqrt{\frac{p}{r}}\cdot\sqrt{\frac{3}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}}$$

$$=\sqrt{3\sqrt{3}\cdot\frac{3}{\frac{3\sqrt{3}}{2}}}=\sqrt{6}\left[\because\sum_{cyc}\cos\frac{A}{2}\leq\frac{3\sqrt{3}}{2}\ and\ p\geq 3\sqrt{3}r\right]$$

again, $\sum_{cyc}\sqrt{\frac{R_a}{h_a}}\overset{CAUCHY-SCHWARZ}{\underset{\smile}{\leq}}\sqrt{\left(\sum_{cyc}R_a\right)\left(\sum_{cyc}\frac{1}{h_a}\right)}$

$$=\sqrt{2R\left(\sum_{cyc}\sin\frac{A}{2}\right)\left(\sum_{cyc}\frac{a}{2\Delta}\right)}=\sqrt{\frac{2R}{r}\left(\sum_{cyc}\sin\frac{A}{2}\right)}\leq\sqrt{\frac{3R}{r}}\left[\because\sum_{cyc}\sin\frac{A}{2}\leq\frac{3}{2}\right]$$

Hence proved

397. In $\Delta\ ABC$:

$$\left(\frac{h_a}{A}+\frac{h_b}{B}+\frac{h_c}{C}\right)\left(\frac{A}{h_a}+\frac{B}{h_b}+\frac{C}{h_c}\right)\geq 10-\frac{2r}{R}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\left(\frac{h_a}{A}+\frac{h_b}{B}+\frac{h_c}{C}\right)\left(\frac{A}{h_a}+\frac{B}{h_b}+\frac{C}{h_c}\right)\geq 10-\frac{2r}{R}$

Supongamos $\sin$ ***pérdida de generalidad***

$$a\geq b\geq c\Leftrightarrow\frac{1}{c}\geq\frac{1}{b}\geq\frac{1}{a},\qquad A\geq B\geq C\Leftrightarrow\frac{1}{C}\geq\frac{1}{B}\geq\frac{1}{A}$$

Aplicando la desigualdad de Chebyshev

$$(aA+bB+cC)\geq\frac{1}{3}(a+b+c)(A+B+C)\ \dots\ \textit{(A)}$$

$$\left(\frac{1}{cC}+\frac{1}{bB}+\frac{1}{aA}\right)\geq\frac{1}{3}\left(\frac{1}{c}+\frac{1}{b}+\frac{1}{a}\right)\left(\frac{1}{C}+\frac{1}{B}+\frac{1}{A}\right)\ \dots\ \textit{(B)}$$

Multiplicando (A) · (B) y aplicando MA ≥ MG

$$(aA+bB+cC)\left(\frac{1}{cC}+\frac{1}{bB}+\frac{1}{aA}\right)\geq$$

$$\geq\frac{1}{9}(A+B+C)\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Por la tanto

$$\left(\frac{h_a}{A}+\frac{h_b}{B}+\frac{h_c}{C}\right)\left(\frac{A}{h_a}+\frac{B}{h_b}+\frac{C}{h_c}\right)=(aA+bB+cC)\left(\frac{1}{cC}+\frac{1}{bB}+\frac{1}{aA}\right)\geq$$

$$\geq(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 10-\frac{2r}{R}\ \textit{(LQQD)}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\left(\sum_{cyc}\frac{h_a}{a}\right)\left(\sum_{cyc}\frac{a}{h_a}\right)=\left(\sum_{cyc}\frac{1}{a^2}\right)\left(\sum_{cyc}a^2\right)\left[\Delta=\frac{ah_a}{2}=\frac{bh_b}{2}=\frac{ch_c}{2}\right]$$

$$\geq\left(\sum_{cyc}\frac{1}{ab}\right)\left(\sum_{cyc}a^2\right)=\left(\frac{a+b+c}{abc}\right)\left(\sum_{cyc}a^2\right)=\frac{2p}{4Rrp}\cdot 2(p^2-r^2-4Rr)$$

$$=\frac{p^2-r^2-4Rr}{Rr}\geq\frac{12Rr-6r^2}{Rr}\ [\because p^2\geq 16Rr-5r^2]$$

$$=\frac{10Rr+2Rr-4r^2-2r^2}{Rr}\geq 10-\frac{2r}{R}\ \textit{(Proved)}$$

398. In $\Delta\ ABC$:

$$\frac{ac+b^2}{b(ra+Rc)}+\frac{ba+c^2}{c(rb+Ra)}+\frac{cb+a^2}{a(rc+Rb)}\geq\frac{4}{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

