solutions of famous integrals by smart techniques

(Part 3)

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$$I=\int\frac{\tan^3 x+\tan x}{\tan^3 x+3\tan^2 x+2\tan x+6}dx$$

$$=\int\frac{\tan x\left(\tan^2 x+1\right)}{\tan^3 x+3\tan^2 x+2\tan x+6}dx$$

$$=\int\frac{\tan x\sec^2 x}{\tan^3 x+3\tan^2 x+2\tan x+6}.\left(\frac{11}{11}\right)dx$$

$$=\frac{1}{11}\int\frac{11\tan x}{\left(\tan^2 x+2\right)\left(\tan x+3\right)}\sec^2 x dx$$

$$=\frac{1}{11}\int\frac{3\tan^2 x+9\tan x-\tan^3 x-3\tan^2 x+\tan^3 x+2\tan x}{\left(\tan^2 x+2\right)\left(\tan x+3\right)}\sec^2 x dx$$

$$=\frac{1}{11}\int\frac{\tan x\left(\left(3-\tan x\right)\left(\tan x+3\right)+\left(\tan^2 x+2\right)\right)}{\left(\tan^2 x+2\right)\left(\tan x+3\right)}\sec^2 x dx$$

$$=\frac{1}{11}\int\left(\frac{\tan x\left(3-\tan x\right)}{\tan^2 x+2}+\frac{\tan x}{\tan x+3}\right)\sec^2 x dx$$

$$=\frac{1}{11}\int\left(-\frac{3\tan x+\tan^2 x+2-2}{\tan^2 x+2}+\frac{\tan x+3-3}{\tan x+3}\right)\sec^2 x dx$$

$$=\frac{1}{11}\int\left(-\frac{\tan^2 x+2}{\tan^2 x+2}-\frac{3\tan x}{\tan^2+2}+\frac{2}{\tan^2 x+2}+\frac{\tan x+3}{\tan x+3}-\frac{3}{\tan x+3}\right)\sec^2 x dx$$

$$=\frac{1}{11}\int\left(\cancel{-\sec^2 x}-\frac{3\tan x\sec^2 x}{\tan^2 x+2}+\frac{2\sec^2 x}{\tan^2 x+2}\cancel{+\sec^2 x}-\frac{3\sec^2 x}{\tan x+3}\right)dx$$

$$=\frac{1}{11}\left(-\frac{3}{2}\ln\left|\tan^2 x+2\right|+\sqrt{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)-3\ln\left|\tan x+3\right|\right)+c$$

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$$I=\int\frac{\sec x}{\sin x+\csc x-1}dx$$

$$=\int\frac{\sec x}{\sin x+\csc x-1}\cdot\left(\frac{36\sin x\cos^2 x}{36\sin x\cos^2 x}\right)dx$$

$$=\frac{1}{36}\int\frac{36\sin x\cos x}{1-\sin x+\sin^3 x-\sin^4 x}dx=-\frac{1}{36}\int\frac{36\sin x}{\sin^4 x-\sin^3 x+\sin x-1}\cos x\,dx$$

$$=-\int\frac{-24\sin^3 x+12\sin^2 x+24\sin x-12+18\sin^3 x+18+6\sin^3 x-12\sin^2 x+12\sin x-6}{36\left(\sin^2 x-\sin x+1\right)\left(\sin x-1\right)\left(\sin x+1\right)}\cos x\,dx$$

$$=-\int\frac{12\left(1-2\sin x\right)\left(\sin x-1\right)\left(\sin x+1\right)+18\left(\sin x+1\right)\left(\sin^2 x-\sin x+1\right)+6\left(\sin^2 x-\sin x+1\right)\left(\sin x-1\right)}{36\left(\sin^2 x-\sin x+1\right)\left(\sin x-1\right)\left(\sin x+1\right)}\cos x\,dx$$

$$=-\int\left(\frac{1-2\sin x}{3\left(\sin^2 x-\sin x+1\right)}+\frac{1}{2\left(\sin x-1\right)}+\frac{1}{6\left(\sin x+1\right)}\right)\cos x\,dx$$

$$=\int\left(\frac{2\sin x\cos x-\cos x}{3\left(\sin^2 x-\sin x+1\right)}-\frac{\cos x}{2\left(\sin x-1\right)}-\frac{\cos x}{6\left(\sin x+1\right)}\right)dx$$

$$=\frac{1}{3}\ln\left|\sin^2 x-\sin x+1\right|-\frac{1}{2}\ln\left|\sin x-1\right|-\frac{1}{6}\ln\left|\sin x+1\right|+c$$

