similarly, $\left(\frac{1}{n^3}, \frac{1}{n^2}\right) \to 0$, *as* $n \to \infty$, *so*, $\frac{1}{(n+1)!} \to 0$, *as* $n \to \infty$

$\frac{1}{((n+1)!)^2} \to 0$, *as* $n \to \infty$; $\frac{1}{((n+1)!)^3} \to 0$, *as* $n \to \infty$

Hence, our limit is, $\Omega = 1$ *(Answer)*

165. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n^4} \sum_{m=1}^{n-1} \left(m \sum_{p=m+1}^{n} p \right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nirapada Pal-Jhargram-India, Solution 2 by Ravi Prakash-New Delhi-India, Solution 3 by Shivam Sharma-New Delhi-India, Solution 4 by Madan Beniwal-India

Solution 1 by Nirapada Pal-Jhargram-India

$$\Omega = \lim_{n\to\infty} \frac{1}{n^4} \sum_{m=1}^{n-1} \left(m \sum_{p=m+1}^{n} p \right) = \lim_{n\to\infty} \frac{1}{n^4} \sum_{m=1}^{n-1} \left(m \left(\sum_{p=1}^{n} p - \sum_{p=1}^{m} p \right) \right)$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{m=1}^{n-1} \left(m \left(\frac{n(n+1)}{2} - \frac{m(m+1)}{2} \right) \right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{1}{n^4} \left(n(n+1) \sum_{m=1}^{n-1} m - \sum_{m=1}^{n-1} m^3 - \sum_{m=1}^{n-1} m^2 \right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{n(n+1)n(n-1)}{2} - \frac{n^2(n-1)^2}{4} \right)$$

As $\lim_{n\to\infty} \frac{1}{n^4} \sum_{m=1}^{n-1} m^2 = 0$

$$=\frac{1}{8}\lim_{n\to\infty}\frac{1}{n^4}n^2(n-1)(n+3)=\frac{1}{8}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1+\frac{3}{n}\right)=\frac{1}{8}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\sum_{p=m+1}^{n}p=\frac{1}{2}(n-m)\{n+m+1\}=\frac{1}{2}\left(n^2-m^2(n-m)\right)$$

$$\Rightarrow m\sum_{p=m+1}^{n}p=\frac{1}{2}(n^2+n)m-\frac{1}{2}m^3-\frac{1}{2}m^2\Rightarrow\sum_{m+1}^{n-1}\left(m\sum_{p=m+1}^{n}p\right)$$

$$=\frac{1}{2}(n^2+n)\sum_{m=1}^{n-1}m-\frac{1}{2}\sum_{m=1}^{n-1}m^3-\frac{1}{2}\sum_{m=1}^{n-1}m^2$$

$$=\frac{1}{2}(n^2+n)\frac{1}{2}(n-1)n-\frac{1}{2}\cdot\frac{1}{4}(n-1)^2n^2-\frac{1}{12}(n-1)(n)(2n-1)$$

$$\lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n-1}\left(m\sum_{p=m+1}^{n}p\right)=\frac{1}{4}-\frac{1}{8}-0=\frac{1}{8}$$

Solution 3 by Shivam Sharma-New Delhi-India

$$\Rightarrow\lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n-1}\left[m\left(\frac{1}{8}(4n^2+4n+1)-\frac{1}{8}(2m+1)^2\right)\right]$$

$$\Rightarrow\lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n-1}\left[m\left(\frac{1}{8}(4n^2+4n+1\right)-\frac{1}{8}(4m^2+1+4m)\right]$$

$$\Rightarrow\lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n-1}\left[\frac{4mn^2+4mn+m-4m^3-4m^2-m}{8}\right]$$

$$\Rightarrow\lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n-1}\left[\frac{mn^2}{2}+\frac{mn}{2}+\frac{m}{8}-\frac{4m^3}{8}-\frac{m^2}{2}-\frac{m}{8}\right]$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n-1}\left[\frac{mn^2}{2}+\frac{mn}{2}-\frac{m^3}{2}-\frac{m^2}{2}\right]$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n-1}\left[\frac{mn^2+mn-m^3-m^2}{2}\right]$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n-1}\left[\left(\frac{mn(n+1)}{2}\right)-\frac{m^3}{2}-\frac{m^2}{2}\right]$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n-1}\left(\frac{m(n^2+n)}{2}\right)-\lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n-1}\left(\frac{m^3}{2}\right)-\lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n-1}\left(\frac{m^2}{2}\right)$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{2n^4}\sum_{m=1}^{n-1}\left[\frac{(n^2+n)(n-1)(n)}{2}\right]-\lim_{n\to\infty}\frac{1}{n^4}\left[\frac{\left(n(n-1)\right)^2}{8}\right]-$$

$$-\lim_{n\to\infty}\frac{1}{n^4}\left(\frac{(n-1)(n)(2(n-1)+1)}{12}\right)$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{2n^4}\left[\frac{(n^2+n)(n^2-n)}{2}\right]-\lim_{n\to\infty}\frac{1}{8n^4}[n^2(n-1)^2]-$$

$$-\lim_{n\to\infty}\frac{1}{n^4}\left[\frac{(n^2-n)(2n-1)}{12}\right]$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{2n^4}\left[\frac{n^4-n^3+n^3-n^2}{2}\right]-\lim_{n\to\infty}\frac{1}{8n^4}[n^2(n^2+1-2n)]-$$

$$-\lim_{n\to\infty}\frac{1}{n^4}\left[\frac{2n^3-3n^2+n}{12}\right]\Rightarrow \frac{1}{4}-\frac{1}{8}-0\Rightarrow \frac{2-1}{8}\ \textit{(OR)}\ \Omega=\frac{1}{8}\ \textit{(Answer)}$$

Solution 4 by Madan Beniwal-India

$$\sum_{p=m+1}^{n} p=[(m+1)+(m+2)+\cdots+m+(n-m-1)+n]$$

$$= m(n-m+1) + \frac{(n-m-1)(n-m-1+1)}{2} + n$$

$$= (n-m-1)\left[\frac{2m+n-m}{2}\right] + n$$

$$\sum_{p=m+1}^{n} p = \frac{(n-m+1)(m+n)}{2} + n = \frac{mn+n^2-m^2-mn-m-n}{2} + n$$

$$\sum_{p=m+1}^{n} p = \frac{n^2-m^2-m-n+2n}{2} = \left(\frac{n^2-m^2-m+n}{2}\right)$$

Then $m\sum_{p=m+1}^{p} = \frac{1}{2}(mn^2-m^3-m^2+nm)$. *Now*

$$\sum_{m=1}^{n-1}\left(m\sum_{p=m+1}^{n} p\right) = \frac{1}{2}\sum_{m=1}^{n-1}(mn^2-m^3-m^2+nm)$$

$$= \frac{1}{2}\left[n^2\cdot\frac{n(n-1)}{2} - \left[\frac{n(n-1)}{2}\right]^2 - \frac{(n-1)(n)(2n-1)}{6} + \frac{n(n-1)n}{2}\right]$$

Then $\lim_{n\to\infty}\frac{1}{n^4}\cdot\frac{1}{2}\left[n^4\left(\frac{1}{2}\right) - \left(\frac{1}{4}\right)n^4 - \left(\frac{2}{6}\right)n^3 + 0\right] = \frac{1}{8}$

166. Find:

$$\Omega = \lim_{n\to\infty}\frac{n(n+1)(n+2)\cdot\ldots\cdot(2n-2)\arctan\frac{\pi}{2^n}}{1\cdot3\cdot5\cdot\ldots\cdot(2n-3)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Yen Tung Chung-Taichung-Taiwan, Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 3 by Ravi Prakash-New Delhi-India

Solution 1 by Yen Tung Chung-Taichung-Taiwan

$$\Omega = \lim_{n\to\infty}\frac{n(n+1)(n+2)\cdots(2n-2)\tan^{-1}\frac{\pi}{2^n}}{1\cdot 3\cdot 5\cdot\ldots\cdot 2n-3}$$

$$= \lim_{n\to\infty}\frac{\big(2\cdot 4\cdot 6\cdot\ldots\cdot(2n-2)\big)\big(n(n+1)(n+2)\cdots(2n-2)\big)\tan^{-1}\frac{\pi}{2^n}}{(1\cdot 3\cdot 5\cdot\ldots\cdot 2n-3)\big(2\cdot 4\cdot 6\cdot\ldots\cdot(2n-2)\big)}$$

$$= \lim_{n\to\infty}\frac{2^{n-1}(2n-2)!\tan^{-1}\frac{\pi}{2^n}}{(2n-2)!} = \frac{1}{2}\underbrace{\lim_{n\to\infty}2^n\tan^{-1}\frac{\pi}{2^n}}_{let\ t=\frac{1}{2^n}} = \frac{1}{2}\lim_{n\to 0^+}\frac{\tan^{-1}\pi t}{t}$$

$$= \frac{1}{2}\underbrace{\lim_{t\to 0^+}\frac{\frac{\pi}{1+\pi^2t^2}}{1}}_{L'Hospital\ Rule} = \frac{1}{2}\cdot\pi = \frac{\pi}{2}$$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$A_n = \frac{n(n+1)\cdot\ldots\cdot(2n-2)\arctan\frac{\pi}{2^n}}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-3)} = \frac{n(n+1)\cdot\ldots\cdot(2n-2)\arctan\frac{\pi}{2^n}}{\frac{(2n-2)!}{2\cdot 4\cdot 6\cdot\ldots\cdot(2n-2)}}$$

$$= \frac{n(n+1)\cdot\ldots\cdot(2n-2)\arctan\frac{\pi}{2^n}}{\frac{(2n-2)!}{2^{n-1}\big(1\cdot 2\cdot\ldots\cdot(n-1)\big)}}$$

$$= \frac{2^{n-1}\cdot 1\cdot 2\cdot\ldots\cdot(n-1)\cdot n(n+1)\cdot\ldots\cdot(2n-2)\arctan\frac{\pi}{2^n}}{(2n-2)!}$$

$$= 2^{n-1}\frac{(2n-2)!}{(2n-2)!}\arctan\frac{\pi}{2^n} = 2^{n-1}\arctan\frac{\pi}{2^n}$$

$$\Omega = \lim_{n\to+\infty}A_n = \lim_{n\to+\infty}2^{n-1}\arctan\frac{\pi}{2^n}$$

$$= \lim_{n\to+\infty}\frac{\pi}{2}\cdot\frac{a\tan\frac{\pi}{2^n}}{\frac{\pi}{2^n}} = \lim_{\frac{\pi}{2^n}\to 0}\frac{\pi}{2}\cdot\frac{a\tan\frac{\pi}{2^n}}{\frac{\pi}{2^n}} = \frac{\pi}{2}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\lim_{n\to\infty}\frac{n(n+1)(n+2)\cdot\ldots\cdot(2n-2)}{(1)(3)(5)\ldots(2n-3)}\,arc\left(\frac{\pi}{2^n}\right)$$

$$=\lim_{n\to\infty}\frac{(2)(4)(6)\cdot\ldots\cdot(2n-2)n(n+1)\cdot\ldots\cdot(2n-2)}{(1)(2)(3)\cdot\ldots\cdot(2n-3)(2n-2)}$$

$$=\lim_{n\to\infty}\frac{2^{n-1}(2n-2)!}{(2n-2)!}arc\left(\frac{\pi}{2^n}\right)=\lim_{n\to\infty}2^{n-1}\left(\frac{\pi}{2^n}\right)\times\frac{arc\left(\frac{\pi}{2^n}\right)}{\frac{\pi}{2^n}}=\frac{\pi}{2}(1)=\frac{\pi}{2}$$

167. Find:

$$\Omega=\lim_{n\to\infty}\frac{n\int_1^{n^2}[\sqrt{x}]dx}{\int_1^{n^3}[\sqrt{x}]dx},\ [*]\text{ - great integer function}$$

Proposed by Daniel Sitaru – Romania

Solution by Francis Fregeau-Quebec-Canada

Let β be a natural number. $\beta^2<x\le(\beta+1)^2\Rightarrow\lceil\sqrt{x}\rceil=\beta+1$

Let n be a perfect square, and $1\le x\le n^2$***. Divide the interval*** $(1,n^2)$ ***into the partition:*** $(1,4]\cup(4,9]\cup\ldots\cup(k^2,(k+1)^2]\cup\ldots\cup((n-1)^2,n^2)$

Now: $\int_{j^2}^{(j+1)^2}\lceil\sqrt{x}\rceil\,dx=\int_{j^2}^{(j+1)^2}(j+1)\,dx=(2j+1)(j+1)=2j^2+3j+1$

$\therefore\int_1^{n^2}\lceil\sqrt{x}\rceil\,dx=\sum\int_{j^2}^{(j+1)^2}\lceil\sqrt{x}\rceil\,dx;\ 1\le j\le n-1$***. Let*** $n-1=m$

$$\sum_1^m\int_{j^2}^{(j+1)^2}\lceil\sqrt{x}\rceil\,dx=\sum_1^m 2j^2+3j+1=\frac{2m^3+3m^2+m}{3}+\frac{3m(m+1)}{2}+m$$

$$=\frac{4m^3+15m^2+17m}{6}$$

$$\therefore n\int_1^{n^2}\lceil\sqrt{x}\rceil\,dx = (m+1)\cdot\frac{4m^3+15m^2+17m}{6} = n\cdot\frac{4(n-1)^3+15(n-1)^2+17(n-1)}{6}$$

But n^3 is also a perfect square since n is a perfect square.

$$\Rightarrow \int_1^{n^3}\lceil\sqrt{x}\rceil\,dx = \sum 2j^2+3j+1;\ 1\le j\le n^{\frac{3}{2}}-1$$

$$\therefore \int_1^{n^3}\lceil\sqrt{x}\rceil\,dx = \frac{4\left(n^{\frac{3}{2}}-1\right)^3+15\left(n^{\frac{3}{2}}-1\right)^2+17\left(n^{\frac{3}{2}}-1\right)}{6} \Rightarrow \lim_{n\to\infty}\frac{n\cdot\int_1^{n^2}\lceil\sqrt{x}\rceil dx}{\int_1^{n^3}\lceil\sqrt{x}\rceil dx} =$$

$$= \lim_{n\to\infty}\frac{n\left(4(n-1)^3+15(n-1)^2+17(n-1)\right)}{4\left(n^{\frac{3}{2}}-1\right)^3+15\left(n^{\frac{3}{2}}-1\right)^2+17\left(n^{\frac{3}{2}}-1\right)}$$

Applying De l'Hôpital's rule four times yields:

$$\lim_{n\to\infty}\frac{n\cdot\int_1^{n^2}\lceil\sqrt{x}\rceil dx}{\int_1^{n^3}\lceil\sqrt{x}\rceil dx} = 0 \text{ when } n \text{ is a perfect square.}$$

But for every real number R, there exists a perfect square n such that

$$R < n \therefore \lim_{R\to\infty}\frac{R\cdot\int_1^{R^2}\lceil\sqrt{x}\rceil dx}{\int_1^{R^3}\lceil\sqrt{x}\rceil dx} = \lim_{n\to\infty}\frac{n\cdot\int_1^{n^2}\lceil\sqrt{x}\rceil dx}{\int_1^{n^3}\lceil\sqrt{x}\rceil dx} = 0$$

168. Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{3n+1}\sum_{k=2}^{n}\frac{1}{\sqrt[k]{k!}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Soumitra Mandal-Chandar Nagore-India;

Solution 2 by Ravi Prakash-New Delhi-India

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{Cauchy-D'Alembert}{\cong}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}$$

$$\Omega=\lim_{n\to\infty}\frac{1}{3n+1}\sum_{k=2}^{n}\frac{1}{\sqrt[k]{k!}}=\lim_{n\to\infty}\frac{\sum_{k=2}^{n+1}\frac{1}{\sqrt[k]{k!}}-\sum_{k=2}^{n}\frac{1}{\sqrt[k]{k!}}}{3n+4-3n-1}=\frac{1}{3}\lim_{n\to\infty}\frac{1}{\sqrt[n+1]{(n+1)!}}$$

$$=\frac{1}{3}\lim_{n\to\infty}\left(\frac{1}{n+1}\cdot\frac{1}{\frac{\sqrt[n+1]{(n+1)!}}{n+1}}\right)=\frac{e}{3}\lim_{n\to\infty}\frac{1}{n+1}=0$$

Solution 2 by Ravi Prakash-New Delhi-India

$\lim_{n\to\infty}\left(\frac{1}{n!}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}=\lim_{n\to\infty}\frac{1}{n+1}=0$. ***Let*** $\epsilon>0$***, there exists***

$m\in\mathbb{N}$ ***s.t.*** $\left|\left(\frac{1}{n!}\right)^{\frac{1}{n}}-0\right|<\epsilon\quad\forall n>m\Rightarrow 0<\left(\frac{1}{n!}\right)^{\frac{1}{n}}<\epsilon\quad\forall\ n>m$

Let $a=\sum_{k=1}^{m}\left(\frac{1}{k!}\right)^{\frac{1}{k}}$***. For*** $\frac{n>m}{n}$

$$\sum_{k=1}^{n}\left(\frac{1}{k!}\right)^{\frac{1}{k}}=\sum_{k=1}^{m}\left(\frac{1}{k!}\right)^{\frac{1}{k}}+\sum_{k=m+1}^{n}\left(\frac{1}{k!}\right)^{\frac{1}{k}}\Rightarrow 0<\sum_{k=1}^{n}\left(\frac{1}{k!}\right)^{\frac{1}{k}}<a+(n-m)\epsilon$$

$$\Rightarrow 0<\frac{1}{3n+1}\sum_{k=1}^{n}\left(\frac{1}{k!}\right)^{\frac{1}{k}}<\frac{a}{3n+1}+\frac{n-m}{3n+1}\epsilon$$

Taking limit as $n\to\infty$***, we get*** $0\le\lim_{n\to\infty}\frac{1}{3n+1}\sum_{k=1}^{n}\left(\frac{1}{k!}\right)^{\frac{1}{k}}\le\frac{\epsilon}{3}$

This is true for each $\epsilon>0$

$$\therefore \lim_{n\to\infty}\frac{1}{3n+1}\sum_{k=1}^{n}\left(\frac{1}{k!}\right)^{\frac{1}{k}}=0$$

169. **Find:**

$$\Omega=\lim_{n\to\infty}\frac{(e+\sqrt{2})\cdot(e+\sqrt{3})\cdot\ldots\cdot(e+\sqrt{n})}{(\pi+\sqrt{2})\cdot(\pi+\sqrt{3})\cdot\ldots\cdot(\pi+\sqrt{n})}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Carlos Suarez-Quito-Ecuador