In $\Delta\, ABC$ then (S = area of ABC); $\frac{ac+b^2}{b(ra+Rc)}+\frac{ba+c^2}{c(rb+Ra)}+\frac{cb+a^2}{a(rc+Rb)}\ge\frac{4}{R}$

$$\Leftrightarrow\sum\frac{ac+b^2}{b\left(\frac{2S}{a+b+c}\cdot a+\frac{abc}{4S}\cdot c\right)}\ge\frac{4}{\frac{abc}{4S}}$$

$$\Leftrightarrow\sum\frac{ac+b^2}{ab\left(\frac{2S}{a+b+c}+\frac{bc^2}{4S}\right)}\ge\frac{16S}{abc}\Leftrightarrow\sum\frac{(ac+b^2)(a+b+c)}{ab\left(8S^2+bc^2(a+b+c)\right)}\ge\frac{4}{abc}$$

$$\Leftrightarrow\sum\frac{c(ac+b^2)(a+b+c)}{\left[\left(\sum a\right)\prod(b+c-a)+2bc^2\left(\sum a\right)\right]}\ge 2$$

$$\Leftrightarrow\sum\frac{c(ac+b^2)}{\prod(b+c-a)+2bc^2}\quad \textbf{(3)}$$

We have: $\prod(b+c-a)\le abc\Rightarrow\sum\frac{c(ac+b^2)}{\prod(b+c-a)+2bc^2}\ge\sum\frac{c(ac+b^2)}{abc+2bc^2}$

$$\Leftrightarrow\sum\frac{c(ac+b^2)}{\prod(b+c-a)+2bc^2}\ge\sum\frac{ac+b^2}{ab+2bc}\ge 3\cdot\sqrt[3]{\frac{\prod(ac+b^2)}{\prod(ab+2bc)}}\quad \textbf{(1)}$$

We prove that: $27\prod(ac+b^2)\ge 8\prod(ab+2bc)$

$$\Leftrightarrow 27\left(2a^2b^2c^2+\sum a^3b^3+abc\sum a^3\right)\ge 8\left(9a^2b^2c^2+2abc\sum ab^2+4abc\sum a^2b\right)$$

$$\Leftrightarrow 27\sum a^3b^3+27abc\sum a^3\ge 18a^2b^2c^2+16abc\sum ab^2+32abc\sum a^2b\quad \textbf{(2)}$$

Because, by AM-GM: $16abc\sum a^3\ge 16abc\sum ab^2$

$$11abc\sum a^3+21\sum a^3b^3\ge 32abc\sum a^2b;\ 6\sum a^3b^3\ge 18a^2b^2c^2$$

Therefore ⇒ (2) true; (1), (2) ⇒ $\sum\frac{c(ac+b^2)}{\prod(b+c-a)+2bc^2}\ge 2\Rightarrow$ ***true ⇒ QED.***

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{ab+c^2}{c(br+aR)}=\sum_{cyc}\frac{ab}{c(br+aR)}+\sum_{cyc}\frac{c^2}{bcr+acR}$$

$$= \frac{1}{abc}\sum_{cyc}\frac{a^2b^2}{br + aR} + \sum_{cyc}\frac{c^2}{bcr + acR}$$

$$\overset{BERGSTORM}{\geq} \frac{(ab + bc + ca)^2}{(R + r)abc(a + b + c)} + \frac{(a + b + c)^2}{(R + r)(ab + bc + ca)}$$

$$= \frac{6}{r+R} \geq \frac{2}{3R} \cdot 6[\because R \geq 2r] = \frac{4}{R} \textit{ (proved)}$$

399. If $n \in \left[\frac{1}{4}, 1\right]$ then in $\Delta\ ABC$:

$$\frac{R}{2r} \geq n + (1 - n) \cdot \frac{a^3 + b^3 + c^3}{3abc}$$

Proposed by Marin Chirciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\frac{R}{2r} \geq n + (1 - n)\frac{a^3 + b^3 + c^3}{3abc} \Leftrightarrow \frac{R}{2r} - 1 \geq (1 - n)\left(\frac{a^3 + b^3 + c^3}{3abc} - 1\right)$$

$$\Leftrightarrow \frac{R - 2r}{2r} \cdot 3abc \geq (1 - n)\left\{\left(\sum_{cyc} a\right)^3 - 3\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\right\}$$

$$\Leftrightarrow 6Rp(R - 2r) \geq (1 - n)2p(p^2 - 3r^2 - 12Rr)$$

$$\Leftrightarrow 3R(R - 2r) \geq \frac{3}{4}(p^2 - 3r^2 - 12Rr)\left[\because \frac{1}{4} \leq n \leq 1 \Rightarrow \frac{3}{4} \geq 1 - n \geq 0\right]$$

$$\Leftrightarrow 4R^2 + 4Rr + 3p^2 \geq p^2, \textit{ which is true } \therefore \frac{R}{2r} \geq n + (1 - n) \cdot \frac{a^3+b^3+c^3}{3abc}$$

