RMM - Calculus Marathon 101 - 200

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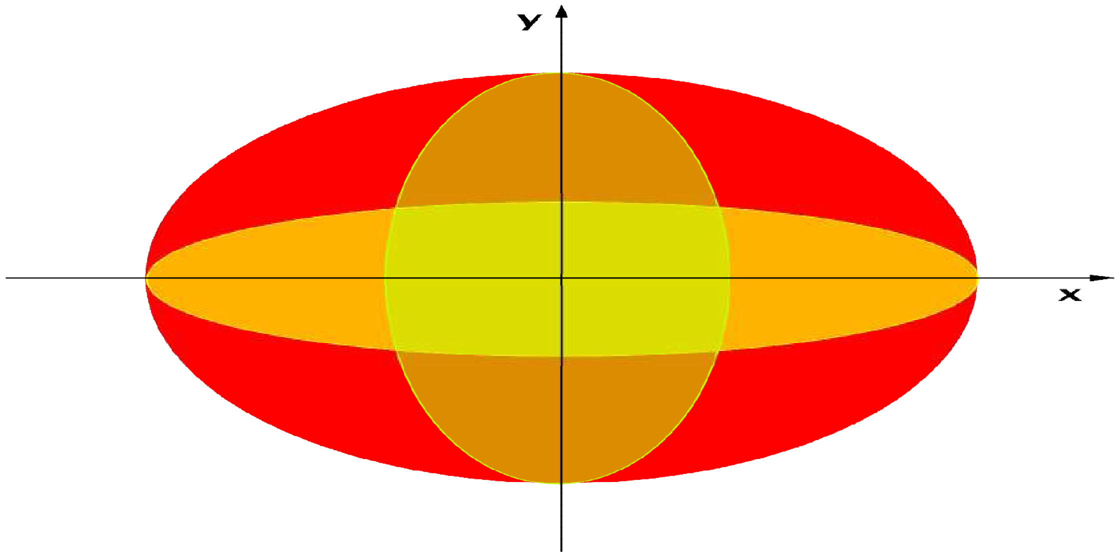
101.

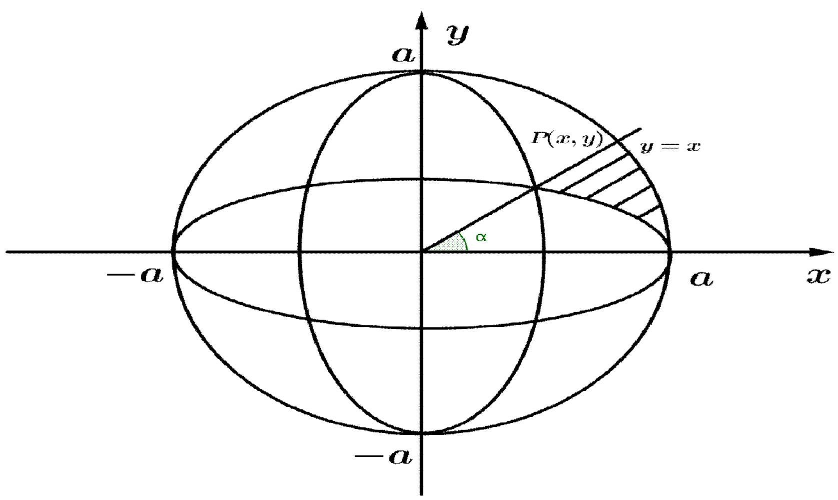


$E_1: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1; E_2: \frac{x^2}{b^2}+\frac{y^2}{a^2}=1, a>b$. **Find yellow area and red area.**

Proposed by Daniel Sitaru – Romania

Solution by Igor Soposki-Skopje

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \wedge \frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \Rightarrow \begin{matrix} x_1=\frac{ab}{\sqrt{a^2+b^2}} \\ y_1=\frac{ab}{\sqrt{a^2+b^2}} \end{matrix}; P(x_1,y_1)=\left(\frac{ab}{\sqrt{a^2+b^2}};\frac{ab}{\sqrt{a^2+b^2}}\right), \alpha=\frac{\pi}{4}$$



$$P_{red}=8\cdot\left[\frac{a^2\pi}{8}-\int_0^{x_1}x\,dx-\int_0^{x_1}b\sqrt{1-\frac{x^2}{a^2}}dx\right]$$

$$I_1=\int_0^{x_1}x\,dx=\frac{x_1^2}{2}=\frac{(ab)^2}{2(a^2+b^2)}$$

$$I_2=b\cdot\int_{x_1}^{a}\sqrt{1-\frac{x^2}{a^2}}=\begin{Bmatrix}x=a\sin t\\ dx=a\cos t\,dt\end{Bmatrix}=ab\int\cos^2 t\,dt=$$

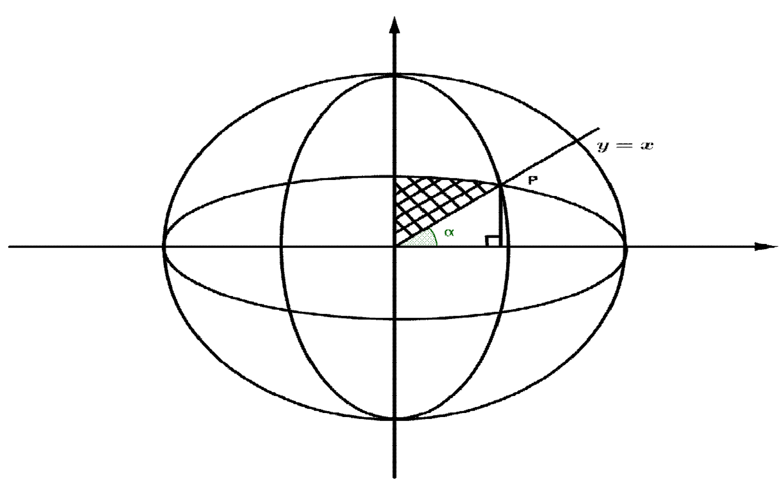
$$=ab\int\frac{1+\cos 2t}{2}dt=ab\left[\frac{t}{2}+\frac{\sin 2t}{4}\right]=\frac{ab}{2}\cdot\left[\arcsin\frac{x}{a}+\frac{x}{a}\cdot\sqrt{1-\frac{x^2}{a^2}}\right]\Big|_{x_1}^{a}=$$

$$=\frac{ab}{2}\cdot\left[\arcsin 1-\arcsin\frac{b}{\sqrt{a^2+b^2}}+0-\frac{ab}{a^2+b^2}\right]=$$

$$=\frac{ab}{2}\cdot\left[\frac{\pi}{2}-\arcsin\frac{b}{\sqrt{a^2+b^2}}-\frac{ab}{a^2+b^2}\right]\Rightarrow$$

$$P_{red}=a^2\pi-\frac{4(ab)^3}{a^2+b^2}-2(ab)\pi+4ab\arcsin\frac{b}{\sqrt{a^2+b^2}}+\frac{4(ab)^2}{a^2+b^2}=$$

$$=a^2\pi-2(ab)\pi+4ab\cdot\arcsin\frac{b}{\sqrt{a^2+b^2}}$$



$$P(x_1, y_1),\ x_1 = \frac{ab}{\sqrt{a^2+b^2}}, y_1 = \frac{ab}{\sqrt{a^2+b^2}}$$

$$P_{yellow} = 8 \cdot \left[\int_0^{x_1} b\sqrt{1-\frac{x^2}{a^2}}\,dx - \int_0^{x_1} x\,dx\right] =$$

$$= 8 \cdot \left[\frac{ab}{2}\left[\arcsin\frac{x}{a} + \frac{x}{a} \cdot \sqrt{1-\frac{x^2}{a^2}}\right]\Big|_0^{x_1} - \frac{x_1^2}{2}\right] =$$

$$= 4ab \cdot \left\{\arcsin\frac{b}{\sqrt{a^2+b^2}} + \frac{ab}{a^2+b^2}\right\} - 4\frac{(ab)^2}{a^2+b^2} =$$

$$= 4ab \cdot \arcsin\frac{b}{\sqrt{a^2+b^2}} + \frac{4(ab)^2}{a^2+b^2} - \frac{4(ab)^2}{a^2+b^2}$$

$$P_{yellow} = 4ab\arcsin\frac{b}{\sqrt{a^2+b^2}}$$

102. Solve in natural numbers the following equation:

$$\frac{1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + n^2(n+1)! - 2}{(n+1)!} = 108$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo, Solution 2 by Ravi Prakash-New Delhi-India, Solution 3 by Rovsen Pirguliyev-Sumgait-Azerbaidian, Solution 4 by Shivam Sharma-New Delhi-India, Solution 5 by Sujeetran Balendran-Sri Lanka, Solution 6 by Kunihiko Chikaya-Tokyo-Japan

Solution 1 by Bedri Hajrizi-Nis-Serbia

Let $S(k) = 1^1 \cdot 2! + 2^2 \cdot 3! + \cdots + k^2(k+1)!\,;\ S(1) = 1^2 \cdot 2 = 2$

$$S(2)=1^2\cdot 2!+2^2\cdot 3!=2+4!$$

$$S(3)=2+4!+3^2\cdot 4!=2+10\cdot 4!=2+2\cdot 5!$$

$$S(4)=2+2\cdot 5!+4^2\cdot 5!=2+18\cdot 5!=2+3\cdot 6!$$

Suppose that $S(n)=2+(n-1)(n+2)!$

We must proof that $S(n+1)=2+n(n+3)!$

Readly: $S(n+1)=S(n)+(n+1)^2(n+2)!=$

$$=2+(n-1)(n+2)!+(n+1)(n+2)!=$$

$$=2+(n^2+2n+1+n-1)(n+2)!=$$

$$=2+(n^2+3n)(n+2)!=2+n(n+3)! \quad \textbf{Q.E.D.}$$

So: $1^2\cdot 2!+2^2\cdot 3!+\cdots+(n-1)^2n!=2+(n-1)(n+2)!$

$$\frac{1^2\cdot 2!+2^2\cdot 3!+\cdots+n^2(n+1)!-2}{(n+1)!}=108$$

$$\frac{2+(n-1)(n+2)!-2}{(n+1)!}=108;\ (n-1)(n+2)=9\cdot 12;\ n=10$$

Solution 2 by Ravi Prakash-New Delhi-India

For $r\ge 1$, ***write*** $r^2\equiv(r+3)(r+2)+A(R+2)+B$

Put $r=-2, 4=B$; ***Put*** $r=-3, 9=-A+B\Rightarrow A=-5$

$$\therefore r^2\equiv(r+3)(r+2)-5(r+2)+4$$

$$\Rightarrow r^2(r+1)!=(r+3)!-5(r+2)!+4(r+1)!$$

$$=\big((r+3)!-(r+2)!\big)-4\big((r+2)!-(r+1)!\big)$$

$$\Rightarrow\sum_{r=1}^{n}r^2(r+1)!=\big((n+3)!-3!\big)-4\big((n+2)!-2!\big)$$

$$=(n+3)!-4(n+2)!+2$$

$$\therefore \sum_{k=1}^{n} k^2(k+1)! - 2 = (n+2)!\,(n+3-4) = (n+2)!\,(n-1)$$

$$\therefore \frac{\sum_{k=1}^{n}(k^2)\,(k+1)! - 2}{(n+1)!} = 108$$

$$\Rightarrow (n+2)(n-1) = 108 \Rightarrow n^2 + n - 110 = 0$$

$$\Rightarrow (n+11)(n-10) = 0. \text{ As } n \in \mathbb{N}, n = 10$$

Solution 3 by Rovsen Pirguliyev-Sumgait-Azerbaidian

$$\frac{\sum_{k=1}^{n} k^2(k+1)! - 2}{(n+1)!} = 108, \sum_{k=1}^{n} k^2(k+1)! = (n-1)(n+2)! + 2,$$

$$\text{then } \frac{(n-1)(n+2)!+2-2}{(n+1)!} = \frac{(n-1)(n+2)!}{(n+1)!} = (n-1)(n+2),$$

$$(n-1)(n+2) = 108 \Rightarrow n = 10$$

Solution 4 by Shivam Sharma-New Delhi-India

$$\frac{[\sum_{j=1}^{n}(j^2)(j+1)!]-2}{(n+1)!} = 108. \text{ Applying partial sum, we get,}$$

$$\frac{\Gamma(n+3)(n-1)+2-2}{(n+1)!} = 108;\ \frac{(n+2)!\,(n-1)+2-2}{(n+1)!} = 108$$

$$\frac{(n+2)(n+1)!\,(n-1)+2-2}{(n+1)!} = 108;\ (n+2)(n-1) = 108$$

$$n^2 + 2n - 2 = 108; n^2 + 2n - 110 = 0;\ n = \frac{-2+\sqrt{4+440}}{2}$$

We get, $n = 10$ ***[Valid];*** $n = -10$ ***[Invalid]. Hence,*** $n = 10$

Solution 5 by Sujeetran Balendran-Sri Lanka

$$\sum_{k=1}^{n}(r+x)!\,(r+x) \text{ [Theory]; } f(n) = (r+x+1)!\,;\ f(r) = (r+x)!$$

$$f(r+1) - f(r) = (r+x+1)! - (r+x)!$$

$$= (r+x)!\,[r+x+1-1] = (r+x)!\,(r+x)$$

$$\sum_{k=1}^{n}(r+x)!\,(r+x)=f(r+1)-f(1)=(r+x+1)!-(x+1)!$$

My solution $108=\frac{\sum_{k=1}^{n} r^2(r+1)!-2}{(n+1)!}$

$$V_r=r^2(r+1)!=(r^2+4r+4)(r+1)!-4(r+1)(r+1)!$$

$$V_r=(r+2)(r+2)!-4(r+1)(r+1)!$$

$$\sum_{k=1}^{n}V_r=\sum_{k=1}^{n}(r+2)(r+2)!-4\sum_{k=1}^{n}(r+1)(r+1)!$$

$$=(n+3)!-6-4(n+2)!+8=(n+3)!-4(n+2)+2$$

$$108=\frac{\sum_{k=1}^{n}V_r-2}{(n+1)!}=\frac{(n+3)!-4(n+2)!}{(n+1)!}=108$$

$$n^2+5n+6-4n-8-108=0;\ n^2+n-110=0$$

$$(n+11)(n-10)=0;\ n=10,n=-11$$

Solution 6 by Kunihiko Chikaya-Tokyo-Japan

Solve in $n\in\mathbb{N}$***; (*)*** $\frac{1^2\cdot 2!+2^2\cdot 3!+\cdots+n^2(n+1)!-2}{(n+1)!}=108$***. Ans.*** $n=10$

$$k^2(k+1)!=\{(k+2)^2-4(k+1)\}(k+1)!$$

Telescopic sum

$$=(k+2)(k+2)!-4(k+2-1)(k+1)!$$

$$=(k+3-1)(k+2)!-4(k+2-1)(k+1)!$$

$$=(k+3)!-(k+2)!-4\{(k+2)!-(k+1)!\}$$

$$\therefore\sum_{k=1}^{n}k^2(k+1)!=(n+3)!-3!-4\{(n+2)!-2!\}$$

$$=(n+3)!-4(n+2)!+2=(n+2)!\,(n+3-4)+2=$$

$$=(n-1)(n+2)(n+1)!+2$$

$\therefore$ ***(*)*** $\Leftrightarrow (n+2)(n-1) = 108$ *increase monotonous*

$n = 11 \ldots = 130x;\ n = 10 \ldots = 108$

103. **Find** $n \in \mathbb{N}, n > 1$**:**

$$\frac{2!\,(2^3-1) + 3!\,(3^3-1) + \cdots + n!\,(n^3-1) - 2}{n^2-2} = 40320$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Carlos Suarez-Quito-Ecuador, Solution 2 by Kunihiko Chikaya-Tokyo-Japan

Solution 1 by Carlos Suarez-Quito-Ecuador

$$\sum_{k=1}^{n}(k^3-1)k! = (n^2-2)(n+1)! + 2;\ \frac{(n^2-2)(n+1)! + 2 - 2}{(n^2-2)} = 40320$$

$$\frac{(n^2-2)(n+1)!}{(n^2-2)} = 40320;\ (n+1)! = 40320;\ n = 7$$

Solution 2 by Kunihiko Chikaya-Tokyo-Japan

Find $n \geq 2$ ***such that (*)*** $\frac{2!(2^3-1)+3!(3^3-1)+\cdots+n!(n^3-1)-2}{n^2-2} = 40320$

$$\sum_{k=1}^{n}(k^3-1)k! = \sum_{k=1}^{n}\{f(k) - f(k-1)\} = f(n) - f(0)$$

$$= (n^2-2)(n+1)! + 2$$

$$f(k) = (k^3 + k^2 - 2k - 2)k! = (k^2-2)(k+1)!$$

$$\therefore (*) \Leftrightarrow (n+1)! = 8!$$

$$\therefore n = 7$$

104. **Find $x, y, z \in \mathbb{N}^*$ such that:**

$$\sqrt{\underbrace{xxxx\ldots xx}_{for\ "2000"\ times} - \underbrace{yyyy\ldots y}_{for\ "1000"\ times}} = \underbrace{zzzz\ldots zz}_{for\ "1000"\ times}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hasan Bostanlik-Sarkisla-Turkey, Solution 2 by Boris Colakovic-Belgrade-Serbia, Solution 3 by Khanh Hung Vu-Ho Chi Minh-Vietnam

Solution 1 by Hasan Bostanlik-Sarkisla-Turkey

$$x \cdot \frac{10^{2000}-1}{9} - y \cdot \frac{10^{1000}-1}{9} = \frac{z^2 \cdot (10^{1000}-1)^2}{81}$$

$$10^{1000} = k \Rightarrow x \cdot \frac{(k^2-1)}{9} - y \cdot \frac{(k-1)}{9} = \frac{z^2(k-1)^2}{81}$$

$$x(k+1) - y = \frac{z^2 \cdot (k-1)}{9};\ 9x(k+1) - 9y = z^2 \cdot k - z^2$$

$$k(z^2 - 9x) = z^2 + 9x - 9y\ \{z^2 \neq 9x \Rightarrow k(z^2 - 9x) > z^2 + 9x - 9y\}$$

$$z^2 = 9x \Rightarrow x = 1, z = 3, y = 2;\ x = 4, z = b, y = 8$$

Solution 2 by Boris Colakovic-Belgrade-Serbia

$$\sqrt{\underbrace{xxxx\ldots xx}_{for\ "2000"\ times} - \underbrace{yyyy\ldots y}_{for\ "1000"\ times}} = \sqrt{x\underbrace{(111\ldots 11)}_{2000} - y\underbrace{(111\ldots 11)}_{1000}} = z\underbrace{(111\ldots 11)}_{1000} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{\frac{10^{2000}-1}{9} \cdot x - \frac{10^{1000}-1}{9} y} = \frac{10^{1000}-1}{9} \cdot z \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{3}\sqrt{x \cdot 10^{2000} - y \cdot 10^{1000} + y - x} = \frac{10^{1000}-1}{9} z \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{3}\sqrt{\left(\sqrt{x}10^{1000} - \frac{y}{2\sqrt{x}}\right)^2 - \frac{(2x-y)^2}{4x}} = \frac{10^{1000}-1}{9} \cdot z \Rightarrow$$

$$\Rightarrow y = 2k^2, x = k^2 \Rightarrow \frac{1}{3}\sqrt{(k \cdot 10^{1000} - k)^2} = \frac{10^{1000}-1}{9} \cdot z \Leftrightarrow$$

$$\Leftrightarrow \frac{k}{3}\cdot\frac{10^{1000}-1}{9}=\frac{10^{1000}-1}{9}\cdot z\Rightarrow z=3k$$

Solutions are $(x,y,z)=(k^2,2k^2,3k)\ \ k\in N$

Solution 3 by Khanh Hung Vu-Ho Chi Minh-Vietnam

$$\sqrt{\overline{\underbrace{xxxx\dots xx}_{for\ "2000"\ times}}-\overline{\underbrace{yyyy\dots y}_{for\ "1000"\ times}}}=\overline{\underbrace{zzzz\dots zz}_{for\ "1000"\ times}}\quad \textbf{(1)}$$

We have $\overline{\underbrace{xxxx\dots xx}_{for\ "2000"\ times}}=x(10^{1999}+10^{1998}+\dots+10+1)=x\cdot\frac{10^{2000}-1}{10-1}$

Similalry, we have $\overline{\underbrace{yyyy\dots y}_{for\ "1000"\ times}}=y\cdot\frac{10^{1000}-1}{10-1}$ ***and***

$$\overline{\underbrace{zzzz\dots zz}_{for\ "1000"\ times}}=z\cdot\frac{10^{1000}-1}{10-1}$$

We have (1) $\Rightarrow\sqrt{x\cdot\frac{10^{2000}-1}{10-1}-y\cdot\frac{10^{1000}-1}{10-1}}=z\cdot\frac{10^{1000}-1}{10-1}$

$$\Rightarrow x\cdot\frac{10^{2000}-1}{10-1}-y\cdot\frac{10^{1000}-1}{10-1}=\left(z\cdot\frac{10^{1000}-1}{10-1}\right)^2\Rightarrow$$

$$\Rightarrow\frac{x(10^{2000}-1)-y(10^{1000}-1)}{9}=\frac{z^2(10^{1000}-1)^2}{81}$$

$$\Rightarrow 9[x(10^{2000}-1)-y(10^{1000}-1)]=z^2(10^{1000}-1)^2\Rightarrow$$

$$\Rightarrow 9[x(10^{1000}+1)-y]=z^2(10^{1000}-1)\Rightarrow(9x-z^2)\cdot 10^{1000}=-z^2-9x+9y\quad \textbf{(2)}$$

We have $-81\le -z^2\le -1,-81\le -9x\le -9$ ***and*** $9\le 9y\le 81$

$$\Rightarrow -153\le -z^2-9x+9y\le 73\Rightarrow -153\le -z^2-9x+9y\le 73\Rightarrow$$

$$\Rightarrow -153\le(9x-z^2)\cdot 10^{1000}\le 73\Rightarrow 9x=z^2$$

On the other hand, we have $1\le x\le 9$ ***and***

$$1\le z\le 9\Rightarrow(x,z)=(1;3);(4;6);(9;9)$$

*** If** $(x;z) = (1;3)$**, we have (2)** $\Rightarrow -9-9+9y=0 \Rightarrow y=2$

*** If** $(x;z) = (4;6)$**, we have (2)** $\Rightarrow -36-36+9y=0 \Rightarrow y=8$

*** If** $(x;z) = (9;9)$**, we have (2)** $\Rightarrow -81-81+9y=0 \Rightarrow y=18$ ***(Absurd)***

So, the equation (1) has 2 roots: $(x;y;z) = (1;2;2);(4;8;6)$

105. Find $n \in \mathbb{N}, n \geq 3$ **such that:**

$$\sum_{k=3}^{n}\binom{n}{k}\binom{k-1}{2} = 21(2^{n-2}-1)$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$\binom{k-1}{2} = \frac{1}{2}(k-1)(k-2) = \frac{1}{2}[k(k-1)-2k+2] = \frac{1}{2}k(k-1)-k+1$$

$$\therefore \sum_{k=3}^{n}\binom{n}{k}\binom{k-1}{2} = \sum_{k=3}^{n}\binom{n}{k}\left[\frac{1}{2}k(k-1)-k+1\right]$$

$$= \frac{1}{2}\sum_{k=3}^{n}k(k-1)\binom{n}{k} - \sum_{k=3}^{n}k\binom{n}{k} + \sum_{k=3}^{n}\binom{n}{k}$$

$$= \frac{1}{2}n(n-1)\sum_{k=3}^{n}\binom{n-2}{k-2} - n\sum_{k=3}^{n}\binom{n-1}{k-1} + \sum_{k=3}^{n}\binom{n}{k}$$

$$= \frac{1}{2}n(n-1)[2^{n-2}-1] - n\left(2^{n-1}-1(n-1)\right) + \left[2^{n}-1-n-\frac{1}{2}n(n-1)\right]$$

$$= n(n-1)2^{n-3} - \frac{1}{2}n(n-1) - n\left(2^{n^{-1}}\right) + n + n(n-1) + 2^{n} - 1 - n - \frac{1}{2}n(n-1)$$

$$= n(n-1)2^{n-3} - (n-2)2^{n-1} - 1$$

$$\therefore n(n-1)2^{n-3} - (n-2)2^{n-1} - 1 = 21(2^{n-1}-1)$$

$$\Rightarrow n(n-1)2^{n-3} - (n-2)2^{n-1} - 21(2^{n-2}) + 20 = 0$$

$$\Rightarrow n(n-1)-4(n-2)-42+20(2^{3-n})=0;$$

$$\Rightarrow n^2-5n-34+5(2^{7-n})=0\Rightarrow 5(2^{7-n})=34+5n-n^2$$

As RHS is an integer, and $n\geq 3,3\leq n\leq 7$.

But $n=3,4,5,6,7$ ***do not satisfy it . So, no solution.***

106. Solve the question in R:

$$\sqrt{x^3-2x^2+2x}+3\cdot\sqrt[3]{x^2-x+1}+2\cdot\sqrt[4]{4x-3x^4}=\frac{x^4-3x^3}{2}+7\quad (1)$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by proposer

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

Solution 1 by proposer

**** We have:*** $\begin{cases}x^3-2x^2+2x\geq 0\\ 4x-3x^4\geq 0\end{cases}\Leftrightarrow\begin{cases}x(x^2-2x+2)\geq 0\\ x(3x^3-4)\leq 0\end{cases}\Leftrightarrow$

$$\Leftrightarrow\begin{cases}x((x-1)^2+1)\geq 0\\ 0\leq x\leq\sqrt[3]{\frac{4}{3}}\end{cases}\Leftrightarrow 0\leq x\leq\sqrt[3]{\frac{4}{3}}$$

**** Because:*** $x^2-x+1=\left(x^2-x+\frac{1}{4}\right)+\frac{3}{4}=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\geq\frac{3}{4}>0$

- Therefore, since inequality AM – GM for $2,3,4$ ***real numbers:***

$$\sqrt{x^3-2x^2+2x}+3\cdot\sqrt[3]{x^2-x+1}+2\cdot\sqrt[4]{4x-3x^4}$$

$$=\sqrt{x(x^2-2x+2)}+3\cdot\sqrt[3]{(x^2-x+1)\cdot 1\cdot 1}+2\cdot\sqrt[4]{x(4-3x^3)\cdot 1\cdot 1}\leq$$

$$\leq\frac{x+x^2-2x+2}{2}+(x^2-x+1)+1+1+\frac{2(x+(4-3x^3)+1+1)}{4}$$

$$\Rightarrow\sqrt{x^3-2x^2+2x}+3\cdot\sqrt[3]{x^2-x+1}+2\cdot\sqrt[4]{4x-3x^4}\leq$$

$$\le \frac{x^2-x+2}{2}+x^2-x+3+\frac{-3x^3+x+6}{2}$$

$$\Leftrightarrow \sqrt{x^3-2x^2+2x}+3\cdot\sqrt[3]{x^2-x+1}+2\cdot\sqrt[4]{4x-3x^4}\le \frac{-3x^3+3x^2-2x+14}{2}\ \ (2)$$

- ***Since (1), (2):***

$$\Rightarrow \frac{x^4-3x^3}{2}+7\le \frac{-3x^3+3x^2-2x+14}{2}\Leftrightarrow \frac{x^4-3x^3+14}{2}\le \frac{-3x^3+3x^2-2x+14}{2}$$

$$\Leftrightarrow x^4-3x^3+14\le -3x^3+3x^2-2x+14\Leftrightarrow x^4-3x^2+2x\le 0$$

$$\Leftrightarrow x(x^3-3x+2)\le 0$$

$$\Leftrightarrow x\left(x^2(x-1)+x(x-1)-2(x-1)\right)\le 0\Leftrightarrow x(x-1)(x^2+x-2)\le 0\Leftrightarrow$$

$$\Leftrightarrow x(x+2)(x-1)^2\le 0\ \ (3)$$

- ***Other,*** $x\ge 0, x(x+2)\ge 0$***. That*** $(x-1)^2\ge 0;\forall x\in R$ ***therefore***

$$x(x+2)(x-1)^2\ge 0\ \ (4)$$

**** Since (3), (4)***: $\Rightarrow x(x+2)(x-1)^2=0\Leftrightarrow \begin{cases} x=x^2-2x+2\\ x^2-x+1=1\\ x=4-3x^3=1\\ x(x+2)(x-1)^2=0\end{cases}\Leftrightarrow$

$$\Leftrightarrow \begin{cases}(x-1)(x-2)=0\\ x(x-1)=0\\ 3x^3+x-4=0; x=1\\ x(x+2)(x-1)^2=0\end{cases}\Leftrightarrow x=1$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sqrt{x^3-2x^2+2x}+3\cdot\sqrt[3]{x^2-x+1}+2\cdot\sqrt[4]{4x-3x^4}=\frac{x^4-3x^3}{2}+7\ \ (*)$$

$$D(x):\begin{cases}x^3-2x^2+2x\ge 0\\ 4x-3x^4\ge 0\end{cases}\Leftrightarrow 0<x\le\sqrt[3]{\frac{4}{3}}\ \ (1)$$

$$D(x): x\in\left]0;\sqrt[3]{\frac{4}{3}}\right]$$

I. $\sqrt{x\cdot(x^2-2x+2)}\le\frac{x^2-x+2}{2}\left[\begin{array}{c}AM=GM\\ x=x^2-2x+2\\ \left.\begin{array}{c}x^2=3x+2=0\\(*)\end{array}\right\}\Rightarrow x=1\end{array}\right]$

II. $3\sqrt[3]{1\cdot1\cdot(x^2-x+1)}\le x^2-x+3\left[\begin{array}{c}AM=GM\\ x^2-x+1=1\\ \left.\begin{array}{c}x^2-x=0\\(*)\end{array}\right\}\Rightarrow x=1\end{array}\right]$

$$\textit{III.}\quad 2\sqrt[4]{4x-3x^4}=2\cdot\sqrt[4]{x\cdot(4-3x^3)\cdot1\cdot1}\le$$

$$\le\frac{6+x-3x^3}{2}\left[\begin{array}{c}AM=GM\\ \left.\begin{array}{l}x=1\\4-3x^3=1\\4-3x^3=x\end{array}\right\}\overset{(1)}{\Rightarrow}x=1\end{array}\right]$$

IV. ***(*)***$\Rightarrow\frac{x^4-3x^3}{2}+7\quad\underset{\Downarrow}{\le}\quad\frac{x^2-x+2}{2}+(x^2-x+3)+\frac{6+x-3x^3}{2}$

$$0\ge(x-1)^2\cdot(x+2)\overset{(1)}{\Rightarrow}$$

$$(x-1)^2\cdot(x+2)=0\Rightarrow x=1$$

I; II; III; IV $\Rightarrow x=1$. ***Done***

107. Solve for real numbers:

$$\arcsin[x]\cdot\arccos[x]=\frac{\pi x}{2}-x^2?$$

Proposed by Rovsen Pirguliev-Sumgait-Azerbaidian

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia, Solution 2 by Ravi Prakash-New Delhi-India, Solution 3 by Soumava Chakraborty-Kolkata-India

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\left.\begin{array}{l}\cos y=[x]\\ \sin y=[x]\end{array}\right\}\Rightarrow -1\le[x]\le+1;\ [x]\in\{-1;0;1\}$$

1) ***If*** $[x] = -1 \Rightarrow$ ***is*** $\arcsin[x] \cdot \arccos[x] = \arcsin(-1) \cdot \arccos(-1) =$

$$= \frac{3\pi}{2} \cdot \pi = \frac{3\pi^2}{2} = LHS$$

$$\frac{3\pi^2}{2} = \frac{\pi}{2} \cdot x - x^2 \Leftrightarrow x^2 - \frac{\pi}{2} \cdot x + \frac{3\pi^2}{2} = 0 \Rightarrow D < 0, x \in \emptyset$$

2) ***If*** $[x] = 0$ ***is*** $\arcsin[x] \cdot \arccos[x] = \arcsin 0 \cdot \arccos 0 =$

$$= 0 \cdot \frac{\pi}{2} = 0 = LHS$$

$$0 = \frac{\pi}{2} \cdot x - x^2 \Rightarrow \left.\begin{array}{c} x_1 = 0 \\ x_2 = \frac{\pi}{2} \Rightarrow [x] \neq 0 \Rightarrow \end{array}\right\} \Rightarrow x = 0$$

3) ***If*** $[x] = +1$ ***is*** $\arcsin 1 \cdot \arccos 1 = \frac{\pi}{2} \cdot 0 = 0$

$$0 = x \cdot \left(\frac{\pi}{2} - x\right) \Rightarrow \left.\begin{array}{c} x = 0 \Rightarrow [x] \neq 1 \\ x = \frac{\pi}{2} \cdot \left[\frac{\pi}{2}\right] = 1 \end{array}\right\} x = \frac{\pi}{2};\ x = 0; x = \frac{\pi}{2}$$

Solution 2 by Ravi Prakash-New Delhi-India

If $[x]$ = ***greatest integer then,*** $[x] = -1, 0, 1$

1. $[x] = -1, -1 \le x < 0$, ***the equation becomes,***

$$\left(-\frac{\pi}{2}\right)\pi = \frac{\pi}{2}x - x^2 \Rightarrow x^2 - \frac{\pi}{2}x - \frac{\pi^2}{2} = 0$$

$$\Rightarrow x = \frac{\frac{\pi}{2} \pm \sqrt{\frac{\pi^2}{4} + 2\pi^2}}{2} = \frac{\pi \pm 3\pi}{4} = \pi, -\frac{\pi}{2}.$$ ***Not possible***

2. ***For*** $[x] = 0, 0 \le x < 1$. ***The equation becomes***

$$0 = \frac{\pi}{2}x - x^2 \Rightarrow x = 0 \text{ or } x = \frac{\pi}{2}$$

3. ***For*** $[x] = 1, 1 \le x < 2$, ***The equation becomes***

$0 = \frac{\pi}{2}x - x^2 \Rightarrow x = 0$ ***or*** $x = \frac{\pi}{2} \therefore$ ***in this case solution is*** $\left\{0, \frac{\pi}{2}\right\}$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\sin^{-1}[x]\cdot\cos^{-1}[x]=\frac{\pi x}{2}-x^2 \rightarrow \textit{Solve } -1\le[x]\le 1 \Rightarrow [x]=-1,0,1$$

Case 1) $[x]=-1 \Rightarrow -\le x<0$

$\therefore$ ***given equlity becomes:*** $\sin^{-1}(-1)\cdot\cos^{-1}(-1)=\frac{\pi x}{2}-x^2$

$$\Rightarrow\left(-\frac{\pi}{2}\right)(\pi)=\frac{\pi x}{2}-x^2\Rightarrow -\pi^2=\pi x-2x^2\Rightarrow 2x^2-\pi x-\pi^2=0$$

$$\Rightarrow x=\frac{\pi\pm\sqrt{\pi^2-4(2)(-\pi^2)}}{4}=\frac{\pi\pm 3\pi}{4}=-\frac{\pi}{2},\pi$$

But $-1\le x<0\Rightarrow$ ***no sol in this case***

Case 2) $[x]=0\Rightarrow 0\le x<1$

$\therefore$ ***given equality becomes:*** $\sin^{-1}(0)\cdot\cos^{-1}(0)=\frac{\pi x}{2}-x^2$

$$\Rightarrow x\left(\frac{\pi}{2}-x\right)=0\Rightarrow x=0\ \left(\because x\ne\frac{\pi}{2} \textit{ as } 0\le x<1\right)$$

Case 3) $[x]=1\Rightarrow 1\le x<2$

$\therefore$ ***given equality becomes:*** $\sin^{-1}(1)\cos^{-1}(1)=\frac{\pi x}{2}-x^2$

$$\Rightarrow x\left(\frac{\pi}{2}-x\right)=0\Rightarrow x=\frac{\pi}{2} \textit{ as } 1\le x<2 \therefore \textit{solutions are: } x=0,\frac{\pi}{2}$$

108. Find $x,y,z\in\mathbb{R}^*$ such that:

$$\frac{x^2}{1+x^2}+\frac{y^2}{(1+x^2)(1+y^2)}+\frac{z^2}{(1+x^2)(1+y^2)(1+z^2)}+\frac{1}{8xyz}=1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 2 by Nguyen Thanh Nho-Tra Vinh-Vietnam

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

$$\frac{x^2(1+y^2)+y^2}{(1+x^2)(1+y^2)}+\frac{z^2}{(1+x^2)(1+y^2)(1+z^2)}+\frac{1}{8xyz}=1$$

$$\Leftrightarrow \frac{(x^2y^2+x^2+y^2)(z^2+1)+z^2}{(1+x^2)(1+y^2)(1+z^2)}+\frac{1}{8xyz}=1 \Leftrightarrow \frac{(x^2+1)(y^2+1)(z^2+1)}{(x^2+1)(y^2+1)(z^2+1)}=1-\frac{1}{8xyz}$$

$$\Leftrightarrow \frac{1}{(x^2+1)(y^2+1)(z^2+1)}=\frac{1}{8xyz} \Leftrightarrow (x^2+1)(y^2+1)(z^2+1)=8xyz$$

By AM-GM $(x^2+1)(y^2+1)(z^2+1)\geq 2x\cdot 2y\cdot 2z=8xyz$

$\Rightarrow$ ***Equality occurs if*** $\Leftrightarrow x=y=z=1$

Solution 2 by Nguyen Thanh Nho-Tra Vinh-Vietnam

$$\left(1-\frac{1}{1+x^2}\right)+\left(\frac{1}{1+x^2}-\frac{1}{(1+x^2)(1+y^2)}\right)+\left(\frac{1}{(1+x^2)(1+y^2)}-\frac{1}{(1+x^2)(1+y^2)(1+z^2)}\right)+\frac{1}{8xyz}=1$$

$$-\frac{1}{(1+x^2)(1+y^2)(1+z^2)}+\frac{1}{8xyz}=0 \Leftrightarrow (1+x^2)(1+y^2)(1+z^2)=8xyz$$

$$\Leftrightarrow \left(\frac{1}{x}+x\right)\left(\frac{1}{y}+y\right)\left(\frac{1}{z}+z\right)=8;\ \frac{1}{x}+x\geq 2, \frac{1}{y}+y\geq 2, \frac{1}{z}+z\geq 2$$

$$\Rightarrow \left(\frac{1}{x}+x\right)\left(\frac{1}{y}+y\right)\left(\frac{1}{z}+z\right)\geq 8;\ \text{“}=\text{”} \Leftrightarrow x=y=z=1$$

109. Find $x,y,z,t\in\mathbb{R}$ such that:

$$5x^2+5y^2+5z^2+5t^2-5xy-5yz-5zt-5t+2=0$$

Proposed by Daniel Sitaru – Romania

Solution by Subhajit Chattopadhyay-Bolpur-India

$$5x^2+5y^2+5z^2+5t^2-5xy-5yz-5zt-5t+2=0$$

$$\textbf{or, } 5\left(x-\frac{y}{2}\right)^2+\frac{15y^2}{4}+5z^2+5t^2-5yz-5zt-5t+2=0$$

$$\textbf{or, } 5\left(x-\frac{y}{2}\right)^2+5\left(\frac{\sqrt{3}y}{2}-\frac{z}{\sqrt{3}}\right)^2+\frac{10z^2}{3}-5zt+5t^2-5t+2=0$$

$$or,\ 5\left(x-\frac{y}{2}\right)^2+5\left(\frac{\sqrt{3}y}{2}-\frac{z}{\sqrt{3}}\right)^2+5\left(\frac{\sqrt{2}z}{\sqrt{3}}-\frac{\sqrt{3}t}{2\sqrt{2}}\right)^2+\frac{25t^2}{8}-5t+2=0$$

$$or,\ 5\left(x-\frac{y}{2}\right)^2+5\left(\frac{\sqrt{3}y}{2}-\frac{z}{\sqrt{3}}\right)^2+5\left(\frac{\sqrt{2}z}{\sqrt{3}}-\frac{\sqrt{3}t}{2\sqrt{2}}\right)^2+\left(\frac{5t}{2\sqrt{2}}-\sqrt{2}\right)^2=0$$

$$t,x,y,z\in\mathbb{R}\Rightarrow x=\frac{y}{2};\frac{\sqrt{3}y}{2}=\frac{z}{\sqrt{3}};\frac{\sqrt{2}z}{\sqrt{3}}=\frac{\sqrt{3}t}{2\sqrt{2}};\frac{5t}{2\sqrt{2}}=\sqrt{2}$$

$$\Rightarrow t=\frac{4}{5},z=\frac{3}{5};y=\frac{2}{5};x=\frac{1}{5}$$

110. From the book “Math Energy”

Find:

$$\Omega=\lim_{x\to\infty}\int_0^x\frac{x^4}{(1+x^3)^2}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Igor Soposki-Skopje, Solution 2 by Togrul Ehmedov-Baku-Azerbaidian, Solution 3 by Carlos Suarez-Quito-Ecuador, Solution 4 by Shivam Sharma-New Delhi-India

Solution 1 by Igor Soposki-Skopje

$$\Omega=\lim_{t\to\infty}\int_0^t\frac{x^4}{(1+x^3)^2}dx;I=\int\frac{x^4}{(1+x^3)^2}dx=\begin{cases}u=x^2\\du=2xdx\end{cases}$$

$$dv=\frac{x^2}{(1+x^3)^2}dx\Rightarrow v=\int\frac{x^2}{(1+x^3)^2}dx=\begin{Bmatrix}1+x^3=t\\3x^2dx=dt\end{Bmatrix}=\frac{1}{3}\int\frac{dt}{t}=$$

$$=-\frac{1}{3t}=-\frac{1}{3(1+x^3)}\}=u\cdot\vartheta-\int\vartheta\cdot du=-\frac{x^2}{8(1+x^3)}+\frac{2}{3}\underbrace{\int\frac{x}{1+x^3}dx}_{I_1}$$

$$I_1 = \int \frac{x}{x^3+1}dx = \int \frac{x}{(x+1)(x^2-x+1)} = \int \frac{A}{x+1}dx + \int \frac{Bx+c}{x^2-x+1}dx$$

$$\frac{x}{x^3+1} = \frac{A}{x+1} + \frac{Bx+c}{x^2-x+1} | \cdot (x+1)(x^2-x+1) \Rightarrow$$

$$\begin{matrix} \Leftrightarrow x = A(x^2-x+1) + (Bx+c)(x+1) \\ \Leftrightarrow x = Ax^2 - Ax + A + Bx^2 + Bx + Cx + c \\ \Leftrightarrow x = (A+B)x^2 + (-A+B+C)x + A + C \end{matrix} \Rightarrow \begin{cases} A + D = 0 \\ -A + B + C = 1 \\ A + C \end{cases} \Leftrightarrow \begin{Bmatrix} A = -\frac{1}{3} \\ B = C = \frac{1}{3} \end{Bmatrix}$$

$$I_2 = \int \frac{A}{x+1}dx = -\frac{1}{3}\ln(x+1); I_3 = \frac{1}{3}\int \frac{x+1}{x^2-x+1}dx = \frac{1}{6}\int \frac{2x+2}{x^2-x+1}dx =$$

$$= \frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{dx}{x^2+x+1} = \frac{1}{6}\cdot I_4 + \frac{1}{2}\cdot I_5$$

$$I_4 = \int \frac{2x-1}{x^2-x+1}dx = \begin{Bmatrix} x^{-x} + x + 1 \\ (2x-1)dx = dt \end{Bmatrix} = \int \frac{dt}{t} = \ln t = \ln(x^2-x+1)$$

$$I_5 = \int \frac{dx}{x^2-x+1} = \int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \begin{Bmatrix} x - \frac{1}{2} = t \\ dx = dt \end{Bmatrix} = \int \frac{dt}{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}}\cdot \arctan\frac{2x-1}{\sqrt{3}}$$

$$I_3 = \frac{1}{6}\cdot \ln(x^2-x+1) + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}; I_1 = I_2 + I_3$$

$$I = -\frac{x^2}{3(1+x^3)} + \frac{2}{3}\cdot I_1 = \frac{1}{9}\cdot \left[\ln\left(\frac{x^2-x+1}{(x+1)^2} - \frac{3x^2}{x^3+1} + 2\sqrt{3}\arctan\frac{2x-1}{\sqrt{3}}\right)\right]\Big|_0^t$$

$$\Omega = \frac{1}{9}\lim_{t\to\infty}\left[\ln\frac{t^2-t+1}{(t+1)^2} - \frac{3t^2}{t^3+1} + e\sqrt{3}\frac{2t-1}{\sqrt{3}} + \frac{2\sqrt{3}\pi}{6}\right] =$$

$$= \frac{1}{9}\cdot\left[2\sqrt{3}\frac{\pi}{2} + 2\sqrt{3}\frac{\pi}{6}\right] = \frac{1}{9}\cdot 2\sqrt{3}\cdot\frac{4\pi}{6} = \frac{4\sqrt{3}\pi}{27}$$

Solution 2 by Togrul Ehmedov-Baku-Azerbaidian

$$\Omega = \lim_{t\to\infty}\int_0^t \frac{x^4}{(1+x^3)^2}dx = \int_0^\infty \frac{x^4}{(1+x^3)^2}dx$$

$$=\left[\frac{1}{3}\int_0^\infty \frac{t^{\frac{2}{3}}}{(1+t)^2}dt\right]_{x^3=t}=\frac{1}{3}\int_0^\infty \frac{t^{\frac{5}{3}-1}}{(1+t)^{\frac{1}{3}+\frac{5}{3}}}dt$$

$$=\frac{1}{3}B\left(\frac{1}{3},\frac{5}{3}\right)=\frac{1}{3}\cdot\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma(2)}=\frac{2}{9}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)=\frac{2}{9}\cdot\frac{\pi}{\sin\left(\frac{\pi}{3}\right)}=\frac{4\pi}{9\sqrt{3}}$$

Solution 3 by Carlos Suarez-Quito-Ecuador

$$\Omega=\lim_{t\to\infty}\int_0^t \frac{x^4}{(1+x^3)^2}dx;\ \Omega=\frac{4\pi}{9\sqrt{3}}=0,80613$$

$$\int_0^t \frac{x^4}{(1+x^3)^2}dx=\frac{1}{9}\left[\ln(x^2-x+1)-\frac{3x^2}{x^3+1}-2\ln(x+1)+2\sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^t$$

$$\int_0^t \frac{x^4}{[(l+x)(1-x+x^2)]^2}=\frac{x^4}{(1+x)^2(1-x+x^2)^2}$$

$$\frac{A}{1+x}+\frac{B}{(1+x)^2}+\frac{Cx+D}{1-x+x^2}+\frac{Ex+F}{(1-x+x^2)^2}=$$

$$\frac{2(x+2)}{9(x^2-x+1)}-\frac{1}{3(x^2-x+1)^2}-\frac{2}{9(x+1)}+\frac{1}{9(x+1)^2}$$

Solution 4 by Shivam Sharma-New Delhi-India

$$\Rightarrow\int_0^\infty \frac{x^4}{(1+x^3)^2}dx\Rightarrow\int_0^\infty \frac{z^4}{(1+z^3)^2}dz\Rightarrow\left(\frac{1}{\sqrt{3}}\right)Resi\left[(2\pi i)\frac{z^4}{(1+z^3)^2};\ -1\right]$$

$$\Rightarrow\left(\frac{1}{\sqrt{3}}\right)(2\pi i^2)\left(-\frac{2}{9}\right)\ \textbf{(OR)}\ I=\frac{4\pi}{9\sqrt{3}}$$

(Q.E.D)

111. If $a, b, c > 0$

$$I(a) = \int_{-\pi}^{\pi} \frac{\cos^2 x}{x + a + \sqrt{x^2 + a^2}} dx$$

then:

$$I(a) + I(b) + I(c) \geq \frac{9\pi}{2(a + b + c)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Togrul Ehmedov-Baku-Azerbaidian, Solution 2 by Shivam Sharma-New Delhi-India

Solution 1 by Togrul Ehmedov-Baku-Azerbaidian

$$I(a) = \int_{-\pi}^{\pi} \frac{\cos^2 x}{a + \sqrt{x^2 + a^2} + x} dx = \int_{-\pi}^{\pi} \frac{\cos^2 x}{a + \sqrt{x^2 + a^2} - x} dx$$

$$2I(a) = \int_{-\pi}^{\pi} \cos^2 x \left(\frac{1}{a + \sqrt{x^2 + a^2} + x} + \frac{1}{a + \sqrt{x^2 + a^2} - x} \right) dx$$

$$2I(a) = \int_{-\pi}^{\pi} \frac{1}{a} \cos^2 x \, dx \Rightarrow I(a) = \frac{1}{a} \int_{0}^{\pi} \cos^2 x \, dx = \frac{1}{2a}$$

$$I(a) + I(b) + I(c) = \frac{9\pi}{2(a + b + c)}$$

Solution 2 by Shivam Sharma-New Delhi-India

$$\int_{-\pi}^{\pi} \frac{\cos^2(-x)}{-x + a + \sqrt{x^2 + a^2}} dx$$

$$2I(a) = 2\int_{0}^{\pi} \left(\frac{\cos^2 x}{\sqrt{x^2 + a^2} + a + x} + \frac{\cos^2 x}{\sqrt{x^2 + a^2} + a - x} \right) dx$$

$$I(a)=\frac{1}{a}\int_0^{\pi}\cos^2(x)\,dx\Rightarrow\frac{1}{a}\int_{-\pi}^{\pi}\frac{1+\cos(2x)}{2}dx\Rightarrow\frac{1}{a}\left[\frac{\pi}{2}-0\right]$$

(OR) $I(a)=\frac{\pi}{2a}$ **. Now,** $\sum_{cyc}(I(a))\overset{AM-GM}{\geq}\frac{9\pi}{2(a+b+c)}$

112. Find:

$$\Omega=\int_{\frac{\pi}{18}}^{\frac{\pi}{12}}\int_{\frac{\pi}{18}}^{\frac{\pi}{12}}\frac{\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right)}{\tan 3x\tan 3y}dx\,dy$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by Togrul Ehmedov-Baku-Azerbaidian, Solution 3 by Shivam Sharma-New Delhi-India

Solution 1 by Ravi Prakash-New Delhi-India

$$\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right)$$

$$=\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left\{\pi-\left(\frac{\pi}{3}-x\right)\right\}$$

$$=\tan x+\tan\left(x+\frac{\pi}{3}\right)-\tan\left(\frac{\pi}{3}-x\right)$$

$$=\tan x+\frac{\tan x+\sqrt{3}}{1-\sqrt{3}\tan x}-\frac{\sqrt{3}-\tan x}{1+\sqrt{3}\tan x}$$

$$=\tan x+\frac{(\sqrt{3}+\tan x)(1+\sqrt{3}\tan x)-(\sqrt{3}-\tan x)(1-\sqrt{3}\tan x)}{1-3\tan^2 x}$$

$$=\frac{\tan x-3\tan^3 x+8\tan x}{1-3\tan^2 x}=3\tan 3x$$

$$\therefore \Omega = \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \frac{3\tan 3x}{\tan 3x \tan 3y} dx\, dy = \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} 3\left(\frac{\pi}{36}\right) \cot 3y\, dy$$

$$= \frac{\pi}{36} \log|\sin(3y)|\Big]_{\frac{\pi}{18}}^{\frac{\pi}{12}} = \frac{\pi}{36}\left\{\log\left(\frac{1}{\sqrt{2}}\right) - \log\left(\frac{1}{2}\right)\right\} = \frac{\pi}{36}\log(\sqrt{2}) = \frac{\pi}{72}\log 2$$

Solution 2 by Togrul Ehmedov-Baku-Azerbaidian

$$A = \frac{\left(\tan x + \tan\left(\frac{\pi}{3} + x\right) + \tan\left(\frac{2\pi}{3} + x\right)\right)}{\tan 3x} = 3$$

$$\int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \frac{3}{\tan 3y} dx\, dy = \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} [\ln 3y]_{\frac{\pi}{18}}^{\frac{\pi}{12}}\, dx = \ln\sqrt{2}\left(\frac{\pi}{12} - \frac{\pi}{18}\right) =$$

$$= \ln\sqrt{2}\,\frac{\pi}{36} = \ln 2\,\frac{\pi}{72}$$

$$\int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \frac{\tan x + \tan\left(\frac{\pi}{3} + x\right) + \tan\left(\frac{2\pi}{3} + x\right)}{\tan 3x \tan 3y} dx\, dy = \ln 2\,\frac{\pi}{72} < \ln 2\,\frac{\pi}{71}$$

Solution 3 by Shivam Sharma-New Delhi-India

$$\int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \frac{\tan(x) + \tan\left(\frac{\pi}{3} + x\right) + \left(-\tan\left(\frac{\pi}{3} - x\right)\right)}{\tan(3x)\tan(3y)} dx\, dy$$

$$\Rightarrow \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \tan(x) + \frac{\tan(x) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan(x)\tan\left(\frac{\pi}{3}\right)} + \frac{\frac{\tan(x) - \tan\left(\frac{\pi}{3}\right)}{1 + \tan(x)\tan\left(\frac{\pi}{3}\right)}}{\tan(3x)\tan(3y)} dx\, dy$$

$$\Rightarrow \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \frac{\tan(x) + \frac{\tan(x)+\sqrt{3}}{1-\sqrt{3}\tan(x)} + \frac{\tan(x)-\sqrt{3}}{1+\sqrt{3}\tan(x)}}{\tan(3x)\tan(3y)} dx\, dy$$

$$\Rightarrow \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \frac{\frac{9\tan(x) - 3\tan^3(x)}{1-\left(\sqrt{3}\tan x\right)^2}}{\tan(3x)\tan(3y)} dx\, dy$$

$$\Rightarrow \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \frac{\frac{3[\tan(x) - \tan^3(x)]}{1-3\tan^2 x}}{\tan(3x)\tan(3y)} dx\, dy \Rightarrow \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \frac{3\tan(3x)}{\tan(3x)\tan(3y)} dx\, dy$$

$$\Rightarrow \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} 3\cot(3y)\, dx\, dy \Rightarrow \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \left(\frac{\pi}{12} - \frac{\pi}{18}\right) 3\cot(3y)\, dy$$

$\Rightarrow \frac{\pi}{12}\int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \cot(3y)\, dy$. **Let** $3y = u \Rightarrow \frac{\pi}{36}\int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \cot(u)\, du \Rightarrow \frac{\pi}{36}[\ln|\sin(u)|]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \Rightarrow$

$\Rightarrow \frac{\pi}{36} \cdot \frac{1}{2} \cdot \ln(2)$ ***(OR)*** $I = \frac{\pi}{72}\ln(2)$ ***(Answer)***

113. If $a \in \left(0, \frac{\pi}{2}\right)$ **find:**

$$\Omega = \int_{\tan a}^{\cot a} \frac{\ln x}{1+x^2} dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Togrul Ehmedov-Baku-Azerbaidian

Solution 2 by Abinash Mohapatra-India

Solution 1 by Togrul Ehmedov-Baku-Azerbaidian

$$\int_{\tan a}^{\cot a} \frac{\ln x}{1+x^2} dx. \textit{ Let } x = \tan b$$

$$I = \int_b^{\frac{\pi}{2}-b} \ln \tan b\, db = \int_b^{\frac{\pi}{2}-b} \ln \cot b\, db$$

$$I = \frac{1}{2}\int_b^{\frac{\pi}{2}-b} [\ln \tan b + \ln \cot b]\, db = 0$$

Solution 2 by Abinash Mohapatra-India

$$\Omega = \int_{\tan a}^{\cot a} \frac{\ln x}{1+x^2} dx, a \in \left(0, \frac{\pi}{2}\right). \textit{ Applying by:}$$

$$\ln x \int_c^c \frac{1}{1+x^2} - \int_c^c \left(\frac{1}{x}\int \frac{1}{1+x^2}\right) dx; \ln x \tan^{-1}\frac{1}{x}\Big|_c^c - \underbrace{\int_c^c \frac{\tan^{-1} x}{x} dx}_{\alpha}$$

$$\therefore \alpha = \int_c^c \frac{\tan^{-1} x}{x} dx = \int_{\tan a}^{\cot a} \frac{1}{x}\cot^{-1}\left(\frac{1}{x}\right) dx; \int_c^c \frac{x}{x^2}\cdot \cot^{-1}\left(\frac{1}{x}\right) dx$$

$$\textit{Let } \frac{1}{x} = t \Rightarrow -\frac{1}{x^2} dx = dt \Rightarrow \alpha = \int_{\cot a}^{\tan a} \frac{\cot^{-1}(t)}{t} dt$$

$$\Rightarrow \alpha = \underbrace{\int_{\tan a}^{\cot a} \frac{\tan^{-1} x}{x} dx}_{(I)} = \underbrace{\int_{\tan a}^{\cot a} \frac{\cot^{-1} x}{x} dx}_{(II)} \textit{ (variable change)}$$

$$\Rightarrow \textit{equating (I) and (II) we get}$$

$$\int_{\tan a}^{\cot a} \frac{\pi}{2x} = 0 \Rightarrow \ln(\cot^2 a) = 0 \Rightarrow \cot a = 1 \Rightarrow a = \frac{\pi}{4}$$

$$\textit{Thus } \Omega = 0$$

114. If $a > 0$, $f:\mathbb{R}\to\mathbb{R}$ continuous one, $f(x)+f(-x)=a\cos x, \forall x\in\mathbb{R}$ then find:

$$\Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x)}{\cos^3 x}dx$$

Proposed by D.M. Bătinețu – Giurgiu & Neculai Stanciu – Romania

Solution 1 by Serban George Florin-Romania, Solution 2 by Lazaros Zachariadis-Thessaloniki-Greece, Solution 3 by Shivam Sharma-New Delhi-India, Solution 4 by Soumava Pal-Kolkata-India, Solution 5 by SK Rejuan-West Bengal-India

Solution 1 by Serban George Florin-Romania

$$x=-t \Rightarrow \Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(-t)}{\cos^3 t}dt$$

$$2\Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x)+f(-x)}{\cos^3 x}dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{a\cos x}{\cos^3 x} = a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\cos^2 x}$$

$$\Omega = \frac{a}{2}\tan x\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{a}{2}\left(\tan\frac{\pi}{4}-\tan\left(-\frac{\pi}{4}\right)\right);\ \Omega = \frac{a}{2}(1+1)=a, \Omega = a$$

Solution 2 by Lazaros Zachariadis-Thessaloniki-Greece

$$f(x)+f(-x)=a\cdot\cos x \Rightarrow \frac{f(x)}{\cos^3 x}+\frac{f(-x)}{\cos^3 x}=\frac{a}{\cos^2 x}$$

$$\Rightarrow \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x)}{\cos^3 x} dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(-x)}{\cos^3 x} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{a}{\cos^2 x} dx = (a \tan x)_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$\Rightarrow \underline{\Omega} + \underline{\Omega} = a(1+1) \Rightarrow 2\underline{\Omega} = 2a \Rightarrow \underline{\Omega} = a$$

$$* \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(-x)}{\cos^3 x} dx \underset{\substack{x = -\frac{\pi}{4}, u = \frac{\pi}{4} \\ x = \frac{\pi}{4}, u = -\frac{\pi}{4}}}{\overset{-x = u, dx = -du}{=\!=\!=\!=}} - \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{f(u)du}{\cos^3 u} = \underline{\Omega}$$

Solution 3 by Shivam Sharma-New Delhi-India

As we know, the following Lemma,

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even function} \\ 0, \text{ if } f(x) \text{ is an odd function} \end{cases}$$

Using this, we get, $\Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(-x)}{\cos^3(-x)} dx$ ***then,*** $2\Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x)+f(-x)}{\cos^3(x)} dx$

$$2\Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{a\cos(x)}{\cos^3(x)} dx;\ \Omega = a\int_0^{\frac{\pi}{4}} \sec^2(x)\, dx \Rightarrow a[\tan(x)]_0^{\frac{\pi}{4}}$$

(OR) $\Omega = a$ ***(Answer)***

Solution 4 by Soumava Pal-Kolkata-India

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x)}{\cos^3 x} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right)}{\cos^3\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right)} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{a\cos x - f(x)}{\cos^3 x} dx$$

$$= a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x\,dx - I \Rightarrow 2I = a(\tan x)_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = a(1-(-1)) = 2a \Rightarrow I = a$$

Solution 5 by SK Rejuan-West Bengal-India

$$\Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x)}{\cos^3 x}\,dx \quad \textbf{(1)}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f\left(\frac{\pi}{4}-\frac{\pi}{4}-x\right)}{\cos^3 x}dx \Rightarrow \Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(-x)}{\cos^3 x}dx \quad \textbf{(2)}$$

Adding (1) & (2) we get $2\Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x)+f(-x)}{\cos^3 x}dx$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{a\cos x}{\cos^3 x}dx \quad [as\ f(x) + f(-x) = a\cos x]$$

$$= a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x\,dx = a[\tan x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = a(1-(1)) = 2a \Rightarrow 2\Omega = 2a \Rightarrow \Omega = a$$

115. Find the integral

$$I = \int \frac{x^2\cos x + x + \sin x\cos x}{x\sin x\,(x+\cos x)}dx$$

Proposed by Abdallah Almalih-Damascus-Syria

Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 3 by Nawar Alasadi-Babylon-Iraq

Solution 1 by Ravi Prakash-New Delhi-India

$$I=\int\frac{x^2\cos x+x+(\sin x)(\cos x)}{x\sin x(x+\cos x)}dx.\text{ Note }\frac{d}{dx}\{x\sin x\,(x+\cos x)\}$$

$$=\frac{d}{dx}\left\{x^2\sin x+\frac{1}{2}x\sin 2x\right\}=2x\sin x+x^2\cos x+x\cos 2x+\frac{1}{2}\sin 2x$$

$$=x^2\cos x+x+\sin x\cos x+2x\sin x-2x\sin^2 x$$

$$=x^2\cos x+x+\sin x\cos x+2x\sin x\,(1-\sin x)$$

$$\therefore I=I_1-2I_2\text{ where}$$

$$I_1=\int\frac{\frac{d}{dx}(x\sin x\,(x+\cos x))}{x\sin x\,(x+\cos x)}dx=\ln|x\sin x\,(x+\cos x)|$$

$$I_2=\int\frac{x\sin x\,(1-\sin x)}{x\sin x\,(x+\cos x)}dx=\ln|x+\cos x|+c$$

$$\text{Thus, }I=\ln|x\sin x\,(x+\cos x)|-2\ln|x+\cos x|+c$$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$\text{Let us denote by }\varphi(x)=x\sin x\,(x+\cos x)$$

$$\Rightarrow\varphi'(x)=x^2\cos x+2x\sin x+\sin x\cos x+x\cos x$$

$$\text{then }\int\frac{x^2\cos x+x+\sin x\cos x}{x\sin x(x+\cos x)}dx=\int\frac{\varphi'(x)}{\varphi(x)}dx+\int\frac{x-2x\sin x-x\cos x}{\varphi(x)}dx$$

$$=\ln|\varphi(x)|-2\int\frac{x\sin x\,(1-\sin x)}{x\sin x\,(x+\cos x)}dx=\ln|\varphi(x)|-2\int\frac{(x+\cos x)'}{x+\cos x}dx$$

$$=\ln|\varphi(x)|-2\ln|x+\cos x|+\lambda=\ln\left|\frac{\varphi(x)}{(x+\cos x)^2}\right|+\lambda,\text{ whith }\lambda\in\mathbb{R}$$

$$\text{Finally we get }\int\frac{x^2\cos x+x+\sin x\cos x}{x\sin x(x+\cos x)}dx=\ln\left|\frac{x\sin x}{x+\cos x}\right|+\lambda$$

Solution 3 by Nawar Alasadi-Babylon-Iraq

$$I=\int\frac{x^2\cos x+x+\sin x\cos x}{x\sin x\,(x+\cos x)}dx$$

$$= \int \frac{x^2\cos x + x(\sin^2 x + \cos^2 x) + \sin x\cos x + x\sin x - x\sin x}{x\sin x\,(x + \cos x)}dx$$

$$= \int \frac{x\sin x + \sin x\cos x + x^2\cos x + x\cos^2 x - x\sin x + x\sin^2 x}{x\sin x\,(x + \cos x)}dx$$

$$= \int \frac{\sin x\,(x + \cos x) + x\cos x\,(x + \cos x) - x\sin x\,(1 - \sin x)}{x\sin x\,(x + \cos x)}dx$$

$$= \int \left(\frac{1}{x} + \frac{\cos x}{\sin x} - \frac{1 - \sin x}{x + \cos x}\right)dx$$

$$= \ln|x| + \ln|\sin x| - \ln|x + \cos x| + c = \ln\left|\frac{x\sin x}{x + \cos x}\right| + c$$

116. Find:

$$\Omega = \int \frac{\cot x\cot 2x\,dx}{(\cot^2 x - \tan^2 x)\sin^3 2x}$$

Proposed by Geanina Tudose – Romania

Solution by proposer

$$\int \frac{\cot 2x\cdot\cot x}{(\cot^2 x - \tan^2 x)\sin^3 2x}dx = \int \frac{\cos 2x\cdot\cos x}{\sin 2x\cdot\sin x}\cdot\frac{1}{\frac{\cos^2 x\cdot\sin^2 x}{\cos^2 x\cdot\sin^2 x}}\cdot\frac{1}{8\sin^2 x\cos^3 x}dx$$

$$= \frac{1}{8}\int \frac{\cos x}{\sin 2x\cdot\sin^2 x\cdot\cos x}dx = \frac{1}{16}\int \frac{1}{\sin^3 x\cos x}dx$$

$$= \frac{1}{16}\int \frac{1}{\sin^3 x\cdot\cos^2 x}\cdot\cos dx = \frac{1}{16}\int \frac{1}{y^3\cdot(1-y^2)}dx = (*),\ y = \sin x,\ dy = \cos x\,dx$$

$$\frac{1}{y^3(1-y^2)} = \frac{1 - y^2 + y^2}{y^3(1-y^2)} = \frac{1}{y^3} + \frac{1}{y(1-y^2)} = \frac{1}{y^3} + \frac{1}{y} + \frac{y}{1-y^2}$$

$$(*) = \frac{1}{16}\left(\int \frac{1}{y^3}dy + \int \frac{1}{y}dy + \int \frac{y}{1-y^2}dy\right) = \frac{1}{16}\left(\frac{y^{-2}}{-2} + \ln y - \frac{1}{2}\ln\underbrace{(1-y^2)}_{+}\right) + C =$$

$$= \frac{1}{16}\left(-\frac{+1}{2y^2} + \ln\frac{y}{\sqrt{1-y^2}}\right) + C = \frac{1}{16}\left(\frac{-1}{2\sin^2 x} + \ln\frac{\sin x}{\cos x}\right) + C = \frac{1}{16}\left(\frac{-1}{2\sin^2 x} + \ln(\tan x)\right) + C$$

117. Find:

$$\Omega = \int \frac{\cos 2x \cot x \, dx}{(\cot^2 x - \tan^2 x)\sin^3 2x}, x \in \left(0, \frac{\pi}{4}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Nguyen Thanh Nho-Tra Vinh-Vietnam

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^4 x - \sin^4 x =$$

$$= \sin^2 x \cos^2 x (\cot^2 x - \tan^2 x); * \sin^3 2x = 8\sin^3 x \cos^3 x$$

$$\Rightarrow \Omega = \int \frac{\sin^2 x \cos^2 x (\cot^2 x - \tan^2 x)\frac{\cos x}{\sin x}}{(\cot^2 x - \tan^2 x)\cdot 8\sin^3 \cos^3 x} dx = \frac{1}{8}\int \frac{1}{\sin^2 x} dx = -\frac{1}{8}\cot x + C$$

118. If $f: [0,1] \to (0,\infty)$ is a continuous function such that

$$\int_0^1 f(x)\,dx = 1 \text{ , then}$$

$$\left(\int_0^1 \sqrt[3]{f(x)}\,dx\right)\left(\int_0^1 \sqrt[5]{f(x)}\,dx\right)\left(\int_0^1 \sqrt[7]{f(x)}\,dx\right) \le 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Chris Kyriazis-Greece ,

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Chris Kyriazis-Greece

It's obvious that $f(x) \ge 0 \ \forall x \in [0,1]$. Because of AM – GM, we take:

$\sqrt[3]{f(x)} = \sqrt[3]{f(x)\cdot 1 \cdot 1} = \frac{f(x)+1+1}{3}$ *so if we integrate, it follows that:*

$$\int_0^1 \sqrt[3]{f(x)}\,dx \le \int_0^1 \frac{f(x)+2}{3}dx = \frac{1}{3}\left(\int_0^1 f(x)dx + 2\right) = \frac{1}{3}\cdot 3 = 1 \quad (1)$$

Doing it the same way we take that: $\int_0^1 \sqrt[5]{f(x)}dx \le 1 \quad (2)$

and $\int_0^1 \sqrt[7]{f(x)}\,dx \le 1 \quad (3)$

Multiplying (1) × (2) × (3) (every party is non negative!)

We have the result we want!

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\int_0^1 \sqrt[3]{f(x)}\,dx \overset{HOLDER'S\ INEQUALITY}{\leq} \sqrt[3]{\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 f(x)dx\right)} = 1$$

$$\int_0^1 \sqrt[5]{f(x)}\,dx \overset{HOLDER'S\ INEQUALITY}{\leq} \sqrt[5]{\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 f(x)dx\right)} = 1$$

$$\int_0^1 \sqrt[7]{f(x)}\,dx \overset{HOLDER}{\leq} \sqrt[7]{\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 f(x)dx\right)} = 1$$

$$\left(\int_0^1 \sqrt[3]{f(x)}\,dx\right)\left(\int_0^1 \sqrt[5]{f(x)}\,dx\right)\left(\int_0^1 \sqrt[7]{f(x)}\,dx\right) \leq 1$$

119. If $f:[a,b] \to (0,\infty), a < b, f$ continuous, increasing then:

$$\left(\int_a^b xf(x)dx\right)\left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b x^3 f(x)dx\right) \geq \frac{a^2b^2}{b-a}\left(\int_a^b f(x)dx\right)^4$$

Proposed by Daniel Sitaru – Romania

Solution by Leonard Giugiuc – Romania

By Chebyshev,

$$\int_a^b xf(x)dx \geq \frac{1}{b-a}\cdot\left(\int_a^b xdx\right)\left(\int_a^b f(x)dx\right) = \frac{a+b}{2}\cdot\int_a^b f(x)\,dx.$$

Similarly,

$$\int_a^b x^3 f(x)\,dx \geq \frac{a^3+a^2b+ab^2+b^3}{4}\cdot\int_a^b f(x)\,dx$$

By Cauchy, $\int_a^b f^2(x)\,dx \geq \frac{1}{b-a}\cdot\left(\int_a^b f(x)\,dx\right)^2$

By AM – GM, $\frac{a+b}{2}\cdot\frac{a^3+a^2b+ab^2+b^3}{4}\geq a^2b^2$. We multiply and get

$$\left(\int_a^b xf(x)dx\right)\left(\int_a^b f^2(x)dx\right)\left(\int_a^b x^3f(x)dx\right)\geq\frac{a^2b^2}{b-a}\left(\int_a^b f(x)dx\right)^4$$

120. From the book: "Math Accent"

$$\Omega=\int_0^1\frac{\ln(1-x^2)^2\ln(1-x)}{x}dx$$

Prove that: $\Omega>\frac{5}{2}\zeta(3)$

Proposed by Daniel Sitaru – Romania

Solution by Shivam Sharma-New Delhi-India

If $I=\int_0^1\frac{\ln(1-x^2)^2\ln(1-x)}{x}dx$. Then, prove that: $I>-\frac{\pi^2}{2}$

$$\Rightarrow 2\int_0^1\frac{[\ln(1-x)+\ln(1+x)]\ln(1-x)}{x}dx$$

$\Rightarrow 2\int_0^1\frac{\ln^2(1-x)}{x}dx+2\int_0^1\frac{\ln(1-x)\ln(1+x)}{x}dx$. Let, $A=\int_0^1\frac{\ln^2(1-x)}{x}dx$

$$\Rightarrow\int_0^1\frac{\ln^2(x)}{1-x}dx\Rightarrow\sum_{n=0}^{\infty}\int_0^1 x^n\ln^2(x)\,dx\Rightarrow\sum_{n=0}^{\infty}\frac{\partial^2}{\partial n^2}\left[\int_0^1 x^n dx\right]$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{\partial^2}{\partial n^2}\left[\frac{x^{n-1}}{n+1}\right]_0^1 \Rightarrow \sum_{n=0}^{\infty}\left[\frac{x^{n-1}\ln^2(x)}{n+1} - 2\frac{x^{n+1}\ln(x)}{(n+1)^2} + 2\frac{x^{n-1}}{(n+1)^3}\right]_0^1$$

$$\Rightarrow 2\sum_{n=0}^{\infty}\left(\frac{1}{n^3}\right) \textit{ (OR) } A = 2\zeta(3)$$

Let $B = \int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}dx$

$$\Rightarrow \frac{1}{4}\left[\frac{1}{2}\int_0^1 \frac{\ln^2(1-x)}{x}dx - 2\int_0^1 \frac{\ln^2(x)}{(1-x)(1+x)}dx\right] \Rightarrow \frac{1}{4}\left[-\frac{1}{2}\int_0^1 \frac{\ln^2(x)}{1-x}dx - \int_0^1 \frac{\ln^2(x)}{1+x}dx\right]$$

Now, applying I.B.P., we get, $\Rightarrow \frac{1}{4}\left[-\int_0^1 \frac{\ln(x)\ln(1-x)}{x}dx + 2\int_0^1 \frac{\ln(x)\ln(1+x)}{x}dx\right]$

Now, again applying I.B.P., we get $\Rightarrow \frac{1}{4}\left[-\int_0^1 \frac{Li_2(x)}{x}dx + 2\int_0^1 \frac{Li_2(-x)}{x}dx\right]$

Let, $x = -u$*, in second integral, we get* $dx = -du$

$$\Rightarrow \frac{1}{4}\left([-Li_3(x)]_0^1 + 2\int_0^1 \frac{Li_2(u)}{u}du\right) \Rightarrow \frac{1}{4}\left(Li_3(1) + 2[Li_3(x)]_0^1\right)$$

$$\Rightarrow \frac{1}{4}\left[-\frac{5}{2}\left(Li_3(1)\right)\right] \Rightarrow \frac{1}{4}\left[-\frac{5}{2}\zeta(3)\right] \textit{ (OR) } B = -\frac{5}{8}\zeta(3).$$

Combining all, we get, $I = 2A + 2B \Rightarrow 2\left(2\zeta(3)\right) + 2\left(-\frac{5}{8}\zeta(3)\right)$

$$\textit{(OR) } I = \frac{11}{4}\zeta(3) > \frac{5}{2}\zeta(3)$$

121. If $[0,1] \to (0,\infty)$ continuous; $\int_0^1 f^3(x)\,dx = \sqrt[7]{2}$ then:

$$\left(\int_0^1 f^5(x)\,dx\right)\left(\int_0^1 f^7(x)\,dx\right)\left(\int_0^1 f^9(x)\,dx\right) \geq 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Solution 2 by Chris Kyriazis-Greece

Solution 1 by proposer

$$\left(\int_0^1 f^5(x)dx\right)\left(\int_0^1 f^7(x)dx\right)\left(\int_0^1 f^9(x)dx\right)\left(\int_0^1 f^3(x)dx\right)=$$

$$=\int_0^1\left(f^2(x)\sqrt{f(x)}\right)^2dx\cdot\int\left(f^3(x)\sqrt{f(x)}\right)^2dx\cdot$$

$$\cdot\int_0^1\left(f^4(x)\sqrt{f(x)}\right)^2dx\cdot\left(\int_0^1 f(x)\sqrt{f(x)}\right)^2dx\overset{CBS}{\geq}$$

$$\geq\left(\int_0^1 f^6(x)dx\right)^2\cdot\left(\int_0^1 f^6(x)dx\right)^2dx=$$

$$=\left(\left(\int_0^1 f^6(x)dx\right)\left(\int_0^1 1^2dx\right)\right)^4\overset{CBS}{\geq}\left(\int_0^1 f^3(x)dx\right)^8=\sqrt[7]{2^8}$$

$$\sqrt[7]{2}\left(\int_0^1 f^5(x)dx\right)\left(\int_0^1 f^7(x)dx\right)\left(\int_0^1 f^9(x)dx\right)\geq\sqrt[7]{2^8}$$

$$\left(\int_0^1 f^5(x)\,dx\right)\left(\int_0^1 f^7(x)\,dx\right)\left(\int_0^1 f^9(x)\,dx\right)\geq 2$$

Solution 2 by Chris Kyriazis-Greece

By Holder's Inequality (only if $f\geq 0$)

$$\left(\int_0^1 f^5(x)dx\right)^{\frac{3}{5}}\cdot\left(\int_0^1 dx\right)^{\frac{1}{5}}\left(\int_0^1 dx\right)^{\frac{1}{5}}\geq\int_0^1 f^3(x)dx$$

$\Rightarrow \int_0^1 f^5(x)dx \ge \left(\sqrt[7]{2}\right)^{\frac{5}{3}} = 2^{\frac{5}{21}}$ ***(1). Working the same way, we have***

$$\left(\int_0^1 f^7(x)\,dx\right)^{\frac{3}{7}} \ge \sqrt[7]{2} \Leftrightarrow \int_0^1 f^7(x)\,dx \ge 2^{\frac{7}{21}} \quad \textit{(2) and}$$

$$\left(\int_0^1 f^9(x)dx\right)^{\frac{3}{9}} \ge \sqrt[7]{2} \Leftrightarrow \int_0^1 f^9(x)\,dx \ge 2^{\frac{9}{21}} \quad \textit{(3)}$$

Multiplying (1) × (2) × (3) we have

$\int_0^1 f^5(x)dx \cdot \int_0^1 f^7(x)dx \cdot \int_0^1 f^9(x)dx \ge 2$ ***as we want!***

122. **In all $\Delta\ ABC$,**

$$\sum_{cyc} \int_0^{\frac{h_a^2}{h_ah_b+h_bh_c+h_ah_c}} e^{-t^2}\,dt \le 3\tan^{-1}\frac{R}{6R}$$

Proposed by Soumitra Mandal-Chandar Nagore-India

Solution by Daniel Sitaru – Romania

$$e^{x^2} \ge x^2+1 \to e^{-x^2} \le \frac{1}{x^2+1} \to$$

$$\int_0^{\frac{h_a^2}{h_ah_b+h_bh_c+h_ah_c}} e^{-t^2}\,dt \le \tan^{-1}\left(\frac{h_a^2}{h_ah_b+h_bh_c+h_ch_a}\right)$$

$$\sum_{cyc} \int_0^{\frac{h_a^2}{h_ah_b+h_bh_c+h_ah_c}} e^{-t^2}\,dt \le \sum \tan^{-1}\left(\frac{h_a^2}{h_ah_b+h_bh_c+h_ch_a}\right) \overset{JENSEN}{\le}$$

$$\le 3\tan^{-1}\left(\frac{1}{3}\sum\frac{h_a^2}{h_ah_b+h_bh_c+h_ch_a}\right) \overset{LEMMA}{\le} 3\tan^{-1}\frac{R}{6r}$$

LEMMA:

$$\frac{h_a^2+h_b^2+h_c^2}{h_ah_b+h_bh_c+h_ah_c}\le\frac{R}{2r}$$

By Adil Abdullayev

We have, $h_a=\frac{2\Delta}{a},h_b=\frac{2\Delta}{b},h_c=\frac{2\Delta}{c},a+b+c=2p$ ***and***

$$ab+bc+ca=p^2+r^2+4Rr$$

$$\frac{h_a^2+h_b^2+h_c^2}{h_ah_b+h_bh_c+h_ah_c}\le\frac{R}{2r}\Leftrightarrow\frac{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}\le\frac{R}{2r}$$

$$\Leftrightarrow\frac{a^2b^2+b^2c^2+a^2c^2}{abc(a+b+c)}\le\frac{R}{2r}\Leftrightarrow\frac{(p^2+r^2+4Rr)^2}{abc(a+b+c)}\le\frac{R+4r}{2r}$$

$$\Leftrightarrow\frac{p^4+r^4+16r^2r^2+2p^2r^2+8Rr^3+8Rrp^2}{8Rrp^2}\le\frac{R+4r}{2r}$$

$$\Leftrightarrow p^4+r^4+16R^2r^2+2p^2r^2+8Rr^3+8Rrp^2\le4R^2p^2+16Rrp^2$$

$$\Leftrightarrow p^4+r^4+16R^2r^2+2p^2r^2+8Rr^3\le4R^2p^2+8Rrp^2$$

We know, $p^2\le4R^2+4Rr+3r^2$***, then we need to prove,***

$$p^2(4R^2+4Rr+3r^2)+(r^2+4Rr)^2+2p^2r^2\le4R^2p^2+8Rrp^2$$

$$\Leftrightarrow p^2(5r^2-4Rr)+(r^2+4Rr)^2\le0\Leftrightarrow p^2\ge\frac{(r^2+4Rr)^2}{4Rr-5r^2}$$

Again, we know, $p^2\ge16Rr-5r^2$***, we will show,*** $16Rr-5r^2\ge\frac{\left(r^2+4Rr\right)^2}{4Rr-5r^2}$

$\Leftrightarrow 4R^2-9Rr+2r^2\ge(R-2r)(4R-r)\ge0$***, which is true.***

Hence Proved

123. **From the book: "Sinergy Math"**

If $x, y, z \in (0, \infty)$

$$\Omega(x) = \lim_{n\to\infty}\left(\frac{(x+3)^{\frac{1}{n}} + x^{\frac{1}{n}}}{(x+2)^{\frac{1}{n}} + (x+1)^{\frac{1}{n}}}\right)^n$$

Then:

$$\Omega^2(x) + \Omega^2(y) + \Omega^2(z) < 3 + 2\sum\frac{1}{x+2}$$

Proposed by Daniel Sitaru – Romania

Solution by Quang Minh Tran-Vietnam

If x in positive real number we have $\Omega^2(x) = \frac{x(x+3)}{(x+2)(x+1)}$

Now we must prove $\sum\left[\frac{x(x+3)}{(x+2)(x+1)} - \frac{2}{x+2}\right] < 3 \Leftrightarrow \sum\frac{x-1}{x+1} < 3 \Leftrightarrow$

$$\Leftrightarrow \sum\left(1 - \frac{2}{x+1}\right) < 3 \Leftrightarrow 3 - 2\sum\frac{1}{x+1} < 3$$

124. $1 < \int_0^1\int_0^1 (x+4)^4 dxdy < \frac{16}{5}$

$$\int_a^b\int_a^b (x+y)^4\,dxdy \le \int_a^b\int_a^b\int_0^1 (tx + (1-t)y)^4\,dxdydt,\ a < b$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

We have, by $R.M \ge A.M \ge G.M$; $8(x^4 + y^4) \ge (x+y)^4 \ge 16x^2y^2$

$$8\left(\int_0^1 dy\right)\left(\int_0^1 x^4\,dx\right) + 8\left(\int_0^1 dx\right)\left(\int_0^1 y^4\,dy\right) \ge \int_0^1\int_0^1 (x+y)^4 dxdy \ge$$

$\geq 16\left(\int_0^1 x^2 dx\right)\left(\int_0^1 y^2 dy\right); \frac{16}{5} \geq \int_0^1 \int_0^1 (x+y)^4\, dxdy \geq \frac{16}{9} > 1$ ***(Proved)***

We have, $0 \leq t \leq 1 \Rightarrow 0 \leq xt \leq x$***, similarly,*** $0 \leq 1-t \leq 1$

$\Rightarrow 0 \leq y(1-t) \leq y$***. Adding we have,*** $0 \leq xt + y(1-t) \leq x + y$

$$\int_a^b \int_a^b \int_0^1 \left(xt + y(1-t)\right)^2 dxdydt \leq \int_a^b \int_a^b \int_0^1 (x+y)^4 dxdydt = \int_a^b \int_a^b (x+y)^4\, dxdy$$

125. If $m, n \in \mathbb{N}, m \geq 2, n \geq 2$ then:

$$\left(\int_0^{\frac{\pi}{2}} \sqrt[m]{\tan x}\, dx\right)\left(\int_0^{\frac{\pi}{2}} \sqrt[n]{\tan x}\, dx\right) \geq \frac{\pi^2}{\left(\cos\frac{\pi}{2m} + \cos\frac{\pi}{2n}\right)^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x\, dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}, -1 < p, q < 1$$

$$\Gamma(p)\Gamma(1-p) = \pi \csc \pi p$$

$$\left(\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{m}} x \cos^{-\frac{1}{m}} x\, dx\right)\left(\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{n}} x \cos^{-\frac{1}{n}} x\, dx\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{m+1}{2m}\right)\Gamma\left(\frac{m-1}{2m}\right) \cdot \frac{1}{2}\Gamma\left(\frac{n+1}{2n}\right)\Gamma\left(\frac{n-1}{2n}\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{m+1}{2m}\right)\Gamma\left\{1 - \frac{m+1}{2m}\right\} \cdot \frac{1}{2}\Gamma\left(\frac{n+1}{2n}\right)\Gamma\left\{1 - \frac{n+1}{2n}\right\}$$

$$= \frac{\pi^2}{4}\csc\frac{\pi(m+1)}{2m}\csc\frac{\pi(n+1)}{2n} = \frac{\pi^2}{4\cos\frac{\pi}{2m}\cos\frac{\pi}{2n}} \overset{AM \geq GM}{\geq} \frac{\pi^2}{\left(\cos\frac{\pi}{2m} + \cos\frac{\pi}{2n}\right)}$$

126. From the book: "Math Accent"

$$\int_1^{\sqrt{3}} \sin^{-1}\left(\frac{2x}{1+x^2}\right)(\tan^{-1}x)^2 dx < \frac{\pi^3}{27}(\sqrt{3}-1)$$

Proposed by Daniel Sitaru – Romania

Solution by Togrul Ehmedov-Baku-Azerbaidian

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x;\ J = \int_1^{\sqrt{3}} (\pi - 2\tan^{-1}x)(\tan^{-1}x)^2 dx$$

$$(\pi - 2tan^{-1}x)(\tan^{-1}x)^2 \overset{AM-GM}{\lesssim} \left(\frac{\pi}{3}\right)^3;\ \max_{[1,\sqrt{3}]}(\pi - 2\tan^{-1}x)(\tan^{-1}x)^2 = \frac{\pi^3}{27}$$

$$(\pi - 2\tan^{-1}x)(\tan^{-1}x)^2 < \frac{\pi^3}{27};\ \int_1^{\sqrt{3}} (\pi - 2\tan^{-1}x)(\tan^{-1}x)^2 dx < \int_1^{\sqrt{3}} \frac{\pi^3}{27} dx$$

$$J < \frac{\pi^3}{27}(\sqrt{3}-1)$$

127. Prove that if $a \in \mathbb{R}$ then:

$$\int_{a+8}^{a+11} e^{x^2} dx + \int_{a+4}^{a+7} e^{x^2} dx \le \int_a^{a+3} e^{x^2} dx + \int_{a+12}^{a+15} e^{x^2} dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Solution 2 by Leonard Giugiuc – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Lemma: Let f be a convex function defined on $I \subseteq \mathbb{R}$ then for any

$x \le y \le z$ in I we have, $f(x-y+z) \le f(x) - f(y) + f(z)$

Now, $\left\{e^{m^2}\right\}'' = 2e^{m^2} + 4m^2e^{m^2} > 0$ ***for all*** $m \in \mathbb{R}$

Let $z = n + 12$ ***and*** $y = n + 8$ ***then from*** $f(x - y + z) \le f(z) - f(y) + f(x) \Rightarrow$

$\Rightarrow f(n+4) + f(n+8) \le f(n) + f(n+12)$ ***where*** $x \in [a, a+3]$ ***then***

$$\int_a^{a+3} f(n+4)\,dn + \int_a^{a+3} f(n+8)\,dn \le \int_a^{a+3} f(n)\,dn + \int_a^{a+3} f(n+12)\,dn$$

$$\Rightarrow \int_{a+4}^{a+7} f(x)\,dx + \int_{a+8}^{a+11} f(x)\,dx \le \int_a^{a+3} f(x)\,dx + \int_{a+12}^{a+15} f(x)\,dx$$

$$\therefore \int_{a+8}^{a+11} e^{x^2}\,dx + \int_{a+4}^{a+7} e^{x^2}dx \le \int_a^{a+3} e^{x^2}dx + \int_{a+12}^{a+15} e^{x^2}dx$$

Solution 2 by Leonard Giugiuc – Romania

Let f ***be an antiderivative of*** e^{x^2} ***on*** R***. Then***

$$f'''(x) = 2(1 + 2x^2)e^{x^2} > 0, \forall x \in R.$$

Let $g: R \to R, g(t) = \int_t^{t+3} e^{x^2}\,dx$***. Then*** $g''(t) = f''(t+3) - f''(t) \ge 0$***,***

because f'' ***strictly increasing. Hence*** g ***is convex.***

We need to prove $g(a+4) + g(a+8) \le g(a) + g(a+12)$***.***

We have: $a < a + 4 < a + 8 < a + 12$ ***and*** $a + a + 12 = a + 4 + a + 8$***,***

hence by Karamata $g(a+4) + g(a+8) \le g(a) + g(a+12)$***.***

128. If $a, b, p, q \in \mathbb{R}, a < b, p > 1, p + q = pq$ **then:**

$$(b-a)^2\sqrt{e^{a+b}} \le \int_a^b \int_a^b e^{\frac{qx+py}{p+q}}\,dx\,dy \le (b-a)(e^b - e^a)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India, Solution 2 by Saptak Bhattacharya-Kolkata-India

Solution 1 by Soumitra Mandal-Chandar Nagore-India

We have, $\frac{1}{p}+\frac{1}{q}=1$ ***and*** $p,q>0$ ***now,*** $e^{\frac{x}{p}}$ ***and*** $e^{\frac{y}{p}}$ ***are convex functions,***

hence by Hermite – Hadamard Inequality

$$\int_a^b e^{\frac{x}{p}}\,dx \ge (b-a)e^{\frac{a+b}{2p}},\ \int_a^b e^{\frac{y}{q}}\,dy \ge (b-a)e^{\frac{a+b}{2q}}$$

$$\therefore \int_a^b\int_a^b e^{\frac{qx+py}{p+q}}\,dx\,dy = \left(\int_a^b e^{\frac{x}{p}}\,dx\right)\left(\int_a^b e^{\frac{y}{q}}\,dy\right) \ge (b-a)^2 e^{\frac{a+b}{2}\left(\frac{1}{p}+\frac{1}{q}\right)}$$

$= (b-a)^2\sqrt{e^{a+b}}$***. Now,*** $e^{\frac{qx+py}{p+q}} \le \frac{e^x}{p}+\frac{e^y}{q}$ $\left[\because e^m \text{ is a convex function and } \frac{1}{p}+\frac{1}{q}=1\right]$

$$\Rightarrow \int_a^b\int_a^b e^{\frac{qx+py}{p+q}}\,dx\,dy \le \frac{1}{p}\int_a^b\int_a^b e^x\,dx\,dy + \frac{1}{q}\int_a^b\int_a^b e^y\,dx\,dy$$

$$= (b-a)\left(e^b-e^a\right)\left(\frac{1}{p}+\frac{1}{q}\right) = (b-a)\left(e^b-e^a\right)$$

$$\therefore (b-a)^2\sqrt{e^{a+b}} \le \int_a^b\int_a^b e^{\frac{qx+py}{p+q}}\,dx\,dy \le (b-a)\left(e^b-e^a\right)$$

Solution 2 by Saptak Bhattacharya-Kolkata-India

$f(t)=e^t$ ***is a convex function, so by first half of Hermite Hadamard***

inequality; (note that $q=\frac{p}{p-1}>0$***)***

$$\int_a^b e^{\frac{py}{b+q}}\int_a^b e^{\frac{dx}{p+q}}\,dx\,dy \ge \int_a^b e^{\frac{py}{b+q}}\cdot(b-a)e^{\frac{q}{p+q}\cdot\left(\frac{a+b}{2}\right)}dy$$

$$\ge (b-a)^2 e^{\frac{a+b}{2}\left(\frac{p+q}{p+q}\right)} = (b-a)^2 e^{\frac{a+b}{2}} \quad \textit{(i)}$$

Since $q > 0$; *let* $\lambda = \frac{q}{p+q} \in (0,1)$. *Then* $1-\lambda = \frac{p}{p+q} \in (0,1)$

Also $f(t) = e^t$ *is convex. Thus; by Jensen*

$$f(2x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

$$\Rightarrow \int_a^b \int_a^b e^{2x+(1-\lambda)y}\, dx\, dy \le \int_a^b \int_a^b (\lambda e^x + (1-\lambda)e^y)\, dx\, dy$$

$$= (\lambda + (1-\lambda))(b-a)(e^b - e^a) = (b-a)(e^b - e^a)$$ *(Proved)* *(ii)*

129. For $a_i \in (0,1], \forall\, i \in [1,n]$

Prove:

$$\frac{1}{\pi^n} \cdot \prod_{i=1}^n a_i^2 \le \int_o^{a_n} \dots \int_o^{a_1} \left(\prod_{i=1}^n \sin x_i \right) dx_1 \dots dx_n \le \frac{1}{2^n} \cdot \left(\prod_{i=1}^n a_i \right)^2$$

Proposed by Uche Eliezer Okeke-Anambra-Nigeria

Solution by Daniel Sitaru – Romania

$$\frac{2x_1}{\pi} < \sin x_1 < x_1 \quad \textit{(Jordan)}$$

$$\frac{2}{\pi} \int_0^{a_1} x_1 dx \le \int_0^{a_1} \sin x_1\, dx_1 \le \int_0^{a_1} x_1 dx_1$$

$$\frac{1}{\pi} \cdot a_1^2 \le \int_0^{a_1} \sin x_1\, dx_1 \le \frac{1}{2} \cdot a_1^2,$$

$$\frac{1}{\pi} \cdot a_2^2 \le \int_0^{a_2} \sin x_2\, dx_1 \le \frac{1}{2} \cdot a_2^2$$

....

$$\frac{1}{\pi}\cdot a_n^2 \le \int_0^{a_n} \sin x_n\, dx_1 \le \frac{1}{2}\cdot a_n^2,$$

$$\frac{1}{\pi^n}\cdot \prod_{i=1}^{n} a_i^2 \le \prod_{i=1}^{n}\int_0^{a_1} \sin x_i\, dx_i \le \frac{1}{2^n}\prod_{i=1}^{n} a_i^2$$

$$\frac{1}{\pi^n}\cdot \prod_{i=1}^{n} a_i^2 \le \int_0^{a_n} \dots \int_0^{a_1}\left(\prod_{i=1}^{n} \sin x_i\right) dx_1 \dots dx_n \le \frac{1}{2^n}\prod_{i=1}^{n} a_i^2$$

130. If $a, b, p, q \in \mathbb{R}, a < b, p > 1, p + q = pq$ then:

$$\frac{a^2+2ab+b^2}{4} < \frac{\int_a^b \int_a^b (px+qy)^2\, dx\, dy}{(b-a)^2(p+q)^2} < \frac{a^2+ab+b^2}{3}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\left(\frac{px+qy}{p+q}\right)^2 = \left(\frac{x}{q}+\frac{y}{p}\right)^2 = \frac{x^2}{q^2}+\frac{2xy}{pq}+\frac{y^2}{p^2}$$

$$\int_a^b\int_a^b \left(\frac{px+qy}{p+q}\right)^2 dx\, dy = \frac{1}{q^2}\int_a^b\int_a^b x^2\, dx\, dy + \frac{2}{pq}\left(\int_a^b x\, dx\right)\left(\int_a^b y\, dy\right) + \frac{1}{p^2}\int_a^b\int_a^b y^2 dy\, dx$$

$$= \frac{(b-a)(b^3-a^3)}{3q^2} + \frac{(b-a)(b^3-a^3)}{3p^2} + \frac{(b^2-a^2)^2}{2pq}$$

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\frac{px+qy}{p+q}\right)^2 dx\, dy = \frac{a^2+ab+b^2}{3}\left(\frac{1}{p^2}+\frac{1}{q^2}\right) + \frac{(b+a)^2}{2pq}$$

$$\ge \frac{(a+b)^2}{4}\left(\frac{1}{p^2}+\frac{1}{q^2}\right) + \frac{(a+b)^2}{2pq}\left[\because a^2+ab+b^2 \ge \frac{3(a+b)^2}{4}\right]$$

$= \frac{(a+b)^2}{4}\left(\frac{1}{p}+\frac{1}{q}\right)^2 = \frac{a^2+2ab+b^2}{4}$. **Similarly,** $\frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\frac{px+qy}{p+q}\right)^2 dx\, dy$

$$\leq \frac{a^2+ab+b^2}{3}\left(\frac{1}{p^2}+\frac{1}{q^2}\right)+\frac{2(a^2+ab+b^2)}{3pq}=\frac{a^2+ab+b^2}{3}\left(\frac{1}{p}+\frac{1}{q}\right)^2=\frac{a^2+ab+b^2}{3}$$

131. Prove:

$$\int_0^1 \ln^2\left(1+\sqrt{\sin x}\right)dx<\frac{1}{2}$$

Proposed by Uche Eliezer Okeke-Anambra-Nigeria

Solution by Daniel Sitaru-Romania

$$e^x \geq 1+x, x\in\mathbb{R}, \log(1+x)\leq x, x>-1\to \log\left(1+\sqrt{\sin x}\right)\leq \sqrt{\sin x}\to$$

$$\log^2\left(1+\sqrt{\sin x}\right)\leq \sin x\leq x;\ \int_0^1 \log^2\left(1+\sqrt{\sin x}\right)<\int_0^1 x\,dx=\frac{1}{2}$$

132. $I(a,b)=\int_a^b\left(\arctan\left(\frac{a\sin x}{b+a\cos x}\right)+\arctan\left(\frac{b\sin x}{a+b\cos x}\right)\right)dx,$

$$0<a<b<c<\frac{\pi}{2}$$

Prove that:

$$\frac{2}{b-a}I(a,b)+\frac{2}{c-b}I(b,c)+\frac{2}{c-a}I(a,c)\geq \sum\left(\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by Saptak Bhattacharya - Kolkata-India, Solution 3 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Ravi Prakash-New Delhi-India

$$\frac{a\sin x}{b+a\cos x}=\frac{a\left(2\tan\frac{x}{2}\right)}{b\left(1+\tan^2\frac{x}{2}\right)+a\left(1-\tan^2\frac{x}{2}\right)}$$

$$=\frac{2a\tan\left(\frac{x}{2}\right)}{(a+b)+(b-a)\tan^2\left(\frac{x}{2}\right)}=\frac{\frac{2a}{b+a}\tan\left(\frac{x}{2}\right)}{1+\frac{b-a}{b+a}\tan^2\frac{x}{2}}=\frac{\tan\frac{x}{2}-\frac{b-a}{b+a}\tan\frac{x}{2}}{1+\frac{b-a}{b+a}\tan^2\left(\frac{x}{2}\right)}$$

Put $\frac{b-a}{b+a}\tan\frac{x}{2}=\tan\theta\therefore\frac{a\sin x}{b+a\cos x}=\frac{\tan\frac{x}{2}-\tan\theta}{1+\tan\frac{x}{2}\tan\theta}=\tan\left(\frac{x}{2}+\theta\right)$

$$\Rightarrow\arctan\left(\frac{a\sin x}{b+a\cos x}\right)=\frac{x}{2}+\theta=\frac{x}{2}+\tan^{-1}\left(\frac{b-a}{b+a}\tan\frac{x}{2}\right)$$

Similarly, $\arctan\left(\frac{b\sin x}{a+b\cos x}\right)$

$$=\frac{x}{2}+\arctan\left(\frac{a-b}{a+b}\tan\frac{x}{2}\right)=\frac{x}{2}-\arctan\left(\frac{b-a}{b+a}\tan\frac{x}{2}\right)$$

$$\therefore I(a,b)=\int_a^b\left(\frac{x}{2}+\frac{x}{2}\right)dx=\frac{1}{2}(b^2-a^2)\Rightarrow\frac{2}{b-a}I(a,b)=b+a$$

Thus, $\frac{2}{b-a}I(a,b)+\frac{2}{c-b}I(b,c)+\frac{2}{c-a}I(c,a)=2(a+b+c)$

Now, $\frac{a+b}{2}\geq\sqrt{ab}\quad[AM\geq GM]$ ***and*** $\frac{a+b}{2}\geq\sqrt{\frac{a^2+b^2}{2}}$

$$\Leftrightarrow(a+b)^2-2(a^2+b^2)\geq 0\Leftrightarrow(a-b)^2\geq 0$$

$$\therefore a+b\geq\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\Rightarrow\sum(a+b)\geq\sum\left(\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\right)$$

Solution 2 by Saptak Bhattacharya-Kolkata-India

$$f(a,b)=\int_a^b\tan^{-1}\left(\frac{\frac{a\sin x}{b+a\cos x}+\frac{b\sin x}{a+b\cos x}}{1-\frac{ab\sin^2 x}{(b+a\cos x)(a+b\cos x)}}\right)dx$$

$$=\int_a^b\tan^{-1}\frac{\sin x\,(a^2+b^2+2ab\cos x)}{(a^2+b^2)\cos x+ab(1+\cos 2x)}dx$$

$$= \int_a^b \tan^{-1}\frac{\sin x(a^2+b^2+2ab\cos x)}{\cos x(a^2+b^2+2ab\cos x)}dx = \int_a^b \tan^{-1}\tan x\,dx = \int_a^b x\,dx = \frac{b^2-a^2}{2}$$

Now, $\frac{1}{b-a}f(a,b) = \frac{a+b}{2} = \frac{\sqrt{(a+b)^2}}{2} = \frac{\sqrt{a^2+b^2+2ab}}{2} = \frac{\sqrt{\left(\frac{a^2+b^2}{2}+ab\right)}}{2}$

$$\overset{Power\ mean}{\geq} \frac{\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}}{2}$$ **. Hence** $\frac{2}{b-a}f(a,b) \geq \sqrt{ab} + \sqrt{\frac{a^2+b^2}{2}}$

$$\Rightarrow \sum \frac{2f(a,b)}{b-a} \geq \sum\left(\sqrt{ab} + \sqrt{\frac{a^2+b^2}{2}}\right)$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$I(a,b) = \int_a^b \left(\tan^{-1}\left(\frac{a\sin x}{b+a\cos x}\right) + \tan^{-1}\left(\frac{b\sin x}{a+b\cos x}\right)\right)dx$$

$$= \int_a^b \left(\tan^{-1}\frac{\tan\frac{x}{2} - \frac{b-a}{b+a}\tan\frac{x}{2}}{1+\frac{b-a}{b+a}\tan^2\frac{x}{2}} + \tan^{-1}\frac{\tan\frac{x}{2} + \frac{b-a}{b+a}\tan\frac{x}{2}}{1-\frac{b-a}{b+a}\tan^2\frac{x}{2}}\right)dx$$

$$= \int_a^b \left(\frac{x}{2} - \tan^{-1}\left(\frac{b-a}{b+a}\tan\frac{x}{2}\right) + \frac{x}{2} + \tan^{-1}\left(\frac{b-a}{b+a}\tan\frac{x}{2}\right)\right)dx$$

$$= \int_a^b x\,dx = \frac{b^2-a^2}{2} \Rightarrow \frac{2}{b-a}I(a,b) = a+b$$

$$\therefore \sum_{cyc}\frac{2}{b-a}I(a,b) = 2(a+b+c) = \sum_{cyc}(a+b) = \sum_{cyc}\sqrt{(1+1)\frac{(a+b)^2}{2}}$$

$$\overset{Cauchy-Schwarz}{\geq} \sum_{cyc}\left(\sqrt{ab} + \sqrt{\frac{a^2+b^2}{2}}\right)$$

133. If $a, b, c > 0, a + b + c = \pi$ then:

$$2\sum\int_0^a \frac{\arctan^2 x}{x}dx + \log(1+a^2)\log(1+b^2)\log(1+c^2) < \pi^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Saptak Bhattacharya-Kolkata-India,

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Saptak Bhattacharya-Kolkata-India

Let $f(x) = x - \tan^{-1} x$; $f'(x) = 1 - \frac{1}{1+x^2} > 0$. **So,** $\forall x > 0$;

$f(x) > f(0) = 0$ ***thus:*** $(\tan^{-1} x)^2 < x^2 \Rightarrow \frac{(\tan^{-1} x)^2}{x} < x$;

$\int_0^a \frac{(\tan^{-1} x)^2 dx}{x} < \frac{a^2}{2}$. ***Thus,*** $2\sum\int_0^a \frac{(\tan^{-1} x)^2 dx}{x} < \sum a^2$ ***(I)***

Now, consider $\phi(x) = x - \ln(1+x^2)$

$$\phi(0) = 0;\ \phi'(x) = 1 - \frac{2x}{1+x^2} = \frac{(x-1)^2}{1+x^2} > 0$$

So, $\phi(x) > 0\ \ \forall x > 0 \Rightarrow \ln(1+x^2) < x$. ***Thus,*** $\prod \ln(1+a^2) < abc$ ***(ii)***

Now, by $AM \ge HM \sum\frac{1}{a} \ge \frac{9}{\pi^2} \Rightarrow \sum\frac{2}{a} \ge \frac{18}{\pi^2} > 1$ $[\because \pi < 4;\ \pi^2 < 16 < 18]$

Thus, $abc < \sum 2ab = 2\sum ab$ ***(iii). Combining (i) & (iii)***

$$LHS < \sum a^2 + 2\sum ab = (a+b+c)^2 = \pi^2$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $t = a\tan\theta$, $dt = a\sec^2\theta\, d\theta$

when $t = 0, \theta = 0$, ***when*** $t = x, \theta = \tan^{-1} x$

$$\Omega(a)=\lim_{x\to\infty}\int_0^x \frac{\log t}{t^2+a^2}dt=\frac{1}{a}\lim_{x\to\infty}\int_a^{\tan^{-1}x}\log(a\tan\theta)\,d\theta$$

$$=\frac{1}{a}\lim_{x\to\infty}\int_0^{\tan^{-1}x}\log(a\tan(\tan^{-1}x-\theta))\,d\theta=\frac{1}{a}\lim_{x\to\infty}\int_0^{\tan^{-1}x}\log\left(a\cdot\frac{x-\tan\theta}{1+x\tan\theta}\right)d\theta$$

$$=\frac{1}{a}\lim_{x\to\infty}\int_0^{\tan^{-1}x}\log\left(a\cdot\frac{1-\frac{\tan\theta}{x}}{\frac{1}{x}+\tan\theta}\right)d\theta=\frac{1}{a}\int_0^{\frac{\pi}{2}}\log\left(\frac{a^2}{a\tan\theta}\right)=\pi\log a^{\frac{1}{a}}-\Omega(a)$$

$$\Rightarrow 2\Omega(a)=\pi\log a^{\frac{1}{a}}\Rightarrow\Omega(a)=\frac{\pi}{2}\log a^{\frac{1}{a}}.\ \textit{So,}\sum_{cyc}\Omega^2(a)=\frac{\pi^2}{4}\log^2\left(a^{\frac{1}{a}}\right)$$

$$\geq\frac{\pi^2}{12}\left(\sum_{cyc}\log\left(a^{\frac{1}{a}}\right)\right)^2=\frac{\pi^2}{12}\log^2\left(a^{\frac{1}{a}}\cdot b^{\frac{1}{b}}\cdot c^{\frac{1}{c}}\right)$$

134. **If $0<a<b$ then:**

$$\frac{2}{\pi}\ln\left(\frac{b}{a}\right)+b-a<\frac{\pi}{2}\int_a^b\frac{dx}{\arctan x}<\frac{\pi}{2}\ln\left(\frac{b}{a}\right)+b-a$$

Proposed by Daniel Sitaru – Romania

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

If $0<a<b$ ***then*** $\frac{2}{\pi}\ln\left(\frac{b}{a}\right)+b-a<\frac{\pi}{2}\int_a^b\frac{dx}{\arctan x}<\frac{\pi}{2}\ln\left(\frac{b}{a}\right)+b-a$

We need to prove that $\frac{2}{\pi x}+1<\frac{1}{2\arctan x}<\frac{\pi}{2x}+1$ ***(1)*** $\forall x>0$

Put $\arctan x=t\Rightarrow 0<t<\frac{\pi}{2}$. ***We have (1)*** $\Rightarrow\frac{2}{\pi\tan t}+1<\frac{\pi}{2t}<\frac{\pi}{2\tan t}+1$

$*\ f(t)=\frac{\pi}{2t}-\frac{2}{\pi\cdot\tan t}-1.$ ***We have*** $f'(t)=\frac{2}{\pi\cdot\sin^2 t}-\frac{\pi}{2t^2}=\frac{4t^2-\pi^2\cdot\sin^2 t}{2t^2\cdot\pi\cdot\sin^2 t}$

On the other hand, by Jordan's inequality, we have

$$\sin t > \frac{2t}{\pi} \Rightarrow \sin^2 t > \frac{4t^2}{\pi^2} \Rightarrow 4t^2 - \pi^2 \cdot \sin^2 t < 4t^2 - \pi^2 \cdot \frac{4t^2}{\pi^2} = 0 \Rightarrow f'(t) < 0$$

$\Rightarrow f(t)$ ***is a decreasing function*** $\Rightarrow$

$$\Rightarrow f(t) > \lim_{t\to\frac{\pi}{2}}\left(\frac{\pi}{2t} - \frac{2}{\pi\cdot\tan t} - 1\right) \Rightarrow f(t) > 0 \Rightarrow \frac{2}{\pi\cdot\tan t} + 1 < \frac{\pi}{2t} \quad \textit{(2)}$$

$$* \, g(t) = \frac{\pi}{2\tan t} + 1 - \frac{\pi}{2t}$$

We have $g'(t) = \frac{\pi}{2t^2} - \frac{\pi}{2\sin^2 t} = \frac{2\pi\cdot\sin^2 t - 2\pi\cdot t^2}{4t^2\cdot\sin^2 t} = \frac{2\pi(\sin t - t)(\sin t + t)}{4t^2\cdot\sin^2 t}$

On the other hand, by Jordan's inequality, we have

$\sin t \le t \Rightarrow g'(t) \le 0 \Rightarrow g(t)$ ***is a decreasing functioin***

$$\Rightarrow f(t) > \lim_{t\to\frac{\pi}{2}}\left(\frac{\pi}{2\tan t} + 1 - \frac{\pi}{2t}\right) \Rightarrow f(t) > 0 \Rightarrow \frac{\pi}{2t} < \frac{\pi}{2\cdot\tan t} + 1 \quad \textit{(3)}$$

(2) and (3) $\Rightarrow \frac{2}{\pi\cdot\tan t} + 1 < \frac{\pi}{2t} < \frac{\pi}{2\cdot\tan t} + 1 \Rightarrow$ ***(1) True*** $\Rightarrow$

$$\Rightarrow \int_a^b \left(\frac{2}{\pi\cdot x} + 1\right)dx < \int_a^b \frac{\pi}{2\arctan x}dx < \int_a^b \left(\frac{\pi}{2x} + 1\right)dx$$

$$\Rightarrow \frac{2}{\pi}\ln\left(\frac{b}{a}\right) + b - a < \frac{\pi}{2}\int_a^b \frac{dx}{\arctan x} < \frac{\pi}{2}\ln\left(\frac{b}{a}\right) + b - a$$

135. If $0 < a < b$ then:

$$\frac{1}{b-a}a\int_a^b\int_a^b \frac{dxdy}{x+y} < \frac{13}{25}\log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Chris Kyriazis-Greece

Using the inequality $(x+y)^2 \ge 4xy, x, y > 0$ ***we have that:***

$$x + y \ge \frac{4xy}{x+y} \Leftrightarrow \frac{1}{x+y} \le \frac{1}{4}\left(\frac{1}{x} + \frac{1}{y}\right) \Rightarrow \int_a^b\int_a^b \frac{1}{x+y}dxdy \le \frac{1}{4}\int_a^b\int_a^b\left(\frac{1}{x} + \frac{1}{y}\right)dx\,dy$$

$$\Leftrightarrow \int_a^b\int_a^b \frac{1}{xy}dx\,dy \le \frac{1}{4}\cdot 2(b-a)\ln\left(\frac{b}{a}\right) \Leftrightarrow \frac{1}{b-a}\int_a^b\int_a^b \frac{1}{x+y}dx\,dy \le \frac{1}{2}\ln\left(\frac{b}{a}\right)$$

so, it suffices to prove that $\frac{1}{2}\ln\left(\frac{b}{a}\right) < \frac{13}{25}\ln\left(\frac{b}{a}\right)$ *or* $25 < 26$ ***which holds!***

136. If:

$$\Omega(a) = \iint_{(x,y)=(0,0)}^{(x,y)=(a,a)} \left(\sqrt{x^2+2xy}+\sqrt{y^2+2xy}\right) dxdy,\ a > 0$$

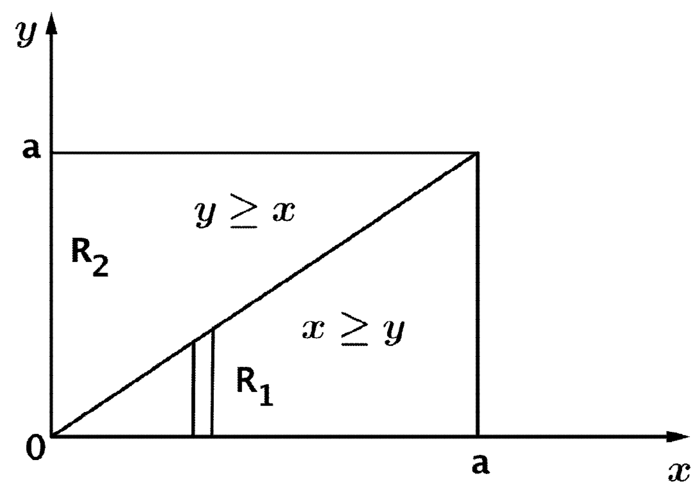
then:

$$\frac{\Omega(a)}{b^3}+\frac{\Omega(b)}{c^3}+\frac{\Omega(c)}{a^3} \geq 2\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by Antonis Anastasiadis-Katerini-Greece

Solution 1 by Ravi Prakash-New Delhi-India



Let $f(x,y) = \sqrt{x^2+2xy}+\sqrt{y^2+2xy},\ x,y \geq 0$

$$\Omega(a) = \int_0^a\int_0^a f(x,y)\,dxdy = \int\int_{R_1} f(x,y)\,dxdy + \int\int_{R_2} f(x,y)dxdy$$

$$\int\int_{R_1} f(x,y)dxdy = \int_0^a \int_{y=0}^{y=x} f(x,y)dxdy \geq \int_0^a \int_{y=0}^{y=x} \left(\sqrt{y^2+2yy}+\sqrt{y^2+2yy}\right)dydx$$

$$= \int_0^a \int_{y=0}^{y=x} 2\sqrt{3}ydydx = \int_0^a \sqrt{3}[y^2]_0^x\, dx = \int_0^a \sqrt{3}\,x^2dx = \frac{1}{\sqrt{3}}a^3$$

Similarly, $\int\int_{R_2} f(x,y)dxdy \geq \frac{1}{\sqrt{3}}a^3 \therefore \Omega(a) \geq \frac{2}{\sqrt{3}}a^3$***. Now***

$$\frac{\Omega(a)}{b^3}+\frac{\Omega(b)}{c^3}+\frac{\Omega(c)}{a^3} \geq \frac{2}{\sqrt{3}}\left(\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\right) \geq 2\sqrt{3}$$

Solution 2 by Antonis Anastasiadis-Katerini-Greece

$$x^2+xy+xy \overset{AM-GM}{\geq} 3\sqrt[3]{x^4y^2} \Leftrightarrow \sqrt{x^2+2xy} \geq \sqrt{3}\sqrt[3]{x^2y}$$

$$y^2+xy+xy \overset{AM-GM}{\geq} 3\sqrt[3]{x^2y^4} \Leftrightarrow \sqrt{y^2+2xy} \geq \sqrt{3}\sqrt[3]{xy^2}$$

$$\text{So}: \sqrt{x^2+2xy}+\sqrt{y^2+2xy} \geq \sqrt{3}\sqrt[3]{x^2y}+\sqrt{3}\sqrt[3]{xy^2} \geq 2\sqrt{\sqrt{3}\sqrt{3}\sqrt[3]{x^2y}\sqrt[3]{xy^2}} = 2\sqrt{3xy} \geq \frac{3}{2}\sqrt{3xy}$$

$$\text{So}: \sqrt{x^2+2xy}+\sqrt{y^2+2xy} \geq \frac{3}{2}\sqrt{3xy}$$

$$\text{and } \Omega(a) \geq \int_0^a\int_0^a \frac{3}{2}\sqrt{3xy}dxdy = \int_0^a \frac{3}{2}\sqrt{3}\frac{a^{\frac{3}{2}}}{\frac{3}{2}}\sqrt{y}dy = \int_0^a \sqrt{3}a^{\frac{3}{2}}y^{\frac{1}{2}}dy = \frac{2\sqrt{3}a^{\frac{3}{2}}a^{\frac{3}{2}}}{3} = \frac{2\sqrt{3}a^3}{3}$$

$$\text{So}: \Omega(a) \geq \frac{2\sqrt{3}a^3}{3}. \text{ Likewise } \Omega(b) \geq \frac{2\sqrt{3}b^3}{3} \text{ and } \Omega(c) \geq \frac{2\sqrt{3}c^3}{3}$$

$$\text{So}: \frac{\Omega(a)}{b^3}+\frac{\Omega(b)}{c^3}+\frac{\Omega(c)}{a^3} \geq \frac{2\sqrt{3}a^3}{3b^3}+\frac{2\sqrt{3}b^3}{3c^3}+\frac{2\sqrt{3}c^3}{3a^3} = \frac{2\sqrt{3}}{3}\left(\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\right) \overset{AM-GM}{\geq} \frac{2\sqrt{3}}{3}3\sqrt[3]{\frac{a^3b^3c^3}{b^3c^3a^3}} = 2\sqrt{3}$$

137. If $n \in \mathbb{N}^*$ then:

$$\int_0^1\int_0^1 \ldots \int_0^1 \prod_{i=1}^{n}\left(1+x_i^2\right)dx_i + \int_0^1\int_0^1 \ldots \int_0^1 \prod_{i=1}^{n}\left(1-x_i^2\right)dx_i \leq 2^n$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Chris Kyriazis-Greece, Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 3 by Kays Tomy-Nador-Tunisia, Solution 4 by Michel Rebeiz-Lebanon, Solution 5 by Hasan Bostanlik-Sarkisla-Turkey

Solution 1 by Chris Kyriazis-Greece

We have that $\int_0^1 \int_0^1 \dots \int_0^1 (1 + x_1^2)\, dx_1 dx_2 \dots dx_2 =$

$$\int_0^1 (1 + x_1^2)\, dx_i \int_0^1 (1 + x_2^2)\, dx_x \dots . \int_0^1 (1 + x_n^2)\, dx_n = \left(\frac{4}{3}\right)^2$$

Doing the same $\int_0^1 \int_0^1 \dots \int_0^1 (1 - x_i^2) dx_1 dx_2 \dots dx_n = \left(\frac{2}{3}\right)^n$

So it suffices to prove that $\left(\frac{4}{3}\right)^n + \left(\frac{2}{3}\right)^n \le 2^n$ ***or*** $2^n + 1 \le 3^n$ ***or***

$1 \le 3^n - 2^n$ ***(*) which clearly holds for every*** $n \in \mathbb{N}^*$

(*) $3^n - 2^n = 3^{n-1} + 3^{n-2} \cdot 2 + \dots + 2^{n-2} \cdot 3 + 2^{n-1} > 1$ ***when*** $n > 1$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$I_n = \int_0^{1^{(n)}} \prod_{i=1}^{n} (1 + x_i^2)\, dx_i + \int_0^{1^{(n)}} \prod_{i=1}^{n} (1 - x_i^2)\, dx_i$$

$$= \prod_{i=1}^{n} \left(\int_0^1 (1 + x_i^2) dx_i \right) + \prod_{i=1}^{n} \left(\int_0^1 (1 - x_i^2) dx_i \right) = \prod_{i=1} \left(\frac{4}{3}\right) + \prod_{i=1} \left(\frac{2}{3}\right)$$

$$= \left(\frac{4}{3}\right)^n + \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^n (2^n + 1) \le \left(\frac{2}{3}\right)^n \times 3^n = 2^n \quad \forall n \in \mathbb{N}^*$$

Solution 3 by Kays Tomy-Nador-Tunisia

Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ ***and*** $\{\beta_1, \beta_2, \dots, \beta_n\}$ ***be two positive sequences of length*** n***. Then*** $\prod_{k=1}^{n} (\alpha_k + b_k) = \prod_{k=1}^{n} \alpha_k + \prod_{k=1}^{n} \beta_k + R_n$ ***with*** $0 < R_n$

it implies $\prod_{k=1}^{n} \alpha_k + \prod_{k=1}^{n} \beta_k \le \prod_{k=1}^{n} (\alpha_k + \beta_k)$ ***(*)***

Let us apply inequality (*) for the case when

$\alpha_k = 1 + x_k^2$ ***and*** $b_k = 1 - x_k^2$ ***with*** $x_k \in (0, 1)$ ***we get***

$$\prod_{k=1}^{n}(1+x_k^2)+\prod_{k=1}^{n}(1-x_k^2)\le\prod_{k=1}^{n}(1+x_k^2+1-x_k^2)$$

$$\Rightarrow\prod_{k=1}^{n}(1+x_k^2)+\prod_{k=1}^{n}(1-x_k^2)\le\prod_{k=1}^{n}2=2^n$$

$$\Rightarrow\int_0^1\int_0^1\dots\int_0^1\prod_{k=1}^{n}(1+x_k^2)\,dx_1\dots dx_n+\int_0^1\int_0^1\dots\int_0^1\prod_{k=1}^{n}(1+x_k^2)\,dx_1\dots dx_n$$

$$\le\int_0^1\int_0^1\dots\int_0^1 2^n dx_1\dots.dx_n=2^n$$

Solution 4 by Michel Rebeiz – Lebanon

Let $a_n=\underbrace{\int_0^1\dots}_{n}\int_0^1\prod_{i=1}^{n}(1+x_i^2)\,dx$ **so** $a_n+b_n\le 2^n$ **??** $a_n>0$ **and** $b_n>0$

and $b_n=\underbrace{\int_0^1\dots}_{n}\int_0^1\prod_{i=1}^{n}(1-x_i^2)\,dx_i$. **For** $n=1$

$a_1+b_1=1+\frac{1}{3}+1-\frac{1}{3}=2\le 2^1$. **Suppose that** $a_n+b_n\le 2^n$

So $a_{n+1}+b_{n+1}=a_n\times\int_0^1(1+x_{x+1}^2)\,dx_{i+1}+b_n\times\int_0^1(1-x_{i+1}^2)\,dx_{i+1}$

$$=\frac{4}{3}a_n+\frac{2}{3}b_n\le\frac{4}{3}a_n+\frac{4}{3}b_n\le\frac{4}{3}\times 2^n\le\frac{2}{3}\times 2^{n+1}\le 2^{n+1}$$

Solution 5 by Hasan Bostanlik-Sarkisla-Turkey

$$\left(x_1+\frac{x_1^3}{3}\right)\Big|_0^1=\frac{4}{3};\ \frac{4}{3}\cdot\left(x_1+\frac{x_2^3}{3}\right)\Big|_0^1=\left(\frac{4}{3}\right)^2;\ \int\int\prod(1+x_1^2)\,dx_1=\left(\frac{4}{3}\right)^n$$

$$\Rightarrow\left(x_1-\frac{x_1^3}{3}\right)\Big|_0^1=\frac{2}{3};\ \frac{2}{3}\cdot\left(x_2-\frac{x_2^3}{3}\right)\Big|_0^1=\left(\frac{2}{3}\right)^2,\int\int\dots\int\prod(1-x_1^2)=\left(\frac{2}{3}\right)^n$$

$$\left(\frac{4}{3}\right)^n+\left(\frac{2}{3}\right)^n\le\left(\frac{4}{3}+\frac{2}{3}\right)^n\le 2^n;\ n\in\mathbb{N}^*$$

138. $\int_0^1 \int_0^1 \int_0^1 \left(x\sqrt{x^2+z^2}+y\sqrt{y^2+z^2}\right)dxdydz \le 1$.

Proposed by Daniel Sitaru – Romania

Solution 1 by Chris Kyriazis-Greece, Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 3 by Anisoara Dudu-Romania, Solution 4 by Hasan Bostanlik-Sarkisla-Turkey

Solution 1 by Chris Kyriazis-Greece

By Cauchy – Schwarz inequality we have that:

$$x\sqrt{x^2+z^2}+y\sqrt{y^2+z^2} \le \sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2+z^2} \Leftrightarrow$$

$$\Leftrightarrow x\sqrt{x^2+z^2}+y\sqrt{y^2+z^2} \le x^2+y^2+z^2 \Rightarrow$$

$$\int_0^1\int_0^1\int_0^1 \left(x\sqrt{x^2+z^2}+y\sqrt{y+z^2}\right)dxdydz \le \int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dxdydz$$

But $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dxdydz = 1$ ***cause***

$$\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dxdydz = \int_0^1\int_0^1\int_0^1 x^2dxdydz + \int_0^1\int_0^1\int_0^1 y^2dxdydz +$$

$$+\int_0^1\int_0^1\int_0^1 z^2dxdydz = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{y^3}{3}\right]_0^1 + \left[\frac{z^3}{3}\right]_0^1 = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$I = \int_0^1\int_0^1\int_0^1 x\sqrt{x^2+z^2}+y\sqrt{y^2+z^2}dxdydz$$

$$= \int_0^1\int_0^1 \left[\frac{1}{3}\sqrt{(x^2+z^2)^3}+xy\sqrt{y^2+z^2}\right]_0^1 dydz$$

$$\leq \int_0^1\int_0^1 \frac{1}{3}\sqrt{(1+z^2)^3}+y\sqrt{y^2+z^2}dydz$$

$$= \int_0^1 \left[\frac{y}{3}\sqrt{(1+z^2)^3}+\frac{1}{3}\sqrt{(y^2+z^2)^3}\right]_0^1 dz$$

$$= \int_0^1 \frac{1}{3}\sqrt{(1+z^2)^3}+\frac{1}{3}\sqrt{(1+z^2)^3}dz$$

$$I = \frac{2}{3}\int_0^1 \sqrt{(1+z^2)^3}\,dz = \frac{2}{3}\int_0^1 \left(\frac{1}{\sqrt{1+z^2}}+z^2\sqrt{1+z^2}\right)dz$$

$$= \frac{2}{3}\left(\left[\ln\left(z+\sqrt{1+z^2}\right)\right]_0^1+\left[1\ \times\frac{1}{3}\sqrt{(1+3^2)^3}\right]_0^1-\int_0^1\frac{1}{3}\sqrt{(1+z^2)^3}\,dz\right)$$

$$= \frac{2}{3}\left(\ln\left(1+\sqrt{2}\right)+\frac{2\sqrt{2}}{3}\right)-\frac{1}{3}I$$

$$\Leftrightarrow 2I = \ln\left(1+\sqrt{2}\right)+\frac{2\sqrt{2}}{3}\Leftrightarrow I\leq\frac{\ln\left(1+\sqrt{2}\right)}{2}+\frac{\sqrt{2}}{3}<1$$

Solution 3 by Anisoara Dudu-Romania

$$x\sqrt{x^2+z^2}+y\sqrt{y^2+z^2} == \sqrt{x^2(x^2+z^2)}+\sqrt{y^2(y^2+z^2)}$$

$$\overset{Means\ Inequality}{\leq}\ \frac{x^2+x^2+z^2}{2}+\frac{y^2+y^2+z^2}{2}\leq\frac{2x^2+2y^2+2z^2}{2}=x^2+y^2+z^2$$

$$\int_0^1\int_0^1\int_0^1\left(x\sqrt{x^2+z^2}+y\sqrt{y^2+z^2}\right)dxdydy\leq\frac{x^3}{3}\Big|_0^1+\frac{y^3}{3}\Big|_0^1+\frac{z^3}{3}\Big|_0^1=1$$

Solution 4 by Hasan Bostanlik-Sarkisla-Turkey

$$A^2\leq(x^2+y^2)(x^2+y^2+2z^2)\quad\{C-S\}$$

$$A^2 \le (x^2+y^2+z^2)^2 - z^4 \le (x^2+y^2+z^2)^2;\ A \le x^2+y^2+z^2$$

$$\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2) = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$$

139. **If $a, b \ge 1$ then:**

$$2\int_1^b \left(y\int_1^a \log\frac{x}{y}dx\right)dy \le (a-1)(b-1)(a-b)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Chris Kyriazis-Greece, Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

Solution 1 by Chris Kyriazis-Greece

From the well – known inequality $\ln a \le a-1, \forall a > 0$ ***we have that:***

$$\ln\frac{x}{y} \le \frac{x}{y} - 1 \Rightarrow 2y\ln\frac{x}{y} \le 2x - 2y = 0;\ 2y\int_1^a \ln\frac{x}{y}dx \le \int_1^a 2x\,dx - \int_1^a 2y\,dx$$

$$\Rightarrow 2\int_1^b\left(y\int_1^a \ln\frac{x}{y}dx\right)dy \le (a^2-1)(b-1)-(a-1)(b^2-1)$$

$$\Rightarrow 2\int_1^b\left(y\int_1^a \ln\frac{x}{y}dx\right)dy \le (a-1)(b-1)(a+1-b-1)$$

$$\Rightarrow 2\int_1^b\left(y\int_1^a \ln\frac{x}{y}dx\right)dy \le (a-1)(b-1)(a-b)$$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$I = 2\int_1^b y\left(\int_1^a \ln\frac{x}{y}dx\right)dy < 2\int_1^b y\left(\int_1^a \left(\frac{x}{y}-1\right)dx\right)dy$$

$$= a^2b - b^2a = ab(a-b) \leq (a-1)(b-1)(a-b)$$

140. **If $0 < a < b$ then:**

$$\frac{\int_a^b e^{x^2}dx}{\int_a^b x^5 e^{x^2}dx} < \frac{1}{4}\left(\frac{1}{ab^4} + \frac{1}{a^2b^3} + \frac{1}{a^3b^2} + \frac{1}{a^4b}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Chris Kyriazis-Greece, Solution 2 by Subhajit Chattopadhyay-Bolpur-India

Solution 1 by Chris Kyriazis-Greece

Set $f(x) = \frac{1}{x^5}, x > 0$ ***and*** $g(x) = x^5 \cdot e^{x^2}, x > 0$

It's $f'(x) = -\frac{5}{x^6} < 0, x > 0$ ***and*** $g'(x) = x^4 e^{x^2}(2x^2+5) > 0, \forall x > 0.$

So f ***strictly decreasing when*** $x > 0$ ***and*** g ***strictly increasing***

Using the Chebyshev's integral inequality, we have that:

$$\int_a^b \frac{1}{x^5}dx \cdot \int_a^b x^5 \cdot e^{x^2}dx > \int_a^b \frac{1}{x^5} \cdot x^5 e^{x^2}dx \cdot (b-a)$$

$$\Rightarrow \left[-\frac{1}{4x^4}\right]_a^b \cdot \int_a^b x^5 e^{x^2}dx > \int_a^b e^{x^2}dx \cdot (b-a)$$

$$\Rightarrow \frac{1}{4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\int_a^b x^5 e^{x^2}dx > \int_a^b e^{x^2}dx\,(b-a)$$

$$\Rightarrow \frac{1}{4}(b-a) \cdot \frac{(b+a)}{a^2b^2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\int_a^b x^5 e^{x^2}dx > \int_a^b e^{x^2}dx\,(b-a)$$

$$\Rightarrow \frac{1}{4}\left(\frac{1}{a^4b}+\frac{1}{a^2b^3}+\frac{1}{a^3b^2}+\frac{1}{ab^4}\right)\int_a^b x^5e^{x^2}dx > \int_a^b e^{x^2}dx$$

Solution 2 by Subhajit Chattopadhyay-Bolpur-India

$$0 < a < b;\ \frac{\int_a^b e^{x^2}dx}{\int_a^b x^5e^{x^2}dx} < \frac{1}{4}\left(\frac{1}{ab^4}+\frac{1}{a^2b^3}+\frac{1}{a^3b^2}+\frac{1}{a^4b}\right)$$

Using Chebyshev's inequality, $\because e^{x^2}$ & x^5 ***are monotone increasing,***

$$L.H.S < \frac{b-a}{\int_a^b x^5dx} = \frac{6(b-a)}{b^6-a^6} = \frac{6}{(b^3+a^3)(b^2+ab+a^2)}$$

$$= \frac{4+2}{2\left(\frac{a^5+b^5}{2}\right)+4\cdot\left(\frac{a^4b+a^3b^2+\cdots}{4}\right)} < \frac{\frac{4}{a^5+b^5}+\frac{16}{a^4b+a^3b^2+a^2b^3+ab^4}}{6}$$

By using $AM > HM$ ***strict inequality.*** $\because a \neq b$.

$$= \frac{1}{3}\left(\frac{2}{a^5+b^5}+\frac{8}{a^4b+a^3b^2+a^2b^3+ab^4}\right).\ \textbf{Now, } \frac{a^5+b^5}{2} > (ab)^{\frac{5}{2}} \Rightarrow \frac{2}{a^5+b^5} < (ab)^{-\frac{5}{2}}$$

$$\frac{1}{4}\left(\frac{1}{ab^4}+\frac{1}{a^2b^3}+\frac{1}{a^3b^2}+\frac{1}{a^4b}\right) = M > (ab)^{-\frac{5}{2}}\ [By, AM > GM]$$

$$\frac{1}{4}\left(\frac{1}{ab^4}+\frac{1}{a^2b^3}+\frac{1}{a^3b^2}+\frac{1}{a^4b}\right) > \frac{4}{a^4b+a^3b^2+a^2b^3+ab^4}$$

Hence, $LHS < \frac{1}{3}(m+2m) = m = \frac{1}{4}\left(\frac{1}{ab^4}+\frac{1}{a^2b^3}+\frac{1}{a^3b^2}+\frac{1}{a^4b}\right)$

141. If $a,b,c \geq 0, m,n \geq 2$

$$\Omega(a) = \sqrt[n]{\int_0^a \sqrt[m]{e^{(m+n)x^2}}\,dx}\cdot\sqrt[m]{\int_0^a \frac{dx}{\sqrt[n]{e^{(m+n)x^2}}}}$$

then: $\Omega^2(a)+\Omega^2(b)+\Omega^2(c) \geq ab+bc+ca$

Proposed by Daniel Sitaru – Romania

Solution 1 by Subhajit Chattopadhyay-Bolpur-India, Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Subhajit Chattopadhyay-Bolpur-India

$$\Omega(a)=\left[\int_0^a e^{\frac{m+n}{m}x^2}dx\right]^{\frac{1}{n}}\left[\int_0^a\left(\frac{1}{e^{x^2}}\right)^{\frac{m+n}{n}}dx\right]^{\frac{1}{m}}$$

Using Hölder's inequality, $[\Omega(a)]^{\frac{mn}{m+n}};\frac{m}{m+n}=1-\frac{n}{m+n}$

$$=\left[\int_0^a e^{\frac{m+n}{m}x^2}dx\right]^{\frac{m}{m+n}}\left[\int_0^a\left(\frac{1}{e^{x^2}}\right)^{\frac{m+n}{n}}dx\right]^{\frac{n}{m+n}}\geq\int_0^a e^{x^2}\cdot\frac{1}{e^{x^2}}dx=a$$

$$\therefore\Omega^2(a)\geq a^{\frac{2(m+n)}{mn}}\because a>0,\Omega(a)>0;\ m,n\geq 2.\ \textbf{Put}\ m=n=2$$

$$\therefore\Omega^2(a)+\Omega^2(b)+\Omega^2(c)\geq a^2+b^2+c^2.$$

Now for any $a,b,c\in\mathbb{R}$

$$(a-b)^2+(b-c)^2+(c-a)^2\geq 0$$

$$\Rightarrow 2(a^2+b^2+c^2)-2(ab+bc+ca)\geq 0$$

$$\Rightarrow a^2+b^2+c^2\geq ab+bc+ca\therefore LHS\geq ab+bc+ca$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\Omega(a)=\sqrt[n]{\int_0^a\sqrt[m]{e^{(m+n)x^2}}dx}\cdot\sqrt[m]{\int_0^a\sqrt[n]{e^{-(m+n)x^2}}dx}$$

$$\overset{HOLDER'S}{\geq}\int_0^a\left|e^{\frac{(m+n)x^2}{mn}}\cdot\frac{1}{e^{\frac{(m+n)x^2}{mn}}}\right|dx=a$$

Similarly, $\Omega(b)\geq b,\Omega(c)\geq c\therefore\sum_{cyc}\Omega^2(a)=\sum_{cyc}a^2\geq\sum_{cyc}ab$

142. If $a > 0$ then:

$$\left(\int_0^a e^{3x^2}dx\right)\left(\int_0^a e^{-3x^2}dx\right) > \frac{1}{a^4}\left(\int_0^a e^{x^2}dx\right)^3\left(\int_0^a e^{-x^2}dx\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution by Chris Kyriazis-Greece

Using Hölder inequality for integrals, I have that.

$$\left(\int_0^a 1^{\frac{3}{2}}dx\right)^{\frac{2}{3}}\cdot\left(\int_0^a e^{3x^2}dx\right)^{\frac{1}{3}} \geq \int_0^a e^{x^2}dx \Rightarrow$$

$$a^{\frac{2}{3}}\left(\int_0^a e^{3x^2}dx\right)^{\frac{1}{3}} \geq \int_0^a e^{x^2}dx \Rightarrow \int_0^a e^{3x^2}dx \geq \frac{1}{a^2}\left(\int_0^a e^{x^2}dx\right)^3 \quad (1)$$

Just the same:

$$\left(\int_0^a 1^{\frac{3}{2}}dx\right)^{\frac{2}{3}}\cdot\left(\int_0^a e^{-3x^2}dx\right)^{\frac{1}{3}} \geq \int_0^a e^{-x^2}dx \Rightarrow\Rightarrow \cdots \int_0^a e^{-3x^2}dx \geq \frac{1}{a^2}\left(\int_0^a e^{-x^2}dx\right)^3 \quad (2)$$

(1) × (2) (everything is positive) we have that

$$\int_0^a e^{3x^2}dx\cdot\int_0^a e^{-3x^2}dx \geq \frac{1}{a^4}\left(\int_0^a e^{x^2}dx\int_0^a e^{-x^2}dx\right)^3$$

143. If $a, b, c > 0, \alpha \in \left(0, \frac{\pi}{2}\right)$

$$\Omega(a,b) = \int_0^b\left(\int_0^a (x\sin^2\alpha + y\cos^2\alpha)(x\cos^2\alpha + y\sin^2\alpha)\,dx\right)dy$$

then:

$$4\Omega(b,c) + 4\Omega(c,a) + 4\Omega(a,b) \geq abc(a+b+c)$$

Proposed by Daniel Sitaru – Romania

Solution by Chris Kyriazis-Greece

We have that $(x\sin^2 a + y\cos^2 a)(x\cos^2 a + y\sin^2 a) =$

$$\left[\left(\sqrt{x}\sin a\right)^2 + \left(\sqrt{y}\cos a\right)^2\right]\cdot\left[\left(\sqrt{x}\cos a\right)^2 + \left(\sqrt{y}\sin a\right)^2\right]$$

$$\overset{B-C-S}{\geq} \left(\sqrt{xy}\sin^2 a + \sqrt{xy}\cos^2 a\right)^2 = xy$$

so $\Omega(a,b) = \int_0^a\int_0^b xy\,dx\,dy = \int_0^a x dx\cdot\int_0^b y\,dy == \frac{(ab)^2}{4}$

Doing exactly the same work, we have that $\Omega(b,c) \geq \frac{(bc)^2}{4}, \Omega(c,a) \geq \frac{(ca)^2}{4}$

So $4\Omega(a,b) + 4\Omega(b,c) + 4\Omega(c,a) \geq 4\frac{(ab)^2}{4} + 4\cdot\frac{(bc)^2}{4} + 4\frac{(ca)^2}{4} =$

$$(ab)^2 + (bc)^2 + (ca)^2 \geq ab^2c + a^2cb + abc^2 = abc(a+b+c)$$

144. $\int_0^{\frac{\pi}{4}}\left(\frac{1-\sin^2 x}{1+\sin^2 x} + \frac{1-\cos^2 x}{1+\cos^2 x}\right)\ln(1+\tan x)\,dx > \frac{\pi\ln 2}{12}$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 2 by Ravi Prakash-New Delhi-India

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$$J = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx \overset{x=\frac{\pi}{4}-t}{=} \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-t\right)\right)dt$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan t}{1+\tan t}\right)dt = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan t}\right)dt$$

$$=\int_0^{\frac{\pi}{4}}\ln(2)\,dt-\int_0^{\frac{\pi}{4}}\ln(1+\tan t)\,dt=\frac{\pi}{4}\ln 2-J\to J=\frac{\pi}{8}\ln 2$$

$$I=\int_0^{\frac{\pi}{4}}\left(\frac{1-\sin^2 x}{1+\sin^2 x}+\frac{1-\cos^2 x}{1+\cos^2 x}\right)\ln(1+\tan x)\,dx$$

$$=\int_0^{\frac{\pi}{4}}\left(\frac{2}{1+\sin^2 x}+\frac{2}{1+\cos^2 x}-2\right)\ln(1+\tan x)\,dx$$

$$=2\int_0^{\frac{\pi}{4}}\left(\frac{1}{1+\sin^2 x}+\frac{1}{2-\sin^2 x}-1\right)\ln(1+\tan x)\,dx$$

$$\therefore \textit{let } f(x)=\frac{1}{1+x^2}+\frac{1}{2-x^2}\quad \forall x\in\left[0;\frac{1}{\sqrt{2}}\right]$$

$$f'(x)=-\frac{2x}{(1+x^2)^2}+\frac{2x}{(2-x^2)^2}=2x\left(\frac{1}{(x^2-2)^2}-\frac{1}{(x^2+1)^2}\right)$$

$$=\frac{2x\left((x^2+1)^2-(x^2-2)^2\right)}{(x^2-2)^2(x^2+1)^2}=\frac{6x(2x^2-1)}{(x^2-2)^2(x^2+1)^2}\le 0\quad\forall x\in\left[0;\frac{1}{\sqrt{2}}\right]$$

$$0\le x\le\frac{\pi}{4}\Rightarrow 0\le\sin x\le\frac{1}{\sqrt{2}}\Rightarrow f(\sin x)\ge f\left(\frac{1}{\sqrt{2}}\right)=\frac{4}{3}$$

$$\Rightarrow\frac{1}{1+\sin^2 x}+\frac{1}{2-\sin^2 x}-1\ge\frac{1}{3}\Rightarrow I\ge 2\int_0^{\frac{\pi}{4}}\frac{1}{3}\ln(1+\tan x)\,dx$$

$$\Leftrightarrow I\ge\frac{2}{3}J\Leftrightarrow I\ge\frac{\pi}{12}\ln 2$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\textit{Let } g(x)\frac{1-\sin^2 x}{1+\sin^2 x}+\frac{1-\cos^2 x}{1+\cos^2 x}\quad 0\le x\le\frac{\pi}{4}=\frac{\cos^2 x(1+\cos^2 x)+\sin^2 x(1+\sin^2 x)}{(1+\sin^2 x)(1+\cos^2 x)}$$

$$=\frac{\cos^2 x+\sin^2 x+\cos^4 x+\sin^4 x}{1+1+\cos^2 x\sin^2 x}$$

$$=\frac{1+1-2\sin^2 x\cos^2 x}{2+\cos^2 x\sin^2 x}=\frac{2(1-\sin^2 x\cos^2 x)}{2+\sin^2 x\cos^2 x}$$

Now, $g(x)\ge\frac{2}{3}\Leftrightarrow\frac{1-\sin^2 x\cos^2 x}{2+\sin^2 x\cos^2 x}\ge\frac{1}{3}$

$$\Leftrightarrow 3-3\sin^2 x\cos^2 x\ge 2+\sin^2 x\cos^2 x\Leftrightarrow 1-4\sin^2 x\cos^2 x\ge 0$$

$\Leftrightarrow 1-\sin^2 2x\ge 0\Leftrightarrow\cos^2 2x\ge 0$, *which is true.*

Note that $g(x)=\frac{2}{3}\Leftrightarrow x=\frac{\pi}{4}$. $\therefore\ g(x)>\frac{2}{3}$ *for* $0\le x<\frac{\pi}{4}$. *Now,*

$$I=\int_0^{\frac{\pi}{4}}\left(\frac{1-\sin^2 x}{1+\sin^2 x}+\frac{1-\cos^2 x}{1+\cos^2 x}\right)\ln(1+\tan x)\,dx$$

$>\frac{2}{3}\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\,dx=\frac{2}{3}I_1$, *where*

$$I_1=\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\,dx=\int_0^{\frac{\pi}{4}}\ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$=\int_0^{\frac{\pi}{4}}\ln\left(1+\frac{1-\tan x}{1+\tan x}\right)dx=\int_0^{\frac{\pi}{4}}\ln 2\,dx-I_1\Rightarrow 2I_1=\frac{\pi}{4}\ln 2\Rightarrow I_1=\frac{\pi}{8}\ln 2$$

$$\therefore I>\frac{2}{3}\left(\frac{\pi}{8}\ln 2\right)\Rightarrow I>\frac{\pi}{12}\ln 2$$

145. If $a>1$ then:

$$\int_a^{2a}\frac{e^x}{x^3}dx\le\frac{3e^a(e^a-1)}{8a^3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Chris Kyriazis-Greece, Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 3 by Dimitris Kastriotis-Greece, Solution 4 by Michel Rebeiz-Lebanon

Solution 1 by Chris Kyriazis-Greece

If we consider the functions

$$f(x) = \frac{1}{x^3}, x \in [a, 2a] \text{ (Strictly decreasing on } [a, 2a])$$

$$g(x) = e^x, x \in [a, 2a] \text{ (Strictly increasing on } [a, 2a])$$

Using Chebyshev integral inequality we have:

$$a \cdot \int_a^{2a} \frac{e^x}{x^3} dx < \int_a^{2a} \frac{1}{x^3} dx \cdot \int_a^{2a} e^x dx = \left[-\frac{1}{2x^2}\right]_a^2 (e^{2a} - e^a)$$

$$\Rightarrow a \int_a^{2a} \frac{e^x}{x^3} dx < \frac{3}{8a^2}(e^{2a} - e^a) \Rightarrow \int_a^{2a} \frac{e^x}{x^3} dx < \frac{3}{8a^3} e^a(e^a - 1)$$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$\text{We have} \left(\int_a^{2a} \frac{e^x}{x^3} dx\right)\left(\int_a^{2a} x^3 dx\right) \overset{Chebyshev}{\leq} (2a - a)\left(\int_a^{2a} e^x\, dx\right)$$

$$\Leftrightarrow \frac{15a^4}{4}\left(\int_a^{2a} \frac{e^x}{x^3} dx\right) \leq ae^a(e^a - 1) \Leftrightarrow \left(\int_a^{2a} \frac{e^x}{x^3} dx\right) \leq \frac{4}{15} \cdot \frac{e^a(e^a - 1)}{a^3}$$

$$\Rightarrow \left(\int_a^{2a} \frac{e^x}{x^3} dx\right) \leq \frac{3}{8} \cdot \frac{e^a(e^a - 1)}{a^3}$$

Solution 3 by Dimitris Kastriotis-Greece

$$\text{Let } (x) = e^x, g(x) = \frac{1}{x^3}. f \uparrow [a, 2a]. g \downarrow [a, 2a]$$

$$\int_a^{2a} e^x \cdot \frac{1}{x^3}dx \overset{Chebyshev}{\leq} \frac{1}{2a-a}\int_a^{2a} e^x\,dx \cdot \int_a^{2a} \frac{1}{x^3}dx$$

$$= \frac{1}{a}(e^{2a}-e^a)\cdot\frac{3}{8a^2} = \frac{3}{8a^3}e^a(e^a-1)$$

Solution 4 by Michel Rebeiz-Lebanon

$$f(a) = \int_a^{2a} \frac{e^x}{x^3}dx - \frac{3e^a(e^a-1)}{8a^3}$$

$$f'(a) = 2\cdot\frac{e^{2a}}{(2a)^3} - \frac{3}{8}\cdot\frac{1}{a^6}[(2e^{2a}-e^a)a^3 - 3a^2(e^{2a}-e^a)]$$

$$= \frac{e^a}{8a^4}[-4ae^a + 3a + 3e^a - 9].\, g(a) = -4ae^a + 3a + 3e^a - 9$$

$$g'(a) = -e^a - 4ae^a + 3$$

$$g''(a) = e^a(-5-4a) < 0 \to g' \downarrow \to [a > 1; g'(a) < g'(1)]$$

$$g'(1) < 0 \to g'(a) < 0 \to g \downarrow \to [a > 1; g(a) < g(1)]$$

$$g(1) < 0 \to g(a) < 0 \to f'(a) < 0 \to f \downarrow$$

$$a > 1 \to f(a) < f(1)\ \ f(1) < 0 \to f(a) < 0 \to \int_a^{2a}\frac{e^x}{x^3}dx < \frac{3a^a(e^a-1)}{8a^3}$$

146. If $1 < a < b$ then:

$$\int_1^a \log^2 x\,dx + \int_1^b \log^2 x\,dx \geq \int_1^{\frac{3a+b}{4}} \log^2 x\,dx + \int_1^{\frac{a+3b}{4}} \log^2 x\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by Abdelhak Maoukouf-Casablanca-Morocco

$$\forall x > 1: f(x) = \int_1^x \log^2 t\, dt \therefore f'(x) = \log^2 x \ \& f''(x) = 2\frac{\log x}{x} \geq 0$$

$\forall x > 1$. ***So by Jensen's inequality:*** $\begin{cases} f\left(\frac{a+3b}{4}\right) \leq \frac{f(a)+3f(b)}{4} \\ f\left(\frac{3a+b}{4}\right) \leq \frac{3f(a)+f(b)}{4} \end{cases}$

$$\Rightarrow f(a) + f(b) \geq f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right)$$

$$\Leftrightarrow \int_1^a \log^2 t\, dt + \int_1^b \log^2 dt \geq \int_1^{\frac{a+3b}{4}} \log^2 t\, dt + \int_1^{\frac{3a+b}{4}} \log^2 t\, dt$$

147. If $0 < a < b; 0 < c < d; f, g$ integrable functions

$f, g: [a, b] \to [c, d]$ then:

$$cd\left(\int_a^b \frac{f(x)}{g(x)} dx + \int_a^b \frac{g(x)}{f(x)} dx\right) < (c^2 + d^2)(b - a)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$c \leq f(x) \leq d$ **and** $c \leq g(x) \leq d; \frac{c}{d} \leq \frac{f}{g} \leq \frac{d}{c}$ **for all** $x \in [a, b]$

$$\Rightarrow \left(\frac{f}{g} - \frac{c}{d}\right)\left(\frac{f}{g} - \frac{d}{c}\right) \leq 0 \Rightarrow \frac{f}{g} + \frac{g}{f} \leq \frac{c}{d} + \frac{d}{c}$$

$$\Rightarrow \int_a^b \frac{f(x)}{g(x)} dx + \int_a^b \frac{g(x)}{f(x)} dx \leq \left(\frac{c}{d} + \frac{d}{c}\right)(b - a)$$

$$\Rightarrow cd\left(\int_a^b \frac{f(x)}{g(x)}dx + \int_a^b \frac{g(x)}{f(x)}dx\right) < (c^2 + d^2)(b - a)$$

148. If $f:[a,b] \to \mathbb{R}, f -$ continuous, $f -$ increasing then:

$$(\sqrt{a} + \sqrt{b})\int_a^{\sqrt{ab}} f(x)dx \le \sqrt{a}\int_a^b f(x)dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Chris Kyriazis-Greece, Solution 2 by Abdallah El Farissi-Bechar-Algerie

Solution 1 by Chris Kyriazis-Greece

First we mention that: $a \le \sqrt{ab} \le b$ ***(supposing that*** $ab \ge 0$***)***

It suffices to prove that

$$\sqrt{b} \cdot \int_a^{\sqrt{ab}} f(x)\,dx \le \sqrt{a}\left(\int_a^b f(x)dx - \int_a^{\sqrt{ab}} f(x)\,dx\right)$$

or $\sqrt{b} \cdot \int_a^{\sqrt{ab}} f(x)\,dx \le \sqrt{a}\int_{\sqrt{ab}}^b f(x)\,dx$

Using the integral mean value theorem it suffices to prove that:

$$\sqrt{b}(\sqrt{ab} - a)f(z_1) \le \sqrt{a}(b - \sqrt{ab})f(z_2)$$

where $z_1 \in [a, \sqrt{ab}]$ ***and*** $z_2 \in [\sqrt{ab}, b]$

or $\sqrt{b}\sqrt{a}(\sqrt{b} - \sqrt{a})f(z_1) \le \sqrt{a}\sqrt{b}(\sqrt{b} - \sqrt{a})f(z_2)$

or $f(z_1) \le f(z_2)$ ***which holds***

due to monotonicity of the function f ***(increasing).***

Solution 2 by Abdallah El Farissi-Bechar-Algerie

***f* is increasing function then for all** $s\in[a,\sqrt{ab}]$ **and** $t\in[\sqrt{ab},b]$ **we have**

$f(s)\le f(t)$ **then** $(b-\sqrt{ab})\int_a^{\sqrt{ab}}f(s)\,dx=\sqrt{b}(\sqrt{b}-\sqrt{a})\int_a^{\sqrt{ab}}f(s)\,ds\le$

$\le\sqrt{a}(\sqrt{b}-\sqrt{a})\int_{\sqrt{ab}}^{b}f(t)dt=(\sqrt{ab}-a)\int_{\sqrt{ab}}^{b}f(t)\,dt$ **it follow that**

$$\sqrt{b}\int_a^{\sqrt{ab}}f(x)\,dx\le\sqrt{a}\int_{\sqrt{ab}}^{b}f(x)dx=\sqrt{a}\left(\int_a^b f(x)\,dx-\int_a^{\sqrt{ab}}f(x)dx\right)$$

then $(\sqrt{b}+\sqrt{a})\int_a^{\sqrt{ab}}f(x)\,dx\le\sqrt{a}\int_a^b f(x)\,dx$

149. For acute triangle ABC

$$\text{If: }\zeta(A)=\int_0^A\frac{1}{\sqrt{\cos x+x\left(1+\frac{2}{\pi}\right)}}dx$$

$$\text{Prove: }\zeta(A)+\zeta(B)+\zeta(C)\le 2\sqrt{3(\pi+3)}-6$$

Proposed by Uche Eliezer Okeke-Anambra-Nigeria

Solution by Daniel Sitaru-Romania

$$\sin\left(\frac{\pi}{2}-x\right)\overset{JORDAN}{\ge}\frac{2}{\pi}\left(\frac{\pi}{2}-x\right)\to\cos x\ge 1-\frac{2}{\pi}x\to\cos x+\frac{2}{\pi}x+x\ge 1+x$$

$$\zeta(A)=\int_0^A\frac{1}{\sqrt{\cos x+\frac{2}{\pi}x+x}}dx\le\int_0^A\frac{1}{\sqrt{1+x}}dx=2\sqrt{1+A}-2$$

$$\sum\zeta(A)\le 2\sum\sqrt{1+A}-6\overset{JENSEN}{\le}2\cdot 3\sqrt{1+\frac{A+B+C}{3}}-6=2\sqrt{3(1+\pi)}-6$$

150. If $x, y, z \in (0,1]$,

$$\Omega(x) = \int_0^x \frac{\ln(1+ax)}{1+a^2} da$$

then:

$$2(\Omega(x) + \Omega(y) + \Omega(z)) \geq 3\ln 2 + \ln(xyz)$$

Proposed by Daniel Sitaru – Romania

Solution by Subhajit Chattopadhyay-Bolpur-India

$$\Omega(x) = \int_0^x \frac{\ln(1+ax)}{1+a^2} da;$$

$$= \int_0^x \int_0^x \frac{a\, da\, dt}{(1+at)(1+a^2)} = \int_0^x \int_0^x \frac{a(1+t^2)dtda}{(1+t^2)(1+at)(1+a^2)}$$

$$= \int_0^x \left[\frac{1}{1+t^2} \int_0^x \frac{a\, da}{1+a^2}\right] dt + \int_0^x \left[\frac{t}{1+t^2} \int_0^x \frac{da}{1+a^2}\right] dt - \int_0^x \left[\frac{t}{1+t^2} \int_0^x \frac{da}{1+at}\right] dt$$

$$= \left(\int_0^x \frac{dt}{1+t^2}\right)\left(\int_0^x \frac{a\, da}{1+a^2}\right) + \left(\int_0^x \frac{t\, dt}{1+t^2}\right)\left(\int_0^x \frac{da}{1+a^2}\right) - \int_0^x \frac{\ln(xt+1)}{1+t^2} dt$$

$$\therefore 2\Omega(x) = \frac{\tan^{-1} x}{2} \ln(1+x^2) + \frac{\ln(1+x^2)}{2} \tan^{-1} x = \tan^{-1} x \ln(1+x^2)$$

Hence, $2(\Omega(x) + \Omega(y) + \Omega(z))$

$$= \tan^{-1} x \ln(1+x^2) + \tan^{-1} y \ln(1+y^2) + \tan^{-1} z \ln(1+z^2)$$

Now, $x \in (0,1)$. *By* $AM \geq GM$ $\ln(1+x^2) \geq \ln(2x)$; $\tan^{-1} x \geq 1$ *for*

$x \in (0,1) \therefore LHS \geq \ln(2x) + \ln(2y) + \ln(2z) = 3\ln 2 + \ln(xyz)$

151. If $a, b, c > 0$ then:

$$\int_a^{2a}\left(\int_b^{2b}\left(\int_c^{2c}\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)dx\right)dy\right)dz \le \ln\sqrt{2^{ab+bc+ca}}$$

Proposed by Daniel Sitaru – Romania

Solution by Chris Kiryazis-Greece

We have that $(x+y)^2 \ge 4xy\,\frac{x+y}{4xy} \ge \frac{1}{x+y} \Rightarrow \frac{1}{x+y} \le \frac{1}{9}\left(\frac{1}{x}+\frac{1}{9}\right)$ *(1)*

So, using (1) (integrating (1)), we have:

$$\int_a^{2a}\int_b^{2b}\int_c^{2c}\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)dx\,dy\,dz \le \frac{1}{2}\int_a^{2a}\int_b^{2b}\int_c^{2c}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)dx\,dy\,dz$$

$$= \frac{1}{2}(bc\ln 2 + ca\ln 2 + ab\ln 2) = \frac{1}{2}\left(\ln 2^{bc+ca+ab}\right) = \ln 2^{\frac{ab+bc+ca}{2}} =$$

$$= \ln\sqrt{2^{ab+bc+ca}}$$

152. Let $f:[1,13]\to\mathbb{R}$ be a convexe and integrable function. Prove that

$$\int_1^3 f(x)\,dx + \int_{11}^{13} f(x)\,dx \ge \int_5^9 f(x)\,dx$$

Proposed by Nitin Gurbani-India

Solution by Daniel Sitaru-Romania

$$1 \le x_n^k \le y_n^k \le z_n^k \le t_n^k \le 13$$

$$x_n^k = 1 + \frac{2k}{n}, y_n^k = 5 + \frac{2k}{n}, z_n^k = 7 + \frac{2k}{n}, t_n^k = 11 + \frac{2k}{n}$$

f – convexe $\rightarrow \frac{f(y_n^k)-f(x_n^k)}{y_n^k-x_n^k} \le \frac{f(t_n^k)-f(z_n^k)}{t_n^k-z_n^k} \rightarrow$

$$\lim_{n\to\infty}\frac{2}{n}\sum_{n=1}^{n} f(y_n^k) + \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n} f(z_n^k) \geq \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n} f(t_n^k) + \lim_{n\to\infty}\frac{2}{n}\sum_{\substack{k=1\\ n\to\infty}}^{n} f(x_n^k)$$

$$\int_5^7 f(x)\,dx + \int_7^9 f(x)dx \leq \int_{11}^{13} f(x)dx + \int_1^3 f(x)dx$$

$$\int_5^9 f(x)dx \leq \int_{11}^{13} f(x)dx + \int_1^3 f(x)dx$$

153. $\int_0^1\int_0^1\int_0^1\int_0^1\left(\sqrt[3]{xyz}+\sqrt[3]{yzt}+\sqrt[3]{ztx}+\sqrt[3]{txy}\right)dxdydzdt \leq 2$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 2 by Lazaros Zachariadis-Thessaloniki-Greece, Solution 3 by Geanina Tudose-Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$$\sum \sqrt[3]{xyz} \overset{AM-GM}{\leq} \sum \frac{x+y+z}{3} \leq (x+y+z+t)$$

$$\to I = \int_0^1\int_0^1\int_0^1\int_0^1\left(\sqrt[3]{xyz}+\sqrt[3]{yzt}+\sqrt[3]{ztx}+\sqrt[3]{txy}\right)dxdydzdt$$

$$\leq \int_0^1\int_0^1\int_0^1\int_0^1 (x+y+z+t)\,dxdydzdt = \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 + \left[\frac{z^2}{2}\right]_0^1 + \left[\frac{t^2}{2}\right]_0^1 = 2$$

Solution 2 by Lazaros Zachariadis-Thessaloniki-Greece

$$\sqrt[3]{xyz}+\sqrt[3]{yzt}+\sqrt[3]{ztx}+\sqrt[3]{txy} \leq \frac{x+y+z+y+z+t+z+t+x+t+x+y}{3}$$

$$= \frac{3(x+y+z+t)}{3} = x+y+z+t$$

$$I=\int_0^1\int_0^1\int_0^1\int_0^1(x+y+z+t)\,dxdydzdt=\int_0^1\int_0^1\int_0^1\left(\frac{x^2}{2}+yx+zx+tx\right)_0^1 dydzdt=$$

$$=\int_0^1\int_0^1\int_0^1\left(\frac{1}{2}+y+z+t\right)dydzdt$$

$$=\int_0^1\int_0^1\left(\frac{1}{2}y+\frac{y^2}{2}+zy+ty\right)_0^1 dzdt=\int_0^1\int_0^1\left(\frac{1}{2}+\frac{1}{2}+z+t\right)dzdt=\int_0^1\left(z+\frac{z^2}{2}+tz\right)_0^1 dt$$

$$=\int_0^1\left(1+\frac{1}{2}+t\right)dt=\left(t+\frac{t}{2}+\frac{t^2}{2}\right)_0^1=1+\frac{1}{2}+\frac{1}{2}=2\Rightarrow I\le 2$$

Solution 3 by Geanina Tudose-Romania

By $GM\le AM$ **we have** $\sqrt[3]{xyz}\le\frac{x+y+z}{3}\Rightarrow\sum_{cyc}\sqrt[3]{xyz}\le x+y+z+t$

$$\int_0^1\int_0^1\int_0^1\int_0^1\left(\sqrt[3]{xyz}+\sqrt[3]{yzt}+\sqrt[3]{ztx}+\sqrt[3]{txy}\right)dxdydzdt$$

$$\le\int_0^1\int_0^1\int_0^1\left(\frac{1}{2}+y+z+t\right)dydzdt=\int_0^1\int_0^1\int_0^1\left(\frac{x^2}{2}+(y+z+t)x\right)\Big|_0^1 dydzdt=$$

$$=\int_0^1\int_0^1\int_0^1\left(\frac{1}{2}+y+z+t\right)dydzdt=\int_0^1\int_0^1\left(\frac{y}{z}+\frac{y^2}{2}+(z+t)y\right)_0^1 dzdx$$

$$=\int_0^1\int_0^1(1+z+t)\,dzdt=\int_0^1\left(z+\frac{z^2}{2}+tz\right)\Big|_0^1 dt=\int_0^1\left(1+\frac{1}{2}+t\right)dt=\frac{3}{2}t+\frac{t^2}{2}\Big|_0^1=2$$

154. **If $0<a<b$ then:**

$$\int_a^b\frac{dx}{(x^3+1)^2}>\frac{5}{9(b^5-a^5)}\ln^2\left(\frac{b^3+1}{a^3+1}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

Applying Cauchy – Schwarz,

$$\left(\int_a^b \frac{x^2}{x^3+1}dx\right)^2 \le \left(\int_a^b \frac{dx}{(x^3+1)^2}\right)\left(\int_a^b x^4 dx\right) = \frac{(b^5-a^5)}{5}\left(\int_a^b \frac{dx}{(x^3+1)^2}\right)$$

$$\Rightarrow \left(\frac{1}{3}[\ln(x^3+1)]_{x=a}^{x=b}\right)^2 \le \frac{b^5-a^5}{5}\left(\int_a^b \frac{dx}{(x^3+1)^2}\right)$$

$$\therefore \ln^2\left(\frac{b^3+1}{a^3+1}\right)\cdot\frac{5}{9(b^5-a^5)} \le \int_a^b \frac{dx}{(x^3+1)^2}$$

155. Evaluate

$$\lim_{x\to 0}\frac{2\sqrt{1+x}+2\sqrt{2^2+x}+\cdots+2\sqrt{n^2+x}-n(n+1)}{x}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Serban George Florin-Romania , Solution 2 by Shivam Sharma-New Delhi-India , Solution 3 by Ravi Prakash-New Delhi-India , Solution 4 by Bedri Hadriji-Mitrovica-Kosovo

Solution 1 by Serban George Florin-Romania

$$l = \lim_{x\to 0}\frac{2\sqrt{1+x}+2\sqrt{2^2+x}+\cdots+2\sqrt{n^2+x}-n(n+1)}{x} = \frac{0}{0}$$

$$l = 2\lim_{x\to 0}\frac{\sqrt{1+x}+\sqrt{2^2+x}+\cdots+\sqrt{n^2+x}-\frac{n(n+1)}{2}}{x},$$

$$1+2+\cdots+n = \frac{n(n+1)}{2}$$

$$l = 2\cdot\lim_{x\to 0}\frac{\left(\sqrt{1+x}-1\right)+\left(\sqrt{2^2+x}-2\right)+\cdots+\left(\sqrt{n^2+x}-n\right)}{x}$$

$$L_n = \lim_{x\to 0}\frac{\sqrt{n^2+x}-n}{x} = \frac{0}{0} = \lim_{x\to 0}\frac{n^2+x-n^2}{x\left(\sqrt{n^2+x}+n\right)} = \lim_{x\to 0}\frac{x}{x\left(\sqrt{n^2+x}+n\right)}$$

$$L_n = \lim_{x\to 0}\frac{1}{\sqrt{n^2+x}+n} = \frac{1}{n+n} = \frac{1}{2n}$$

$$l = 2\cdot(L_1+L_2+\cdots+L_n) = 2\cdot\left(\frac{1}{2\cdot 1}+\frac{1}{2\cdot 2}+\cdots+\frac{1}{2n}\right);\ l = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}$$

Solution 2 by Shivam Sharma-New Delhi-India

$\Rightarrow \lim_{x\to 0}\frac{2\sum_{k=1}^{n}\sqrt{k^2+x}-n(n+1)}{x}$***. Applying L. Hospital's rule, we get,***

$\Rightarrow \lim_{x\to 0} 2\sum_{k=1}^{n}\frac{1}{2\sqrt{k^2+x}} \Rightarrow \lim_{x\to 0}\sum_{k=1}^{n}\frac{1}{\sqrt{k^2+x}}$ ***(OR)*** $L = \sum_{k=1}^{n}\frac{1}{k}$

(OR) $L = H_n$

Solution 3 by Ravi Prakash-New Delhi-India

For $1 \le k \le n$***,***

$$\lim_{x\to 0}\frac{\sqrt{k^2+x}-k}{x} = \lim_{x\to 0}\frac{k^2+x-k^2}{x\left(\sqrt{k^2+x}+k\right)} = \lim_{x\to 0}\frac{1}{\sqrt{k^2+x}+k} = \frac{1}{2k}$$

$$\Rightarrow \lim_{x\to 0}\frac{2\sqrt{k^2+x}-2k}{x} = \frac{1}{k} \Rightarrow \sum_{k=1}^{n}\lim_{x\to 0}\frac{2\sqrt{k^2+x}-2k}{x} = \sum_{k=1}^{n}\frac{1}{k}$$

$$\Rightarrow \lim_{x\to 0}\frac{\sum_{k=1}^{n}2\sqrt{k^2+x}-n(n+1)}{x} = \sum_{k=1}^{n}\frac{1}{k}$$

Solution 4 by Bedri Hadriji-Mitrovica-Kosovo

$$L = 2\lim_{x\to 0}\sum_{k=1}^{n}\frac{\sqrt{k^2+x}-k}{x} = 2\sum_{k=1}^{n}\lim_{x\to 0}\frac{x}{x\left(\sqrt{k^2+x}+k\right)}$$

$$= 2\sum_{k=1}^{n}\frac{1}{2k} = \sum_{k=1}^{n}\frac{1}{k} = H_n$$

156. **Evaluate**

$$\lim_{n\to\infty}\int_n^{n+1} e^{\frac{1}{x}}\,dx$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by Shivam Sharma-New Delhi-India

Solution 1 by Ravi Prakash-New Delhi-India

For $n \le x \le n+1;\ \frac{1}{n+1} \le \frac{1}{x} \le \frac{1}{n} \Rightarrow e^{\frac{1}{(n+1)}} \le e^{\frac{1}{x}} \le e^{\frac{1}{n}}$

$$\Rightarrow \int_n^{n+1} e^{\frac{1}{(n+1)}}\,dx \le \int_n^{n+1} e^{\frac{1}{x}}\,dx \le \int_n^{n+1} e^{\frac{1}{n}}\,dx \Rightarrow e^{\frac{1}{n+1}} \le \int_n^{n+1} e^{\frac{1}{x}}\,dx \le e^{\frac{1}{n}}$$

Since $e^{\frac{1}{n}} \to e^0 = 1$ as $n \to \infty$; $e^{\frac{1}{(n+1)}} \to e^0 = 1$ as $n \to \infty$

we get $\lim_{n\to\infty}\int_n^{n+1} e^{\frac{1}{x}}\,dx = 1$

Solution 2 by Shivam Sharma-New Delhi-India

$\lim_{n\to\infty}\int_n^{n+1} e^{\frac{1}{x}}\,dx$. Let, $h(x) = e^{\frac{1}{n+1}}$. And, $g(x) = e^{\frac{1}{n}}$. So, $h(x) \le L \le g(x)$

$e^{\frac{1}{n+1}} \le e^{\frac{1}{x}} \le e^{\frac{1}{n}}$. Then, $\int_n^{n+1} e^{\frac{1}{n-1}}\,dx \le \int_n^{n-1} e^{\frac{1}{x}}\,dx \le \int_n^{n-1} e^{\frac{1}{n}}\,dx$

$$e^{\frac{1}{n+1}}[n+1-n] \le \int_n^{n-1} e^{\frac{1}{x}}\,dx \le e^{\frac{1}{n}}[n+1-n]$$

$$\lim_{n\to\infty}\left(e^{\frac{1}{n-1}}\right) \le \lim_{n\to\infty}\int_n^{n+1} e^{\frac{1}{x}}\,dx \le \lim_{n\to\infty}\left(e^{\frac{1}{n}}\right)$$

$1 \le L \le 1$. Then, by Squeeze theorem, we get, $L = 1$

(Q.E.D)

157. If

$$\Omega(a) = \lim_{n\to\infty} n^2\left(\sqrt[n+5]{e^{a^2+a+1}} - \sqrt[n+7]{e^{a^2+a+1}}\right), a > 0$$

then:

$$\frac{\Omega(a)}{b+c} + \frac{\Omega(b)}{c+a} + \frac{\Omega(c)}{a+b} > a + b + c$$

Proposed Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Solution 2 by Subhajit Chattopadhyay-Bolpur-India

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = e^{x(a^2+a+1)}$ ***for all*** $x \in \left[\frac{1}{n+7}, \frac{1}{n+5}\right]$

$\therefore$ ***by Lagrange's Mean Value Theorem;***

$$\frac{\sqrt[n+5]{e^{a^2+a+1}} - \sqrt[n+7]{e^{a^2+a+1}}}{\frac{1}{n+5} - \frac{1}{n+7}} = (a^2 + a + 1)e^{\xi_n(a^2+a+1)} \text{ where } \xi \in \left[\frac{1}{n+7}, \frac{1}{n+5}\right]$$

$$\sqrt[n+5]{e^{a^2+a+1}} - \sqrt[n+7]{e^{a^2+a+1}} = \frac{2(a^2 + a + 1)}{(n+5)(n+7)} e^{\xi_n(a^2+a+1)}$$

Now, $\frac{1}{n+7} \le \xi_n \le \frac{1}{n+5} \Rightarrow \frac{a^2+a+1}{n+7} \le \xi_n(a^2 + a + 1) \le \frac{a^2+a+1}{n+5}$

$$\sqrt[n+7]{e^{a^2+a+1}} \le e^{\xi_n(a^2+a+1)} \le \lim_{n\to\infty} \sqrt[n+5]{e^{a^2+a+1}}$$

$$\lim_{n\to\infty} \sqrt[n+7]{e^{a^2+a+1}} \le e^{\xi_n(a^2+a+1)} \le \lim_{n\to\infty} \sqrt[n+5]{e^{a^2+a+1}}$$

So, by Sandwich Theorem, $\lim_{n\to\infty} e^{\xi_n(a^2+a+1)} = 1$

$$\therefore \lim_{n\to\infty} n^2\left(\sqrt[n+5]{e^{a^2+a+1}} - \sqrt[n+7]{e^{a^2+a+1}}\right) = \lim_{n\to\infty} \frac{2(a^2 + a + 1)}{\left(1 + \frac{5}{n}\right)\left(1 + \frac{7}{n}\right)} \cdot \lim_{n\to\infty} e^{\xi_n(a^2+a+1)}$$

$$= 2(a^2 + a + 1)$$

$$\therefore \sum_{cyc} \frac{\Omega(a)}{b+c} = 2\sum_{cyc} \frac{a^2}{b+c} + \sum_{cyc} \frac{2a}{b+c} + 2\sum_{cyc} \frac{1}{b+c}$$

$$\geq a+b+c+3+\frac{9}{a+b+c} > a+b+c$$

Solution 2 by Subhajit Chattopadhyay-Bolpur-India

$$\Omega(a) = \lim_{n\to\infty} n^2 \left(\sqrt[n+5]{c^{a^2+a+1}} - \sqrt[n+7]{e^{a^2+a+1}}\right), a > 0$$

Expanding by $c^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \sqrt[n+5]{c^{a^2+a+1}} - \sqrt[n+7]{c^{a^2+a+1}}$

$$= 1 + \frac{a^2+a+1}{n+5} + \frac{(a^2+a+1)^2}{2(n+5)^2} + \cdots$$

$$-1 - \frac{a^2+a+1}{n+7} - \frac{(a^2+a+1)^2}{2(n+7)^2} - \cdots$$

$$= \frac{2(a^2+a+1)}{(n+5)(n+7)} + \frac{(a^2+a+1)^2}{2} + 0\left(\frac{1}{n^4}\right)$$

$$\therefore \lim_{n\to\infty} n^2\left(e^{\frac{a^2+a+1}{n+5}} - e^{\frac{a^2+a+1}{n+7}}\right) = \lim_{n\to\infty} \frac{2(a^2+a+1)}{\left(1+\frac{5}{n}\right)\left(1+\frac{7}{n}\right)} + 0 = 2(a^2+a+1)$$

Now, $\frac{\Omega(a)}{b+c} + \frac{\Omega(b)}{c+a} + \frac{\Omega(c)}{a+b} = \frac{2(a^2+a+1)}{b+c} + \frac{2(b^2+b+1)}{c+a} + \frac{2(c^2+c+1)}{a+b}$

without loss of generality assume, $a \geq b \geq c$***, Apply Chebyshev***

inequality, LHS $\geq \frac{2}{3}(a^2+b^2+c^2+a+b+c+3)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$

By AM $\geq$ ***HM*** $\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \geq \frac{9}{2(a+b+c)}$

158. Find:

$$\Omega = \lim_{n\to\infty} \frac{\sum_{k=2}^{n}\left(k^2 \cdot \sqrt[k]{\binom{2k}{k}}\right)}{n(n+1)(2n+1)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by Shivam Sharma-New Delhi-India

Solution 1 by Ravi Prakash-New Delhi-India

$$\textit{Let } a_n = \binom{2n}{n};\ \lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty}\binom{2n+2}{n+1}\Big/\binom{2n}{n} = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4\lim_{n\to\infty}\left(-\frac{1}{2n+2}\right) = 4$$

Let $0 < \epsilon < 1$***, there exists a positive integer*** m ***such that***

$$\left|(a_n)^{\frac{1}{n}} - 4\right| < \epsilon \quad \forall n > m \Rightarrow 4 - \epsilon < (a_n)^{\frac{1}{n}} < 4 + \epsilon \quad \forall n > m$$

$$\textit{Let } b_k = k^2\binom{2k}{k}^{\frac{1}{k}} = k^2(a_k)^{\frac{1}{k}}$$

Let $A = b_1 + b_2 + \cdots + b_m - (1^2 + 2^2 + \cdots + m^2)(4 - \epsilon)$ ***and***

$B = b_1 + b_2 + \cdots + b_m - (1^2 + 2^2 + \cdots + m^2)(4 + \epsilon)$ ***Now, for*** $n > m$

$$(2^2 + 3^2 + \cdots + n^2)(4 - \epsilon) + A <$$

$$< b_2 + b_3 + \cdots + b_n < (2^2 + \cdots + n^2)(4 + \epsilon) + B \Rightarrow \frac{\left[\frac{1}{6}n(n+1)(2n+1) - 1\right](4-\epsilon) + A}{n(n+1)(2n+1)}$$

$$< \frac{\sum_{k=2}^{n} b_k}{n(n+1)(2n+1)} < \frac{\left(\frac{1}{6}n(n+1)(2n+1) - 1\right)(4 + \epsilon) + B}{n(n+1)(2n+1)}$$

Taking limit as $n \to \infty$***, we get*** $\frac{1}{6}(4 - \epsilon) \le \lim_{n\to\infty}\frac{\sum_{k=2}^{n} b_k}{n(n+1)(2n+1)} \le \frac{1}{4}(4 + \epsilon)$

$\Rightarrow \frac{2}{3} - \epsilon \le \Omega \le \frac{2}{3} + \epsilon$***. Its true for each*** $\epsilon > 0$, $\therefore\ \Omega = \frac{2}{3}$

Solution 2 by Shivam Sharma-New Delhi-India

$$\Rightarrow \lim_{n\to\infty}\frac{\sum_{k=2}^{n}\left(k^2\left(\frac{(2k)!}{(k!)^2}\right)^{\frac{1}{k}}\right)}{n(2n+1)(n+1)}$$. *As we know, the Stirling's formula,*

$n! = \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$. *Using this, we get,* $\Rightarrow \lim_{n\to\infty}\frac{\sum_{k=2}^{n}\left(\frac{k^2\left(\left(\frac{2k}{e}\right)^k\sqrt{2\pi k}\right)^{\frac{1}{k}}}{\left(\left(\frac{k}{e}\right)^k\sqrt{2\pi k}\right)^{\frac{2}{k}}}\right)}{n(n+1)(2n+1)}$

$$\Rightarrow \lim_{n\to\infty}\frac{\sum_{k=2}^{n}\left(\frac{k^3}{2k}\right)}{n(n+1)(2n+1)}\Rightarrow\frac{2}{3}\lim_{n\to\infty}\frac{2n^3+3n^2+n-6}{2n^3+3n^2+n}$$

$$\Rightarrow\frac{2}{3}\lim_{n\to\infty}\frac{2n^3+3n+n-6}{2n^3+3n^2+n}\Rightarrow\frac{2}{3}\left(\frac{2+0+0-0}{2+0+0}\right)\text{ (OR) }\Omega=\frac{2}{3}$$

159. $f:\mathbb{R}\to[a,b],\ a<b$

Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(n-k+1)^2f(k)}{k(1^2+2^2+\dots+n^2)}$$

Proposed by Daniel Sitaru – Romania

Solution by Saptak Bhattacharya-Kolkata-India

$$\lim_{n\to\infty}6\sum_{k=1}^{n}\frac{(n-k+1)^2(k)}{kn((n+1)(2n+1)}=\lim_{n\to\infty}\frac{6}{n(n+1)(2n+1)}\sum_{k=1}^{n}\frac{(n-k+1)^2f(k)}{k}$$

Now, $a\le f(k)\le b$, *And,* $\lim_{n\to\infty}\frac{6}{n(n+1)(2n+1)}\sum_{k=1}^{n}\frac{(n-k+1)^2a}{k}$

$$=\lim_{n\to\infty}\frac{6a}{n(n+1)(2n+1)}\cdot\sum_{k=1}^{n}\frac{n^2+k^2+1-2k-2nk+2n}{k}$$

$$=\lim_{n\to\infty}\frac{6a}{n(n+1)(2n+1)}\cdot\left[(n+1)^2H_n+\frac{n(n+1)}{2}-2n(n+1)\right]$$

$$= \lim_{n\to\infty} \frac{6a}{n(n+1)(2n+1)}\left((n+1)^2 H_n - \frac{3n(n+1)}{2}\right)$$

$$= a \lim_{n\to\infty} 6\left(\frac{(n+1)H_n}{n(2n+1)} - \frac{3}{2(2n+1)}\right) = 6a\left(\frac{1}{2}\lim_{n\to\infty}\frac{H_n}{n}\right)$$

$= 3a\lim_{n\to\infty}\frac{H_n}{n} = 0$ *(Cauchy first theorem). Similarly,*

$$\lim_{n\to\infty} \frac{6b}{n(n+1)(2n+1)} \sum_{k=1}^{n} \frac{(n-k+1)^2}{k} = 0$$

Thus by squezze theorem, the given limit is 0

160. Evaluate

$$\frac{\pi}{2}\left(1+\frac{1}{2}\left(1+\frac{3}{4}\left(1+\frac{5}{6}(1+\cdots)\right)\right)\right) - \left(1+\frac{2}{3}\left(1+\frac{4}{5}\left(1+\frac{6}{7}(1+\cdots)\right)\right)\right)$$

Proposed by Vidyamanohar Sharma Astakala-Hydebarad-India

Solution by proposer

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}dx = \int_0^{\frac{\pi}{2}} (1+\cos x)^{-1}dx$$

$$= \int_0^{\frac{\pi}{2}} \left[\sum_{\gamma}^{\infty} (\cos x)^{2\gamma} - \sum_{\gamma=0}^{\infty} (\cos x)^{2\gamma+1}\right] dx$$

$$= \frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2} + \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}$$

$$\dots$$

$$-\left(1 + \frac{2}{3} + \frac{4}{5}\cdot\frac{2}{3} + \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot\dots\right)$$

$\therefore$ *Given sum* $= 1$

161. Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(n-k+1)e^{-k^2}}{1+2+\cdots+n}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$\text{Let } a_k = \frac{(n-k+1)e^{-k^2}}{1+2+\cdots+n} = \frac{2}{n(n+1)}[(n+1)-k]e^{-k^2} = \frac{2}{n}\left(1-\frac{k}{n+1}\right)e^{-k^2}$$

$$\text{Let } b_k = e^{-k^2}, c_k = ke^{-k^2}; \lim_{n\to\infty} b_k = 0, \lim_{n\to\infty} c_k = 0$$

$$\Rightarrow \lim_{n\to\infty}\frac{b_1+b_2+\cdots+b_n}{n} = 0 \text{ and } \lim_{n\to\infty}\frac{c_1+c_2+\cdots+c_n}{n} = 0$$

$$\text{Now, } \lim_{n\to\infty}\sum_{k=1}^{n} a_k = 2\lim_{n\to\infty}\frac{b_1+b_2+\cdots+b_n}{n} - \lim_{n\to\infty}\frac{2}{n+1}\lim_{n\to\infty}\frac{c_1+c_2+\cdots+c_n}{n} = 0$$

162. Find:

$$\Omega = \lim_{n\to\infty} n^2\int_0^1\frac{x^2+\arctan x}{e^{nx}}dx$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\text{For } x\ge 0, 0\le x^2n^2e^{-nx} < \frac{4!}{x^2n^2}, \text{ since, } e^{nx} > \frac{(nx)^4}{4!}$$

$$\text{Similarly, for } x\ge 0, 0\le n^2e^{-nx}\tan^{-1}x < \frac{4!\tan^{-1}x}{x^4n^2}$$

$$0\le \lim_{n\to\infty} n^2\int_0^1\frac{x^2+\tan^{-1}x}{e^{nx}}dx < \lim_{n\to\infty}\int_0^1\left(\frac{4!}{x^2n^2}+\frac{4!\tan^{-1}x}{x^4n^2}\right)dx =$$

$$\int_0^1\lim_{n\to\infty}\left(\frac{4!}{x^2n^2}+\frac{4!\tan^{-1}x}{x^4n^2}\right)dx = 0$$

so, by sandwich theorem $\lim_{n\to\infty}\int_0^1 \frac{x^2+\tan^{-1}x}{e^{nx}}dx = 0$

163. **If $a_n > 0, n \geq 1, \lim_{n\to\infty} a_n = a, b, c > 0$ then find:**

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{a_k}{b + ca_k}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Nirapada Pal-Jhargram-India, Solution 2 by Ravi Prakash-New Delhi-India, Solution 3 by Abdallah El Farissi-Bechar-Algerie, Solution 4 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Nirapada Pal-Jhargram-India

By Cauchy's limit theorem $\lim_{n\to\infty} A_n = A \Rightarrow \lim_{n\to\infty}\frac{A_1+A_2+A_3+\cdots+A_n}{n} = A$

Now, we have $\lim_{n\to\infty}\frac{f(n)}{g(n)} = \frac{\lim_{n\to\infty} f(n)}{\lim_{n\to\infty} g(n)}$ ***provided*** $\lim_{n\to\infty} g(n) \neq 0$

Given $\lim_{n\to\infty} a_n = a$***. So*** $\lim_{n\to\infty}\frac{a_n}{b+ca_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty}(b+ca_n)} = \frac{a}{b+ca}$

So by Cauchy's limit theorem we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{a_k}{b + ca_k} = \lim_{n\to\infty}\frac{a_n}{b + ca_n} = \frac{a}{b + ca}$$

Solution 2 by Ravi Prakash-New Delhi-India

As $a > 0$***, we choose*** $\epsilon > 0$ ***such that*** $0 < a < \epsilon$***. Since*** $\lim_{n\to\infty} a_n = a$

these exists $k \in \mathbb{N}$ ***such that*** $|a_n - a| < \epsilon \quad \forall n > k$

$\Rightarrow 0 < a - \epsilon < a_n < a + \epsilon \quad \forall n > k$ ***(1)***

Let $A = \sum_{j=1}^{k}\frac{a_j}{b+ca_j}$***. From (1)*** $\forall n > k$

$b + c(a - \epsilon) < b + ca_n < b + c(a + \epsilon) \Rightarrow \frac{1}{b+c(a+\epsilon)} < \frac{1}{b+ca_n} < \frac{1}{b+c(a-\epsilon)}$ ***(2)***

From (1), (2) we get $\frac{a-\epsilon}{b+c(a+\epsilon)} < \frac{a_n}{b+ca_n} < \frac{a+\epsilon}{b+c(a+\epsilon)} \quad \forall n > k$

$$\Rightarrow (n-k)\frac{a-\epsilon}{b+c(a+\epsilon)} < \sum_{j=k}^{n}\frac{a_j}{b+ca_j} < (n-k)\frac{a+\epsilon}{b+c(a-\epsilon)} \quad \forall\, n > k$$

$$\Rightarrow \frac{1}{n}\left\{A + (n-k)\frac{a-\epsilon}{b+c(a+\epsilon)}\right\} < \sum_{j=1}^{n}\frac{a_j}{b+ca_j} < \frac{1}{n}\left\{A + (n-k)\frac{a+\epsilon}{b+c(a-\epsilon)}\right\} \quad \forall n > k$$

Taking limit as $n \to \infty$**, we get** $0 + (1-0)\frac{a-\epsilon}{b+c(a+\epsilon)} \le \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{a_j}{b+ca_j}$ $\le 0 + (1-0)\frac{a+\epsilon}{b+c(a-\epsilon)}$. **Taking limit as** $\epsilon \to 0_+$**, we get**

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{a_j}{b+ca_j} = \frac{a}{b+ca}$$

Solution 3 by Abdallah El Farissi-Bechar-Algerie

Theorem of Cesaro: If $u_n \to l$ **in** $\mathbb{R}$**, then** $\frac{\sum_{k=1}^{n} u_n}{n} \to l$

Let $u_n = \frac{a_n}{b+ca_n}$**, we have** $u_n \to \frac{a}{b+ca}$ **then** $\frac{\sum_{k=1}^{n}\frac{a_n}{b+ca_n}}{n} \to \frac{a}{b+ca}$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty} a_n = a \text{ now, } \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{a_k}{b+ca_k}$$

$$\overset{Caesaro-Stolz}{=} \lim_{n\to\infty}\frac{1}{n+1-n}\left(\sum_{k=1}^{n+1}\frac{a_k}{b+ca_k} - \sum_{k=1}^{n}\frac{a_k}{b+ca_k}\right)$$

$$= \lim_{n\to\infty}\frac{a_{n+1}}{b+ca_{n+1}} = \frac{a}{b+ca}$$

164. Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k}{(k+1)!}\right)\left(\sum_{k=1}^{n}\frac{k(k+2)}{\left((k+1)!\right)^2}\right)\left(\sum_{k=1}^{n}\frac{k(k^2+3k+3)}{\left((k+1)!\right)^3}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 2 by Khanh Hung Vu-Ho Chi Minh-Vietnam, Solution 3 by Ravi Prakash-New Delhi-India Solution 4 by Șerban George Florin-Romania, Solution 5 by Shivam Sharma-New Delhi-India

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$$\sum_{h=1}^{n}\frac{h}{(h+1)!}=\sum_{h=1}^{n}\left(\frac{h+1}{(h+1)!}-\frac{1}{(h+1)!}\right)$$

$$=\sum_{h=1}^{n}\left(\frac{1}{h!}-\frac{1}{(h+1)!}\right)=\frac{1}{1!}-\frac{1}{(n+1)!}=1-\frac{1}{(n+1)!}$$

$$\sum_{h=1}^{n}\frac{k(h+1)}{\left((h+1)!\right)^2}=\sum_{h=1}^{n}\frac{k^2+2k}{\left((h+1)!\right)^2}$$

$$=\sum_{h=1}^{n}\left(\frac{(h+1)^2}{\left((h+1)!\right)^2}-\frac{1}{\left((h+1)!\right)^2}\right)=\sum_{h=1}^{n}\left(\frac{1}{(h!)^2}-\frac{1}{\left((h+1)!\right)^2}\right)$$

$$=\frac{1}{(1!)^2}-\frac{1}{\left((n+1)!\right)^2}=\left(1-\frac{1}{\left((n+1)!\right)^2}\right)$$

$$\sum_{h=1}^{n}\frac{h(h^2+8h+3)}{\left((h+1)!\right)^3}=\sum_{h=1}^{n}\frac{h^3+3h^2+3k}{\left((h+1)!\right)^3}=\sum_{h=1}^{n}\frac{(h+1)^3}{\left((h+1)!\right)^3}-\frac{1}{\left((k+1)!\right)^3}$$

$$=\sum_{h=1}^{n}\frac{1}{(h!)^3}-\frac{1}{\left((h+1)!\right)^3}=1-\frac{1}{\left((n+1)!\right)^3}$$

$$\Rightarrow \Omega=h_{+\infty}\left(1-\frac{1}{(n+1)!}\right)\left(1-\frac{1}{(n+1)!^2}\right)\left(1-\frac{1}{(n+1)!^3}\right)=1$$

Solution 2 by KHanh Hung Vu-Ho Chi Minh-Vietnam

$$s_1=\sum_{k=1}^{n}\frac{k}{(k+1)!}=\sum_{k=1}^{n}\frac{k+1-1}{(k+1)!}=\sum_{k=1}^{n}\frac{1}{k!}-\frac{1}{(k+1)!}=1-\frac{1}{(n+1)!}$$

$$s_2=\sum_{k=1}^{n}\frac{k(k+2)}{[(k+1)!]^2}=\sum_{k=1}^{n}\frac{(k+1)^2-1^2}{[(k+1)!]^2}=\sum_{k=1}^{n}\frac{1}{(k!)^2}-\frac{1}{[(k+1)!]^2}=1-\frac{1}{[(n+1)!]^2}$$

$$s_3=\sum_{k=1}^{n}\frac{k(k^2+3k+3)}{[(k+1)!]^3}=\sum_{k=1}^{n}\frac{(k+1)^3-1^3}{[(k+1)!]^3}=\sum_{k=1}^{n}\frac{1}{(k!)^3}-\frac{1}{[(k+1)!]^3}=1-\frac{1}{[(n+1)!]^3}$$

$$\Omega=\lim_{n\to\infty}\left[1-\frac{1}{(n+1)!}\right]\left[1-\frac{1}{[(n+1)!]^2}\right]\left[1-\frac{1}{[(n+1)!]^3}\right]$$

$$=\lim_{t\to 0}[1-t][1-t^2][1-t^3]=1$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\textit{Let } a_n=\sum_{k=1}^{n}\frac{k}{(k+1)!}=\sum_{k=1}^{n}\frac{k+1-1}{(k+1)!}=\sum_{k=1}^{n}\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right)=\left(1-\frac{1}{(n+1)!}\right)$$

$$b_n=\sum_{k=1}^{n}\frac{k(k+2)}{\left((k+1)!\right)^2}=\sum_{k=1}^{n}\frac{(k+1)^2-1}{\left((k+1)!\right)^2}$$

$$=\sum_{k=1}^{n}\left(\frac{1}{(k!)^2}-\frac{1}{\left((k+1)!\right)^2}\right)=\left(1-\frac{1}{\left((n+1)!\right)^2}\right)$$

$$c_n=\sum_{k=1}^{n}\frac{k(k^2+3k+3)}{\left((k+1)!\right)^3}=\sum_{k=1}^{n}\frac{(k+1)^3-1}{\left((k+1)!\right)^3}$$

$$=\sum_{k=1}^{n}\left(\frac{1}{(k!)^3}-\frac{1}{\left((k+1)!\right)^3}\right)=1-\frac{1}{\left((n+1)!\right)^3}$$

$$\textit{We have } \lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=\lim_{n\to\infty}c_n=1$$

$$\therefore \Omega=\lim_{n\to\infty}a_nb_nc_n=1$$