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$$I=\int \frac{dx}{(x^2+a^2)\sqrt{x^2+b^2}}$$

$$=\int \frac{1}{(x^2+a^2)\sqrt{x^2+b^2}}\cdot\left(\frac{-b^2}{-b^2}\right)dx$$

$$=\int \frac{-b^2}{\sqrt{x^2+b^2}\left(a^2x^2-x^2b^2-a^2x^2-a^2b^2\right)}dx$$

$$=\int \frac{x^2-x^2-b^2}{\sqrt{x^2+b^2}\left(x^2(a^2-b^2)-a^2(x^2+b^2)\right)}\cdot\left(\frac{\frac{1}{\sqrt{x^2+b^2}}}{\frac{1}{\sqrt{x^2+b^2}}}\right)dx$$

$$=\int \frac{\frac{x^2}{\sqrt{x^2+b^2}}-\sqrt{x^2+b^2}}{x^2(a^2-b^2)-a^2(x^2+b^2)}\cdot\left(\frac{a\sqrt{a^2-b^2}}{a\sqrt{a^2-b^2}}\right)dx$$

$$=\frac{1}{a\sqrt{a^2-b^2}}\int \frac{\frac{ax^2\sqrt{a^2-b^2}}{\sqrt{x^2+b^2}}-a\sqrt{x^2+b^2}\sqrt{a^2-b^2}}{\left(1-\frac{a^2(x^2+b^2)}{x^2(a^2-b^2)}\right)x^2(a^2-b^2)}dx$$

$$=\frac{1}{a\sqrt{a^2-b^2}}\tanh^{-1}\left(\frac{a\sqrt{x^2+b^2}}{x\sqrt{a^2-b^2}}\right)+c$$

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$$I=\int\frac{\sin(2x)\left(\cos^2(2x)+2\cos(2x)-3\right)}{4\cos^8x+12\cos^4x+4}dx$$

$$=\int\frac{2\sin x\cos x\left[\left(2\cos^2x-1\right)^2+2\left(2\cos^2x-1\right)-3\right]}{4\left(\cos^8x+3\cos^4x+1\right)}dx$$

$$=\frac{1}{2}\int\frac{\sin x\cos x\left(4\cos^4x-4\cos^2x+1+4\cos^2x-2-3\right)}{\cos^8x+3\cos^4x+1}dx$$

$$=\frac{1}{2}\int\frac{\sin x\cos x\left(4\cos^4x-4\right)}{\cos^8x+3\cos^4x+1}dx=\frac{4}{2}\int\frac{\sin x\left(\cos^5x-\cos x\right)}{\cos^8c+3\cos^4x+1}dx$$

$$=2\int\frac{\sin x\left(\cos^5x-\cos x\right)}{\cos^8x+2\cos^4x+1+\cos^4x}dx=2\int\frac{\sin x\left(\cos^5x-\cos x\right)}{\left(\cos^4x+1\right)^2+\cos^4x}\cdot\frac{\left(\cos^4x+1\right)^{-2}}{\left(\cos^4x+1\right)^{-2}}dx$$

$$=2\int\frac{\dfrac{\sin x\left(\cos^5x-\cos x\right)}{\left(\cos^4x+1\right)^2}}{1+\left(\dfrac{\cos^2x}{\cos^4x+1}\right)^2}dx=-\tan^{-1}\left(\frac{\cos^2x}{\cos^4x+1}\right)+c$$

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$$I=\int \frac{\tan^{-1}x}{x^2}dx$$

$$=\int \frac{\tan^{-1}x}{x^2}\cdot\left(\frac{x^2+1}{x^2+1}\right)dx$$

$$=\int \frac{-x^3+x^3+x-x+(x^2+1)\tan^{-1}x}{x^2(x^2+1)}dx$$

$$=\int \frac{-x^3+x(x^2+1)-x+(x^2+1)\tan^{-1}x}{x^2(x^2+1)}dx$$

$$=\int\left(\frac{-x}{x^2+1}+\frac{1}{x}-\frac{x-(x^2+1)\tan^{-1}x}{x^2(x^2+1)}\cdot\frac{(x^2+1)^{-1}}{(x^2+1)^{-1}}\right)dx$$

$$=\int\left(\frac{-x}{x^2+1}+\frac{1}{x}-\frac{\frac{x}{x^2+1}-\tan^{-1}x}{x^2}\right)dx$$

$$=\int\left(\frac{-x}{x^2+1}+\frac{1}{x}\right)dx-\int d\left(\frac{\tan^{-1}x}{x}\right)$$

$$=-\frac{1}{2}\ln(x^2+1)+\ln|x|-\frac{\tan^{-1}x}{x}+c$$

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$$I=\int \sec(x-a)\csc(x-b)\,dx$$

$$=\int \frac{1}{\cos(x-a)\sin(x-b)}\cdot\left(\frac{\cos(a-b)}{\cos(a-b)}\right)dx$$

$$=\frac{1}{\cos(a-b)}\int \frac{\cos((x-b)-(x-a))}{\cos(x-a)\sin(x-b)}dx$$

$$=\sec(a-b)\int \frac{\cos(x-b)\cos(x-a)+\sin(x-b)\sin(x-a)}{\cos(x-a)\sin(x-b)}dx$$

$$=\sec(a-b)\int\left(\frac{\cos(x-b)}{\sin(x-b)}+\frac{\sin(x-a)}{\cos(x-a)}\right)dx$$

$$=\sec(a-b)\left(\ln\left|\sin(x-b)\right|-\ln\left|\cos(x-a)\right|\right)+c$$

$$=\sec(a-b)\ln\left|\frac{\sin(x-b)}{\cos(x-a)}\right|+c$$

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$$I=\int \frac{x^2}{x^3-6x-4}dx$$

$$=\frac{1}{3}\int \frac{2x^2-4x-4+x^2+4x+4}{(x+2)(x^2-2x-2)}dx$$

$$=\frac{1}{3}\int \frac{2(x^2-2x-2)+(x+2)^2}{(x+2)(x^2-2x-2)}dx$$

$$=\frac{1}{3}\int \left(\frac{2\cancel{(x^2-2x-2)}}{(x+2)\cancel{(x^2-2x-2)}}+\frac{(x+2)\cancel{(x+2)}}{\cancel{(x+2)}(x^2-2x-2)}\right)dx$$

$$=\frac{1}{3}\int \left(\frac{2}{x+2}+\frac{1}{2}\frac{2x-2+6}{x^2-2x-2}\right)dx$$

$$=\frac{1}{3}\int \left(\frac{2}{x+2}+\frac{1}{2}\frac{2x-2}{x^2-2x-2}+\frac{1}{2}\frac{6}{(x-1)^2-3}\right)dx$$

$$=\frac{1}{3}\left(2\ln|x+2|+\frac{1}{2}\ln|x^2-2x-2|-\sqrt{3}\tanh^{-1}\left(\frac{x-1}{\sqrt{3}}\right)\right)+c$$

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$$I=\int \frac{2x^3-1}{x^6+4x^4+2x^3+5x^2+4x+1}dx$$

$$=\int \frac{2x^3-1}{\left(x^3+2x+1\right)^2+x^2}\cdot\frac{\left(x^3+2x+1\right)^{-2}}{\left(x^3+2x+1\right)^{-2}}dx$$

$$=\int \frac{\dfrac{2x^3-1}{\left(x^3+2x+1\right)^2}}{1+\dfrac{x^2}{\left(x^3+2x+1\right)^2}}dx=-\tan^{-1}\left(\frac{x}{x^3+2x+1}\right)+c$$

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$$I=\int\frac{2}{x\left(x^2+x+1\right)}dx$$

$$=2\int\frac{x^2+x+1-x^2-x}{x\left(x^2+x+1\right)}dx$$

$$=2\int\left(\frac{\cancel{\left(x^2+x+1\right)}}{x\cancel{\left(x^2+x+1\right)}}-\frac{\cancel{x}\left(x+1\right)}{\cancel{x}\left(x^2+x+1\right)}\right)dx$$

$$=2\int\left(\frac{1}{x}-\frac{1}{2}\frac{2x+1+1}{x^2+x+1}\right)dx$$

$$=2\int\left(\frac{1}{x}-\frac{1}{2}\frac{2x+1}{x^2+x+1}-\frac{1}{2}\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\right)dx$$

$$=2\ln|x|-\ln\left|x^2+x+1\right|-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

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$$I=\int \frac{\sqrt{x}-\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[4]{x}}dx$$

$$=\int \frac{\sqrt{x}-\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[4]{x}}\cdot\left(\frac{\sqrt[4]{x}-\sqrt[3]{x}}{\sqrt[4]{x}-\sqrt[3]{x}}\right)dx$$

$$=\int \frac{\sqrt[12]{x^7}\left(1-\sqrt[6]{x}\right)\left(\sqrt[12]{x}-1\right)}{\sqrt{x}\left(1-\sqrt[6]{x}\right)}dx$$

$$=\int\left(\sqrt[6]{x}-\sqrt[12]{x}\right)dx=\frac{6}{7}\sqrt[6]{x^7}-\frac{12}{13}\sqrt[12]{x^{13}}+c$$

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$$I=\int\frac{\sqrt{x}}{1+\sqrt[4]{x}}dx$$

$$=\int\frac{\sqrt{x}}{1+\sqrt[4]{x}}\cdot\left(\frac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}}\right)dx=\int\frac{\sqrt[4]{x^5}-\sqrt[4]{x^3}+\sqrt[4]{x^3}}{1+\sqrt[4]{x}}\cdot\frac{1}{\sqrt[4]{x^3}}dx$$

$$=\int\left(\frac{\sqrt[4]{x^5}-\sqrt[4]{x^3}}{1+\sqrt[4]{x}}+\frac{\sqrt[4]{x^3}+1-1}{1+\sqrt[4]{x}}\right)\frac{1}{\sqrt[4]{x^3}}dx$$

$$=\int\left(\frac{\sqrt[4]{x^3}\left(\sqrt[4]{x}-1\right)\left(\sqrt[4]{x}+1\right)}{\left(1+\sqrt[4]{x}\right)}+\frac{\left(\sqrt[4]{x}+1\right)\left(\sqrt{x}-\sqrt[4]{x}+1\right)}{\left(1+\sqrt[4]{x}\right)}-\frac{1}{1+\sqrt[4]{x}}\right)\frac{1}{\sqrt[4]{x^3}}dx$$

$$=\int\left(x-\sqrt[4]{x^3}+\sqrt{x}-\sqrt[4]{x}+1-\frac{1}{1+\sqrt[4]{x}}\right)\frac{1}{\sqrt[4]{x^3}}dx$$

$$=\int\left(\sqrt[4]{x}-1+\frac{1}{\sqrt[4]{x}}-\frac{1}{\sqrt{x}}+\frac{1}{\sqrt[4]{x^3}}-\frac{\frac{1}{\sqrt[4]{x^3}}}{1+\sqrt[4]{x}}\right)dx$$

$$=\frac{4}{5}\sqrt[4]{x^5}-x+\frac{4}{3}\sqrt[4]{x^3}-2\sqrt{x}+4\sqrt[4]{x}-4\ln\left|1+\sqrt[4]{x}\right|+c$$

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$$I=\int\frac{x^3-2x^2+5x+6}{x+1}dx$$

$$=\int\frac{x^3+x^2-3x^2-3x-2+8x+8}{x+1}dx$$

$$=\int\frac{x^2(x+1)-3x(x+1)-2+8(x+1)}{x+1}dx$$

$$=\int\left(x^2-3x-\frac{2}{x+1}+8\right)dx$$

$$=\frac{x^3}{3}-\frac{3}{2}x^2-2\ln|x+1|+8x+c$$

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$$I=\int\frac{\sqrt{x^3-1}}{x}dx$$

$$=\int\frac{\sqrt{x^3-1}}{x}.\left(\frac{x^2\sqrt{x^3-1}}{x^2\sqrt{x^3-1}}\right)dx=\int\frac{x^3-1}{x^3}.\frac{x^2}{\sqrt{x^3-1}}dx$$

$$=\int\left(1-\frac{1}{x^3-1+1}\right)\frac{x^2}{\sqrt{x^3-1}}dx=\int\left(\frac{x^2}{\sqrt{x^3-1}}-\frac{\frac{x^2}{\sqrt{x^3-1}}}{\left(\sqrt{x^3-1}\right)^2+1}\right)dx$$

$$=\frac{2}{3}\sqrt{x^3-1}-\frac{2}{3}\tan^{-1}\left(\sqrt{x^3-1}\right)+c$$

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$$I=\int \sin\left(\frac{1}{x}\right)dx$$

$$=\int\left(\sin\left(\frac{1}{x}\right)-\frac{1}{x}\cos\left(\frac{1}{x}\right)+\frac{1}{x}\cos\left(\frac{1}{x}\right)\right)dx$$

$$=\int d\left(x\sin\left(\frac{1}{x}\right)\right)+\int\frac{1}{x}\cos\left(\frac{1}{x}\right)dx$$

$$=x\sin\left(\frac{1}{x}\right)-Ci\left(\frac{1}{x}\right)+c$$

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Q. Show that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{n\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

Sol. put $x^n = \sin^2\theta \Rightarrow x = (\sin\theta)^{\frac{2}{n}} d\theta \Rightarrow dx = \frac{2}{n}(\sin\theta)^{\frac{2}{n}-1}\cos\theta d\theta$

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{2}{n}\int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{\frac{2}{n}-1}\cos\theta}{\sqrt{1-\sin^2\theta}} d\theta$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{2}{n}\int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{2}{n}-1} d\theta$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{2}{n}\frac{\Gamma\left(\frac{1}{2}\left(\frac{2}{n}-1+1\right)\right)\Gamma\left(\frac{1}{2}(0+1)\right)}{2\Gamma\left(\frac{1}{2}\left(\frac{2}{n}-1+0+2\right)\right)}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{n\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$$

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$$I=\int x^{x}dx$$

$$=\int e^{\ln x^{x}}dx=\int e^{x\ln x}dx=\int\sum_{n=0}^{\infty}\frac{(x\ln x)^{n}}{n!}dx$$

$$\text{Let } u=\ln x\Rightarrow x=e^{u}\Rightarrow dx=e^{u}du$$

$$I=\sum_{n=0}^{\infty}\frac{1}{n!}\int e^{un}u^{n}e^{u}du=\sum_{n=0}^{\infty}\frac{1}{n!}\int e^{un+u}u^{n}du$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{\infty}\frac{(n+1)^{k}}{k!}\int u^{k}u^{n}du=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+1)^{k}}{n!k!}\int u^{k+n}du$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+1)^{k}}{n!k!}\frac{u^{k+n+1}}{k+n+1}+c=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+1)^{k}}{n!k!}\frac{(\ln x)^{k+n+1}}{k+n+1}+c$$

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$$I=\int e^{\frac{1}{x}}dx$$

$$Let\ y=\frac{1}{x}\Rightarrow dy=-\frac{dx}{x^2}\Rightarrow dy=-y^2dx$$

$$I=\int\frac{-e^y}{y^2}dy$$

$$Let\ u=e^y\Rightarrow du=e^y\ ,dv=-\frac{dy}{y^2}\Rightarrow v=\frac{1}{y}$$

$$I=\int udv=uv-\int vdu$$

$$I=\frac{e^y}{y}-\int\frac{e^y}{y}dy=\frac{e^y}{y}-Ei(y)+c$$

$$\therefore I=\int e^{\frac{1}{x}}dx=xe^{\frac{1}{x}}-Ei\left(\frac{1}{x}\right)+c$$

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