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1. Prove that in any triangle ABC,

$$\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\geq 2\sqrt{3}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

Proof.

We use the following lemma:

Lemma 1.
In ΔABC

$$\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)^2\geq\frac{108}{5p^2-3r^2-12Rr}$$

Proof.

Using the inequality $(x+y+z)^2\geq 3(xy+yz+zx)$, *with* $x=\frac{1}{m_a}, y=\frac{1}{m_b}, z=\frac{1}{m_c}$ *we obtain*

$$\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)^2\geq 3\left(\frac{1}{m_am_b}+\frac{1}{m_bm_c}+\frac{1}{m_cm_a}\right)\overbrace{\geq}^{(1)}3\left(\frac{4}{2a^2+bc}+\frac{4}{2b^2+ca}+\frac{4}{2c^2+ab}\right)\geq$$

$$\overbrace{\geq}^{Bergstrom}\ 12\cdot\frac{9}{2a^2+bc+2b^2+ca+2c^2+ab}=\frac{108}{2(a^2+b^2+c^2)+ab+bc+ca}=$$

$$=\frac{108}{2\cdot 2(p^2-r^2-4Rr)+p^2+r^2+4Rr}=\frac{108}{5p^2-3r^2-12Rr}$$

where inequality (1) follows from $4m_bm_c\leq 2a^2+bc$ *and analogs.*

Equality holds if and only if the triangle is equilateral.

□

Let's pass to solving the inequality from enuntiation.

Using ***Lemma 1*** *and* $\sum a=2p, \sum a^2=2(p^2-r^2-4Rr)$ *it is enough to prove that*

$$\left(\frac{2(p^2-r^2-4Rr)}{2p}\right)^2\cdot\frac{108}{5p^2-3r^2-12Rr}\geq 12\Leftrightarrow 9(p^2-r^2-4Rr)^2\geq p^2(5p^2-3r^2-12Rr)\Leftrightarrow$$

$$\Leftrightarrow p^2(4p^2-15r(4R+r))+9r^2(4R+2)^2\geq 0.$$ *We distinguish the cases:*

Case 1. If $4p^2-15r(4R+r)\geq 0$ *the inequality is obvious.*

Case 2. If $4p^2-15r(4R+r)<0$ *we write the inequality*

$p^2(15r(4R+r)-4p^2) \le 9r^2(4R+r)^2$, *which follows from Gerretsen's inequality* $16Rr-5r^2 \le p^2 \le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$(4R^2+4Rr+3r^2)(15r(4R+r)-4(16Rr-5r^2)) \le 9r^2(4R+r)^2 \Leftrightarrow$$

$$\Leftrightarrow (4R^2+4Rr+3r^2)(-4Rr+35r^2) \le 9r^2(4R+r)^2 \Leftrightarrow 4R^3+5R^2r-14Rr^2-24r^3 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)(4R^2+13Rr+12r^2) \ge 0,$$ *obviously from Euler's inequality* $R \ge 2r$

Equality holds if and only if the triangle is equilateral.

□

Remark 1.

*Inequality **1** can be developed:*

3. In $\boldsymbol{\Delta ABC}$

$$\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{1}{m_a+m_b}+\frac{1}{m_b+m_c}+\frac{1}{m_c+m_a}\right) \ge \sqrt{3}$$

Proof.

Using the following lemma:

Lemma 2.

4. In $\boldsymbol{\Delta ABC}$

$$\left(\frac{1}{m_a+m_b}+\frac{1}{m_b+m_c}+\frac{1}{m_c+m_a}\right)^2 \ge \frac{36}{7p^2-5r^2-20Rr}.$$

Proof.

Using the inequality $(x+y+z)^2 \ge 3(xy+yz+zx)$,

with $x=\dfrac{1}{m_a+m_b}, y=\dfrac{1}{m_b+m_c}, z=\dfrac{1}{m_c+m_a}$ *we obtain*

$$\left(\sum\frac{1}{m_b+m_c}\right)^2 \ge 3\left(\sum\frac{1}{(m_a+m_b)(m_a+m_c)}\right) \overbrace{\ge}^{Bergstrom} 3\cdot\frac{9}{\sum(m_a+m_b)(m_a+m_c)} =$$

$$= \frac{27}{\sum(m_a^2+m_am_b+m_bm_c+m_cm_a)} = \frac{27}{\sum m_a^2+3\sum m_bm_c} \overbrace{\ge}^{(1)} \frac{27}{\frac{3}{4}\sum a^2+\frac{3}{4}\sum(2a^2+bc)} =$$

$$= \frac{36}{3\sum a^2+\sum bc} = \frac{36}{3\cdot 2(p^2-r^2-4Rr)+p^2+r^2+4Rr} = \frac{36}{7p^2-5r^2-20Rr}$$

where inequality (1) follows from $4m_bm_c \le 2a^2+bc$ *and analogs.*

Equality holds if and only if the triangle is equilateral.

□

*Let's pass to solving inequality **3.***

*Using **Lemma 2** and* $\sum a = 2p, \sum a^2 = 2(p^2-r^2-4Rr)$ *it is enough to prove that*

$$\left(\frac{2(p^2-r^2-4Rr)^2}{2p}\right)^2\cdot\frac{36}{7p^2-5r^2-20Rr} \ge 3 \Leftrightarrow 12(p^2-r^2-4Rr)^2 \ge p^2(7p^2-5r^2-20Rr) \Leftrightarrow$$

$\Leftrightarrow p^2(5p^2-19r(4R+r))+12r^2(4R+r)^2 \ge 0$. *We distinguish the following cases:*

Case 1. *If* $5p^2 - 19r(4R+r) \geq 0$ *inequality is obvious.*
Case 2. *If* $5p^2 - 19r(4R+r) < 0$ *inequality can be rewritten*
$p^2(19r(4R+r) - 4p^2) \leq 12r^2(4R+r)^2$, *which follows from Gerretsen's inequality*

$16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$(4R^2 + 4Rr + 3r^2)(19r(4R+r) - 5(16Rr - 5r^2)) \leq 9r^2(4R+r)^2 \Leftrightarrow$$

$$\Leftrightarrow (4R^2+4Rr+3r^2)(-4Rr+44r^2) \leq 12r^2(4R+r)^2 \Leftrightarrow 4R^3+8R^2r-17Rr^2-30r^3 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2R)(4R^2 + 16Rr + 15r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark 2.

Inequality ***3.*** *can be developed:*

5. In ΔABC

$$\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{1}{m_a+\lambda m_b}+\frac{1}{m_b+\lambda m_c}+\frac{1}{m_c+\lambda m_a}\right) \geq \frac{2\sqrt{3}}{1+\lambda}, \text{ where } \lambda \geq 0$$

Proposed by Marin Chirciu - Romania

Proof.

We use the following lemma:

Lemma 3.
6. In ΔABC, for $\lambda \geq 0$,

$$\left(\frac{1}{m_a+\lambda m_b}+\frac{1}{m_b+\lambda m_c}+\frac{1}{m_c+\lambda m_a}\right)^2 \geq$$

$$\geq \frac{108}{(5\lambda^2+11\lambda+5)p^2 - (3\lambda^2+9\lambda+3)r^2 - (12\lambda^2+36\lambda+12)Rr}$$

Proof.

Using the inequality $(x+y+z)^2 \geq 3(xy+yz+zx)$,

with $x = \dfrac{1}{m_a+\lambda m_b}, y = \dfrac{1}{m_b+\lambda_c}, z = \dfrac{1}{m_c+\lambda m_a}$ *we obtain*

$$\left(\sum \frac{1}{m_b+\lambda m_c}\right)^2 \geq 3\left(\sum \frac{1}{(m_a+\lambda m_b)(\lambda m_a+m_c)}\right) \overbrace{\geq}^{Bergstrom} 3\cdot\frac{9}{\sum(m_a+\lambda m_b)(\lambda m_a+m_c)} =$$

$$= \frac{27}{\sum(\lambda m_a^2 + \lambda^2 m_a m_b + \lambda m_b m_c + m_c m_a)} = \frac{27}{\lambda \sum m_a^2 + (\lambda^2+\lambda+1)\sum m_b m_c} \overbrace{\geq}^{(1)}$$

$$\overbrace{\geq}^{(1)} \frac{27}{\frac{3\lambda}{4}\sum a^2 + \frac{\lambda^2+\lambda+1}{4}\sum(2a^2+bc)} = \frac{108}{(2\lambda^2+5\lambda+2)\sum a^2 + (\lambda^2+\lambda+1)\sum bc} =$$

$$= \frac{108}{(2\lambda^2+5\lambda+2)\cdot 2(p^2-r^2-4Rr) + (\lambda^2+\lambda+1)(p^2+r^2+4Rr)} =$$

$$= \frac{108}{(5\lambda^2+11\lambda+5)p^2 - (3\lambda^2+9\lambda+3)r^2 - (12\lambda^2+36\lambda+12)Rr}$$

where inequality (1) follows from $4m_b m_c \leq 2a^2 + bc$ *and analogs.*

Equality holds if and only if the triangle is equilateral.

$\square$

Let's pass to solve inequality ***5.***

Using ***Lemma 3*** *and* $\sum a = 2p, \sum a^2 = 2(p^2 - r^2 - 4Rr)$ *it is enough to prove that*

$$\left(\frac{2(p^2 - r^2 - 4Rr)}{2p}\right)^2 \cdot \frac{108}{(5\lambda^2 + 11\lambda + 5)p^2 - (3\lambda^2 + 9\lambda + 3)r^2 - (12\lambda^2 + 36\lambda + 12)Rr} \geq \frac{12}{1+\lambda}$$

$$\Leftrightarrow 9(\lambda+1)^2(p^2 - r^2 - 4Rr)^2 \geq p^2((5\lambda^2+11\lambda+5)p^2 - (36\lambda^2+9\lambda+3)r^2 - (12\lambda^2+36\lambda+12))$$

$$\Leftrightarrow p^2((4\lambda^2 + 7\lambda + 4)p^2 - 3(5\lambda^2 + 9\lambda + 5)r(4Rr + r)) + 9(\lambda + 1)^2 r^2 (4R + r)^2 \geq 0$$

We distinguish the following cases:

Case 1. If $(4\lambda^2 + 7\lambda + 4)p^2 - 3(5\lambda^2 + 9\lambda + 5)r(4R + r) \geq 0$ *inequality is obvious.*

Case 2. If $(4\lambda^2 + 7\lambda + 4)p^2 - 3(5\lambda^2 + 9\lambda + 5)r(4R + r) < 0$ *we write the following inequality:*

$$p^2(3(5\lambda^2 + 9\lambda + 5)r(4R + r) - (4\lambda^2 + 7\lambda + 4)p^2) \leq 9(\lambda + 1)^2 r^2 (4R + r)^2$$

which follows from Blundon-Gerretsen's inequality $16Rr - 5r^2 \leq p^2 \leq \dfrac{R(4R + r)^2}{2(2R - r)}$

It remains to prove that:

$$\frac{R(4R + r)^2}{2(2R - r)} \cdot (3(5\lambda^2 + 9\lambda + 5)r(4R + r) - (4\lambda^2 + 7\lambda + 4)p^2) \leq 9(\lambda + 1)^2 r^2 (4R + r)^2 \Leftrightarrow$$

$$\Leftrightarrow (4\lambda^2 + 4\lambda + 4)R^2 + (\lambda^2 + 10\lambda + 1)Rr - (18\lambda^2 + 36\lambda + 18)r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R - 2r)((4\lambda^2 + 4\lambda + 4)R + (9\lambda^2 + 18\lambda + 9)r) \geq 0$, *obviously from* $R \geq 2r$ *(Euler).*

Equality holds if and only if the triangle is equialteral.

$\square$

Note.

For $\lambda = 0$ *in inequality* ***5.*** *we obtain inequality* ***1.***, *and for the case* $\lambda = 1$ *we obtain* ***3.***

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