400. In any scalene $\Delta\ ABC$:

$$\sum \frac{a^4 + b^2c^2}{(a - b)(a - c)} < 96R^2$$

Proposed by Daniel Sitaru – Romania

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Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un $\Delta\, ABC$ ***escaleno:*** $\frac{a^4+b^2c^2}{(a-b)(a-c)}+\frac{b^4+c^2a^2}{(b-c)(b-a)}+\frac{c^4+a^2b^2}{(c-a)(c-b)}<96R^2$

Como el $\Delta\, ABC$ ***es escaleno*** $\rightarrow a\neq b\neq c \Leftrightarrow a^2+b^2+c^2<9R^2$***. Recordar las siguiente identidades*** $(a-b)(b-c)(c-a)=ab(b-a)+bc(c-b)+ca(a-c)$.

Ahora bien $\frac{(a^4+b^2c^2)(c-b)}{(a-b)(a-c)(c-b)}+\frac{(b^4+c^2a^2)(a-c)}{(b-c)(b-a)(a-c)}+\frac{(c^4+a^2b^2)(b-a)}{(c-a)(c-b)(b-a)}=\frac{\sum(a^4+b^2c^2)(c-b)}{(a-b)(b-c)(c-a)}$***. Es suficiente probar*** $(a^4+b^2c^2)(c-b)+(b^4+c^2a^2)(a-c)+(c^4+a^2b^2)(b-a)=$

$=(a+b+c)^2(a-b)(b-c)(c-a)$***. Demostración***

$$(a+b+c)\big(ab(b-a)+bc(c-b)+ca(a-c)\big)=$$

$$=ab(b^2-a^2)+abc(b-a)+bc(c^2-b^2)+bca(c-b)+ca(a^2-c^2)+cab(a-c)$$

$$(a+b+c)(a-b)(b-c)(c-a)=ab(b^2-a^2)+bc(c^2-b^2)+ca(a^2-c^2)$$

$$(a+b+c)^2(a-b)(b-c)(c-a)=(a+b+c)\left(ab(b^2-a^2)+bc(c^2-b^2)+ca(a^2-c^2)\right)$$

$$(a+b+c)^2(a-b)(b-c)(c-a)=\sum ab\,(b^2-a^2)(a+b)+abc(b^2-a^2)+$$

$$+bca(c^2-b^2)+cab(a^2-c^2)$$

$$(a+b+c)^2(a-b)(b-c)(c-a)=ab(b^2-a^2)(a+b)+bc(c^2-b^2)(b+c)+$$

$+ca(a^2-c^2)(c+a)$ ***. Por último***

$$E=ab(b^2-a^2)(a+b)+bc(c^2-b^2)(b+c)+ca(a^2-c^2)(c+a)$$

$$E=ab(b^2a+b^3-a^3-a^2b)+bc(c^2b+c^3-b^3-b^2c)+ca(a^2c+a^3-c^3-c^2a)$$

$$E=b^3a^2+b^4a-a^4b-a^3b^2+c^3b^2+c^4b-b^4c-b^3c^2+a^3c^2+a^4c-$$

$$-c^4a-c^3a^2$$

$$E=a^4c-a^4b+c^3b^2-b^3c^2+b^4a-b^4c+a^3c^2-c^3a^2+c^4b-c^4a+b^3a^2-a^2b^2$$

$$E=(a^4+b^2c^2)(c-b)+(b^4+c^2a^2)(a-c)+(c^4+a^2b^2)(b-a)$$

(LQQD). Finalmente

$$\frac{a^4+b^2c^2}{(a-b)(a-c)}+\frac{b^4+c^2a^2}{(b-c)(b-a)}+\frac{c^4-a^2b^2}{(c-a)(c-b)}=(a+b+c)^2<$$

$$<3(a^2+b^2+c^2)<27R^2<96R^2$$

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Its nice to be important but more important its to be nice!

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru