Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

A retrospective of the Ionescu-Nesbitt inequality

By D. M. Bătineţu-Giurgiu, Daniel Sitaru and Neculai Stanciu

1.Introduction

In Romanian Mathematical Gazette, Volume XXXII (September 15, 1926 - August 15, 1927), at page 120 *Ion Ionescu* - one of the founders and pillars of Mathematical Gazette published the problem.

3478. ***If*** x, y, z ***are positive, show that*:**

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}, \qquad (*).$$

In the same volume, at pp. 194-196, are presented two solutions to this problem, as well as a generalization.

From [17] yields that *Nesbitt* published the inequality (1) in 1903.

It is appropriate to say here that the problem.

In any triangle ABC***, with usual notations holds the inequality***

$$a^2+b^2+c^2\geq 4\sqrt{3}S\ , \qquad \text{(I-W)},$$

was published first by *Ion Ionescu* in 1897 (see [1]), and also published by *Roland Weitzenböck* in 1919. However, this inequality has long been called ’’*Weitzenböck*’s inequality’’, and after the appearance of paper [1] is called *Ionescu-Weitzenböck* inequality (IW)

Compared to the above we suggest that inequality (*) to be called *Nesbitt-Ionescu* inequality.

In the next, we will do a retrospective of our results on *Nesbitt-Ionescu* inequality and we shall give some generalizations of problem 752 from The Pentagon journal (see [19]).

2. Ionescu-Nesbitt type inequality for two variables

- **If** $x_1, x_2 \in R_+^*$**, then:**

$$A(x_1,x_2)=\frac{x_1}{x_2}+\frac{x_2}{x_1}\geq 2, \text{ (see for e.g. [7, 12])}$$

- **If** $a, b, x_1, x_2 \in R_+^*, X_2 = x_1 + x_2$ **and** $aX_2 > b\max\{x_1, x_2\}$**, then:**

$$B(x_1, x_2) = \frac{x_1}{aX_2 - bx_1} + \frac{x_2}{aX_2 - bx_2} \geq \frac{2}{2a - b}$$ **, (see [7, 12]).**

3. Ionescu-Nesbitt type inequality for three variables

- **If** $x_1, x_2, x_3 \in R_+^*$**, then:**

$$C(x_1, x_2, x_3) = \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_1} + \frac{x_3}{x_1 + x_2} \geq \frac{3}{2}$$ **, (Nesbitt,1903, see [17, 12]).**

- **If** $a, b, x_1, x_2, x_3 \in R_+^*$**, then:**

$$D(x_1, x_2, x_3) = \frac{x_1}{ax_2 + bx_3} + \frac{x_2}{ax_3 + bx_1} + \frac{x_3}{ax_1 + bx_2} \geq \frac{3}{a + b}$$ **, (see [4, 12]).**

- **If** $a_k, b_k, x_k \in R_+^*, k = \overline{1,3}$ **and** $a_1 + b_2 = a_2 + b_3 = a_3 + b_1 = t$**, then:**

$$E(x_1, x_2, x_3) = \frac{x_1}{a_1x_2 + b_1x_3} + \frac{x_2}{a_2x_3 + b_2x_1} + \frac{x_3}{a_3x_1 + b_3x_2} \geq \frac{3}{t}$$ **, (see [4, 12]).**

- **If** $a, b, x_1, x_2, x_3 \in R_+^*$**,** $x_1 + x_2 + x_3 = X_3$ **and** $aX_3 > b\max\{x_1, x_2, x_3\}$**, then:**

$$F(x_1, x_2, x_3) = \frac{x_1}{aX_3 - bx_1} + \frac{x_2}{aX_3 - bx_2} + \frac{x_3}{aX_3 - bx_3} \geq \frac{3}{3a - b}$$ **, (see [4, 12]).**

- **If** $a, b, x_1, x_2, x_3 \in R_+^*$**,** $x_1 + x_2 + x_3 = X_3$**, such that** $aX_3 > b\max\{x_1, x_2, x_3\}$**,**

and $m, p \in [1, \infty)$ **, then:**

$$G(x_1, x_2, x_3) = \frac{x_1^m}{(aX_3 - bx_1)^p} + \frac{x_2^m}{(aX_3 - bx_2)^p} + \frac{x_3^m}{(aX_3 - bx_3)^p} \geq \frac{3^{-m+p+1}}{(3a - b)^p} \cdot X_3^{m-p}$$ **, (see [4, 12]).**

- **If** $a, b, x_1, x_2, x_3 \in R_+^*$**,** $x_1 + x_2 + x_3 = X_3$ **and** $m \in R_+$**, then:**

$$H(x_1, x_2, x_3) = \frac{x_1}{(aX_3 - bx_1)^m} + \frac{x_2}{(aX_3 - bx_2)^m} + \frac{x_3}{(aX_3 - bx_3)^m} \geq \frac{3^m}{(3a - b)^m \cdot X_3^{m-1}}$$ **, (see [4]).**

- **If** $a, b, c, x, y, z \in R_+^*$**, then:**

$$I(x,y,z)=\frac{ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y}\geq\frac{(ax+by+cz)^2}{(a+b)xy+(b+c)yz+(c+a)zx}\geq$$

$$\geq\frac{3(abxy+bcyz+cazx)}{(a+b)xy+(b+c)yz+(c+a)zx}$$ **, (see [2, 5]).**

- **If** $a,b,x,y,z\in R_+^*$ **and** $m\in R_+$**, then:**

$$J(x,y,z)=\frac{x^{m+1}}{(ay+bz)^{2m+1}}+\frac{y^{m+1}}{(az+bx)^{2m+1}}+\frac{z^{m+1}}{(ax+by)^{2m+1}}\geq\frac{3^{m+1}}{(a+b)^{2m+1}(x+y+z)^m}$$ **, (see [3]).**

4. Ionescu-Nesbitt type inequality for n variables

- **If** $x_k\in R_+^*, \forall k=\overline{1,n}, X_n=\sum_{k=1}^{n}x_k$ **, then:**

$$N_n=\sum_{k=1}^{n}\frac{x_k}{X_n-x_k}\geq\frac{n}{n-1}$$ **, (see for e.g [16], the case** $n=4$ **and also [12]).**

- **If** $n\in N^*-\{1,2\}, m,p\in[1,\infty), a,b,x_k\in R_+^*, \forall k=\overline{1,n}$**,** $X_n=\sum_{k=1}^{n}x_k$ **, and**

$aX_n>b\max_{1\leq k\leq n}x_k$ **, then:**

$$K_n=\sum_{k=1}^{n}\frac{x_k^n}{(aX_n-bx_k)^p}\geq\frac{n^{-m+p+1}}{(an-b)^p}\cdot X_n^{m-p}$$ **, (see [3]).**

- **If** $a,b,x_k\in R_+^*, c,y_k\in R_+, \forall k=\overline{1,n}, n\in N^*-\{1\}$,

$X_n=\sum_{k=1}^{n}x_k$ **and** $aX_n>b\max_{1\leq k\leq n}x_k$, $y_k\in\left[0,\frac{1}{n}X_n\right], \forall k=\overline{1,n}$ **and** $m\in R_+$**, then:**

$$L_n=\sum_{k=1}^{n}\frac{x_k^{m+1}}{(aX_n-bx_k+cy_k)^{2m+1}}\geq\frac{n^{m+1}}{(an-b+c)^{2m+1}X_n^m}$$ **, (see [3]).**

- **If** $n\in N^*-\{1,2\}, a\in R_+, b,c,d,x_k\in R_+^*, \forall k=\overline{1,n}$**,** $X_n=\sum_{k=1}^{n}x_k$ **, and**

$cX_n>d\max_{1\leq k\leq n}x_k$ **, then:**

$$M_n = \sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n - dx_k} \geq \frac{(an+b) \cdot n}{cn-d}$$ **, (see [6] and [18]).**

- **If** $a \in R_+$**,** $b, c, d, x_k \in R_+^*$**,** $\forall k = \overline{1,n}$**,** $X_n = \sum_{k=1}^{n} x_k$ **,** $m \in [1, \infty)$

and $cX_n^m > d \max_{1 \leq k \leq n} x_k^m$**, then:**

$$U_n = \sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n^m - dx_k^m} \geq \frac{(an+b)n^m}{cn^m - d} X_n^{1-m}$$ **, (see [8] and also [11]).**

- **If** $a, m \in R_+$**,** $b, c, d, x_k \in R_+^*$**,** $\forall k = \overline{1,n}$**,** $X_n = \sum_{k=1}^{n} x_k$ **,** $p \in [1, \infty)$

and $cX_n^m > d \max_{1 \leq k \leq n} x_k^m$**, then:**

$$V_n = \sum_{k=1}^{n} \frac{aX_n + bx_k}{(cX_n^m - dx_k^m)^p} \geq \frac{(an+b)n^{mp}}{(cn^m - d)^p} X_n^{1-mp}$$ **, (see [9]).**

- **If** $n \in N^* - \{1\}$, $a \in R_+$, $b, c, d, x_k \in R_+^*$, $\forall k = \overline{1,n}$, $X_n = \sum_{k=1}^{n} x_k$,

$m, p, r, s \in [1, \infty)$**, such that** $cX_n^m > d \max_{1 \leq k \leq n} x_k^m$**, then:**

$$W_n = \sum_{k=1}^{n} \frac{(aX_n^r + bx_k^r)^s}{(cX_n^m - dx_k^m)^p} \geq \frac{(an^r + b)^s}{(cn^m - d)^p} n^{mp-rs+1} X_n^{rs-mp}$$ **, (see [10, 13]).**

- **If** $n \in N^* - \{1\}$**,** $m, p \in R_+^*$**,** $x_k \in R_+^*$**,** $\forall k = \overline{1,n}$**, and we denotes** $X_{n,m} = \sum_{k=1}^{n} x_k^m$ **,** $X_{n,p} = \sum_{k=1}^{n} x_k^p$ **, then:**

$$Y_n = \sum_{k=1}^{n} \frac{x_k^m}{X_{n,p} - x_k^p} \geq \frac{n}{n-1} \cdot \frac{X_{n,m}}{X_{n,p}}$$ **, (see [15]).**

- **If** $n \in N^* - \{1\}$**,** $a \in R_+$**,** $b, c, d, m, p \in R_+^*$**,** $x_k \in R_+^*$**,** $k = \overline{1,n}$**,** $X_{n,m} = \sum_{k=1}^{n} x_k^m$ **,** $X_{n,p} = \sum_{k=1}^{n} x_k^p$ **, such that** $c \cdot X_{n,p} > d \cdot \max_{1 \leq k \leq n} x_k^p$**, then**

$$Z_n = \sum_{k=1}^{n} \frac{a \cdot X_{n,m} + b \cdot x_k^m}{c \cdot X_{n,p} - d \cdot x_k^p} \geq \frac{n \cdot (an+b)}{cn-d} \cdot \frac{X_{n,m}}{X_{n,p}}$$ **, (see [15]).**

- **If** $n \in N^* - \{1\}$**,** $a, v \in R_+$**,** $b, c, d, m, p \in R_+^*$**,** $x_k \in R_+^*$**,** $k = \overline{1, n}$**,** $t \in [1, \infty)$**,** $X_{n,m} = \sum_{k=1}^{n} x_k^m$ **,**

$X_{n,p} = \sum_{k=1}^{n} x_k^p$ **, such that** $c \cdot X_{n,p} > d \cdot \max_{1 \le k \le n} x_k^p$**, then**

$$\sum_{k=1}^{n} \frac{(a \cdot X_{n,m} + b \cdot x_k^m)^t}{(c \cdot X_{n,p} - d \cdot x_k^p)^v} \ge \frac{n^{v-t} \cdot (an+b)^t}{cn-d} \cdot \frac{X_{n,m}^t}{X_{n,p}^v}$$ **, (see [14]).**

See also [16].

New results

- **If** $a \in R_+$ **and** $b, c, d, x, y, z \in R_+^*$**,** $X = x + y + z$ **,** $cX > d \max\{x, y, z\}$**, then:**

$$\sum_{cyc} \frac{aX + bx}{cX - dx} \ge \frac{3(3a+b)}{3c-d}.$$

- **If** $a \in R_+, m \in [1, \infty)$ **and** $b, c, d, x, y, z \in R_+^*$**,** $X = x + y + z$ **,** $cX > d \max\{x, y, z\}$**, then**

$$\frac{aX + bx}{(cX - dx)^m} + \frac{aX + by}{(cX - dy)^m} + \frac{aX + bz}{(cX - dz)^m} \ge \frac{3^m (3a+b)}{(3c-d)^m X^{m-1}}.$$

- **If** $a, m \in R_+$ **and** $b, c, d, x, y, z \in R_+^*$**,** $X = x + y + z$ **,** $cX > d \max\{x, y, z\}$**, then:**

$$\left(\frac{aX + bx}{cX - dx}\right)^{m+1} + \left(\frac{aX + by}{cX - dy}\right)^{m+1} + \left(\frac{aX + bz}{cX - dz}\right)^{m+1} \ge \frac{3(3a+b)^{m+1}}{(3c-d)^{m+1}}.$$

- **If** $n \in N^* - \{1\}$**,** $a \in R_+ = [0, \infty)$**,** $b, c, d, x_k \in R_+^* = (0, \infty), k = \overline{1, n}$**,** $X_n = \sum_{k=1}^{n} x_k$ **and**

$cX_n > d \max_{1 \le k \le n} x_k$ **, then**

$$\sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n - dx_k} \ge \frac{(an+b)n}{cn-d}.$$

- **If** $n \in N^* - \{1\}$**,** $a, m \in R_+ = [0, \infty)$**,** $b, c, d, x_k \in R_+^* = (0, \infty), k = \overline{1, n}$**,** $X_n = \sum_{k=1}^{n} x_k$ **and**

$cX_n > d \max_{1 \le k \le n} x_k$ **, then**

$$\sum_{k=1}^{n} \frac{(aX_n + bx_k)^{m+1}}{(cX_n - dx_k)^{2m+1}} \ge \frac{(an+b)^{m+1} n^{m+1}}{(cn-d)^{2m+1} X_n^m}.$$

For proofs of the above results we used the following inequalities:

- **The inequality of** ***H. Bergström*** **is**

If $x_k \in R, y_k \in R_+^*, \forall k = \overline{1,n}, n \in N^* - \{1\}$**, then** $\sum_{k=1}^{n} \frac{x_k^2}{y_k} \geq \frac{\left(\sum_{k=1}^{n} x_k\right)^2}{\sum_{k=1}^{n} y_k}$**.**

- **The inequality of** ***J. Radon*** **is**

If $x_k \in R_+^*, y_k \in R_+^*, \forall k = \overline{1,n}, n \in N^* - \{1\},\ m \in R_+$**, then**

$$\sum_{k=1}^{n} \frac{x_k^{m+1}}{y_k^m} \geq \frac{\left(\sum_{k=1}^{n} x_k\right)^{m+1}}{\left(\sum_{k=1}^{n} y_k\right)^m}.$$

References

[1] Bătineţu-Giurgiu, D.M., Stanciu, N., ***Inegalităţi de tip Ionescu-Weitzenböck,*** **Gazeta Matematică, 118(1), 2013.**

[2] Bătineţu-Giurgiu, D.M., Stanciu, N., ***O extindere şi o rafinare a inegalităţii lui Nesbitt,*** **Articole și Note Matematice, 4(1), 2011.**

[3] Bătineţu-Giurgiu, D.M., Stanciu, N., ***Noi generalizări ale inegalității lui Nesbitt*** **Articole și Note Matematice, 4(1), 2011.**

[4] Bătineţu-Giurgiu, D.M., Bencze, M., Stanciu, N., ***New generalizations and new approaches for Nesbitt's inequality,*** **Mathematical Education in the Current European Context, 2(1), 2011.**

[5] Bătineţu-Giurgiu, D.M., Stanciu, N., ***O extindere şi o rafinare a inegalităţii lui Nesbitt, R.M.T.,*** **91(1), 2012.**

[6] Bătineţu-Giurgiu, D.M., Stanciu, N., ***Încă patru demonstraţii ale problemei L:155 din Sclipirea minţii nr. VII 2011*****, Sclipirea Minţii, 5(9), 6, 2012.**

[7] Bătineţu-Giurgiu, D.M., Stanciu, N., ***Inegalitatea lui Nesbitt,*** **Didactica Matematică, 2(1), 2012.**

[8] Bătineţu-Giurgiu, D.M., Stanciu, N., ***A generalization of some remarkable inequalities*****, Experienţe Utile în Predarea-Învăţarea Matematicii, 1(1), 2012.**

[9] Bătineţu-Giurgiu, D.M., Stanciu, N., ***Problem 11634,*** **The American Mathematical Monthly, 119(3), 2012.**

[10] Bătineţu-Giurgiu, D.M., Stanciu, N., ***Nuevas generalizaciones y aplicaciones de la desigualdad de Nesbitt,*** **Revista Escolar de la Olimpiada Iberoamericana de Matematicá, 11(47), 2012.**

[11] Bătineţu-Giurgiu, D.M., Stanciu, N., ***Generalizations of some remarkable inequalities,*** **The Teaching of Mathematics, 16(1), 2013.**

[12] Bătineţu-Giurgiu, D.M., Stanciu, N., ***About Nesbitt's inequality,*** **Octogon Mathematical Magazine, 20(2), 2012.**

[13] Bătineţu-Giurgiu, D.M., Stanciu, N., *New generalizations and new applications for Nesbitt's inequality*, Journal of Science and Arts, 12(4), 2012.
[14] Bătineţu-Giurgiu, D.M., Stanciu, N., *A New generalization of Nesbitt's inequality*, Journal of Science and Arts, 13(3), 2013.
[15] Bătineţu-Giurgiu, D.M., Stanciu, N., Zvonaru, T., *Generalizarea problemei VIII. 169 din RecMat nr. 2/2013*, Recreaţii Matematice, 15(1), 2014.
[16] Bătineţu-Giurgiu, D.M., Stanciu, N., *Una nota sobre una desigualdad de Nesbitt-Ionescu*, Revista Escolar de la Olimpiada Iberoamericana de Matematica, Numero 53, Julio-Diciembre, 2015.
[17] Nesbitt, A.M., *Problem 15114*, Educational Times, 3(2), 1903.
[18] Zvonaru, T., Stanciu, N., *Şase soluţii pentru problema L:155 din Sclipirea Minţii, nr. VII, 2011*, Sclipirea Minţii, 4(8), 2011.
[19] Problem 725, *The Pentagon*, Fall 2014, p.