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1. In ΔABC

$$\left(1+\frac{1}{m_a}\right)\left(1+\frac{1}{m_b}\right)\left(1+\frac{1}{m_c}\right)\geq\left(1+\frac{2}{3R}\right)^3.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

Using Huygens' inequality we obtain

$$\left(1+\frac{1}{m_a}\right)\left(1+\frac{1}{m_b}\right)\left(1+\frac{1}{m_c}\right)\geq\left(1+\sqrt[3]{\frac{1}{m_am_bm_c}}\right)^3\geq\left(1+\frac{2}{3R}\right)^3,$$

where the last inequality is equivalent with:

$$\sqrt[3]{\frac{1}{m_am_bm_c}}\geq\frac{2}{3R}\Leftrightarrow\frac{1}{m_am_bm_c}\geq\left(\frac{2}{3r}\right)^3\Leftrightarrow m_am_bm_c\leq\left(\frac{3R}{2}\right)^3$$

which follows from means inequality and the known inequality in triangle $\sum m_a\leq 4R+r\leq\frac{9R}{2}$;

indeed: $m_am_bm_c\leq\left(\frac{m_a+m_b+m_c}{3}\right)^3\leq\left(\frac{4R+r}{3}\right)^3\leq\left(\frac{3R}{2}\right)^3$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same way it can be proposed:

2. In ΔABC

$$\left(1+\frac{1}{m_a+m_b}\right)\left(1+\frac{1}{m_b+m_c}\right)\left(1+\frac{1}{m_c+m_a}\right)\geq\left(1+\frac{1}{3R}\right)^3$$

Proof.

Using Huygens' inequality we obtain

$$\left(1+\frac{1}{m_a+m_b}\right)\left(1+\frac{1}{m_b+m_c}\right)\left(1+\frac{1}{m_c+m_a}\right)\geq\left(1+\sqrt[3]{\frac{1}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}}\right)^3\geq\left(1+\frac{1}{3R}\right)^3$$

where the last inequality is equivalent with:

$$\sqrt[3]{\frac{1}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}}\geq\frac{1}{3R}\Leftrightarrow\frac{1}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}\geq\left(\frac{1}{3R}\right)^3\Leftrightarrow$$

$$(m_a+m_b)(m_b+m_c)(m_c+m_a)\leq(3R)^3$$

which follows from means inequality and the known inequality in triangle

$$\sum m_a \leq 4R + r \leq \frac{9R}{2}; \textit{ indeed:}$$

$$(m_a + m_b)(m_b + m_c)(m_c + m_a) \leq \Big(\frac{2(m_a + m_b + m_c)}{3}\Big)^3 \leq \Big(\frac{2(4R + r)}{3}\Big)^3 \leq (3R)^3$$

Equality holds if and only if the triangle is equilateral.

□

3. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{\Big(1+\frac{1}{m_a + \lambda m_b}\Big)\Big(1+\frac{1}{m_b + \lambda m_c}\Big)\Big(1+\frac{1}{m_c + \lambda m_a}\Big) \geq \Big(1+\frac{2}{3(\lambda + 1)R}\Big)^3}, \textit{\textbf{ where }} \boldsymbol{\lambda \geq 0}$$

Proposed by Marin Chirciu - Romania

Proof.

Using Huygens' inequality we obtain

$$\Big(1+\frac{1}{m_a + \lambda m_b}\Big)\Big(1+\frac{1}{m_b + \lambda m_c}\Big)\Big(1+\frac{1}{m_c + \lambda m_a}\Big) \geq \Big(1+\sqrt[3]{\frac{1}{(m_a + \lambda m_b)(m_b + \lambda m_c)(m_c + \lambda m_a)}}\Big)^3 \geq$$

$$\geq \Big(1 + \frac{2}{3(\lambda + 1)R}\Big)^3$$

where the last inequality is equivalent with:

$$\sqrt[3]{\frac{1}{(m_a + \lambda m_b)(m_b + \lambda m_c)(m_c + \lambda m_a)}} \geq \frac{1}{3R} \Leftrightarrow \frac{1}{(m_a + \lambda m_b)(m_b + \lambda m_c)(m_c + \lambda m_a)} \geq \Big(\frac{2}{3(\lambda + 1)R}\Big)^3 \Leftrightarrow$$

$$(m_a + \lambda m_b)(m_b + \lambda m_c)(m_c + \lambda m_a) \leq \Big(\frac{3(\lambda + 1)R}{2}\Big)^3$$

which follows from means inequality and the known inequality in triangle

$$\sum m_a \leq 4R + r \leq \frac{9R}{2}; \textit{ indeed:}$$

$$(m_a+\lambda m_b)(m_b+\lambda m_c)(m_c+\lambda m_a) \leq \Big(\frac{(1 + \lambda)(m_a + m_b + m_c)}{3}\Big)^3 \leq \Big(\frac{(1 + \lambda)(4R + r)}{3}\Big)^3 \leq \Big(\frac{3(\lambda + 1)R}{2}\Big)^3$$

Equality holds if and only if the triangle is equilateral.

□

Note

*For $\lambda = 0$ we obtain inequality **1.**, and for $\lambda = 1$ we obtain inequality **2**.*

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