Solution 2 by Boris Colakovic-Belgrade-Serbia

Solution 1 by Carlos Suarez-Quito-Ecuador

$$\Omega=\lim_{n\to\infty}\prod_{i=1}^{n}\left(\frac{(e+\sqrt{i})}{(\pi+\sqrt{i})}\right)=\lim_{n\to\infty}\frac{\prod_{i=1}^{n}(e+\sqrt{i})}{\prod_{i=1}^{n}(\pi+\sqrt{i})}$$

$$\Omega=\lim_{n\to\infty}\prod_{i=1}^{n}\left(\frac{e}{\pi}+\frac{\sqrt{i}\left(1-\frac{e}{\pi}\right)}{\sqrt{i}+\pi}\right)$$

$$\Omega=\lim_{n\to\infty}\prod_{i=1}^{n}\left(\frac{e}{\pi}\right)+\lim_{n\to\infty}\prod_{i=1}^{n}\left(1-\frac{e}{\pi}\right)\left[\frac{1}{1+\frac{\pi}{\sqrt{i}}}\right]=0$$

Solution 2 by Boris Colakovic-Belgrade-Serbia

$$\textit{Find } \Omega=\lim_{n\to\infty}\frac{(e+\sqrt{2})(e+\sqrt{3})\cdot\ldots\cdot(e+\sqrt{n})}{(\pi+\sqrt{2})(\pi+\sqrt{3})\cdot\ldots\cdot(\pi+\sqrt{n})}$$

$$\ln\Omega=\lim_{n\to}\frac{\ln(e+\sqrt{2})^n+\ln(e+\sqrt{3})^n+\cdots+\ln(e+\sqrt{n})^n}{n}-$$

$$-\lim_{n\to\infty}\frac{\ln(\pi+\sqrt{2})^n+\ln(\pi+\sqrt{3})^n+\cdots+\ln(\pi+\sqrt{n})^n}{n}\overset{Stolz-Cesaro}{=}$$

$$= \lim_{n\to\infty} \ln\left(e + \sqrt{n}\right)^n - \lim_{n\to\infty} \ln\left(\pi + \sqrt{n}\right)^n = \lim_{n\to\infty} \ln\left(\frac{e+\sqrt{n}}{\pi+\sqrt{n}}\right)^n =$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{e+\sqrt{n}}{\pi+\sqrt{n}}\right)}{\frac{1}{n}} \overset{L'Hospital}{\cong}$$

$$= \lim_{n\to\infty} \frac{\frac{\pi+\sqrt{n}}{e+\sqrt{n}} \cdot \frac{\pi - e}{2\sqrt{n}\left(\pi+\sqrt{n}\right)^2}}{-\frac{1}{n^2}} = -\frac{\pi-e}{2} \lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{\left(e+\sqrt{n}\right)\left(\pi+\sqrt{n}\right)} =$$

$$= -\frac{\pi-e}{2} \cdot \frac{3}{2} \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{\frac{1}{2}n^{-\frac{1}{2}}\left(\pi+\sqrt{n}\right) + \frac{1}{2}n^{-\frac{1}{2}}\left(e+\sqrt{n}\right)} =$$

$$= -\frac{3}{2}(\pi - e) \lim_{n\to\infty} \frac{n}{\pi + e + 2\sqrt{n}} = -\frac{3}{2}(\pi - e) \lim_{n\to\infty} \frac{1}{2 \cdot \frac{1}{2} n^{-\frac{1}{2}}} =$$

$$= -\frac{3}{2}(\pi - e) \lim_{n\to\infty} \sqrt{n} = -\infty \Rightarrow \ln \Omega = -\infty \Rightarrow -\Omega = \frac{1}{e^{\infty}} = 0$$

170. If $a, b, c, x, y, z > 0, a + b + c = 1$

$$\Omega(a) = \lim_{n\to\infty} n\left(a^{\sqrt[n]{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}-1} - 1\right)$$

then: $\Omega(ax + by + cz) \geq \Omega\left(x^a y^b z^c\right)$

Proposed by Daniel Sitaru – Romania

Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam, Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

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Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam

If $\boldsymbol{a,b,c,x,z} > 0, a+b+c=1. \Omega(\boldsymbol{a}) = \lim_{n\to\infty} n\left(a^{\sqrt[n]{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}-1}-1\right)$

then $\Omega(ax+by+cz) \ge \Omega(x^a y^b z^c)$***. We have***

$$\left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2}\cdot\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\cdot\left(\frac{1}{n}\right)^3+\cdots+\left(\frac{1}{n}\right)^n$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n < 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!} \Rightarrow \sqrt[n]{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}-1 > \frac{1}{n} \Rightarrow n\left(a^{\sqrt[n]{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}-1}-1\right) < n^{\left(a^{\frac{1}{n}}-1\right)} \quad (1)$$

Lemma: $1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!} < e$

On the other hand, using the lemma, we have $1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!} < e \Rightarrow$

$$\Rightarrow \sqrt[n]{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}-1 < \sqrt[n]{e}-1 \Rightarrow n\left(a^{\sqrt[n]{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}-1}-1\right) > n\left(a^{\sqrt[n]{e}-1}-1\right) \quad (2)$$

(1) and (2) $\Rightarrow n\left(a^{\sqrt[n]{e}-1}-1\right) < n\left(a^{\sqrt[n]{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}-1}-1\right) < n\left(a^{\frac{1}{n}-1}\right)$ ***(3)***

On the other hand, we have

$$+)\lim_{n\to\infty} n\left(a^{\frac{1}{n}}-1\right) = \lim_{n\to\infty}\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\ln a = \ln a \quad (4)$$

$$+)\lim_{n\to\infty} n\left(a^{\sqrt[n]{e}}-1\right)$$

Put $x=\sqrt[n]{e}-1 \Rightarrow n=\ln(x+1)$ ***and*** $\lim_{n\to\infty} x = 0$***. We have***

$$\lim_{n\to\infty} n\left(a^{\sqrt[n]{e}}-1\right) = \lim_{x\to 0}\frac{a^x-1}{\ln(x+1)} = \lim_{x\to 0}\frac{a^x-1}{x}\cdot\frac{x}{\ln(x+1)} =$$

$$= \ln a\cdot 1 = \ln a \quad (5)$$

(3), (4) and (5) $\Rightarrow \Omega(a) = \ln a$

By Becnuli inequality, we have $x^a < e^{a(x-1)}$

Similarly, we have $y^b < e^{b(y-1)}$ ***and*** $c^z < e^{c(z-1)}$

$$\Rightarrow x^a y^b c^z < e^{a(x-1)+b(y-1)+c(z-1)} \Rightarrow x^a y^b z^c < e^{ax+by+cz-1} \Rightarrow \ln(x^a y^b z^c) < ax + by + cz - 1 \quad (6)$$

By Becnulli inequality, we have $e^{ax+by+cz} \geq ax + by + cz + 1 \quad$ ***(7)***

(6) and (7) $\Rightarrow$ ***QED***

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$\Omega(a) = \lim_{n\to+\infty} n\left(a^{\sqrt[n]{\sum_{h=0}^{n}\frac{1}{h!}}-1} - 1\right) = \lim_{n\to+\infty}\left(a^{\sqrt[n]{U_n}-1} - 1\right);\ U_n = \sum_{h=0}^{n}\frac{1}{h!}$$

$\lim_{+\infty} U_n = e$ ***; on pose,*** $V_n = U_n - e; \lim_{n\to+\infty} V_n = 0$

$$\Omega(a) = \lim_{n\to+\infty} n\left(e^{\left(\sqrt[n]{U_n}-1\right)\ln a} - 1\right) = \lim_{+\infty}\frac{e^{\left(U_n^{\frac{1}{n}}-1\right)\ln a} - 1}{\left(U_n^{\frac{1}{n}} - 1\right)\ln a}\cdot n\left(U_n^{\frac{1}{n}} - 1\right)\ln a$$

$$= \lim_{+\infty}\frac{e^{\left((V_n+e)^{\frac{1}{n}}-1\right)\ln a} - 1}{\left((V_n + e)^{\frac{1}{n}} - 1\right)\ln a}\cdot\frac{e^{\frac{\ln(V_n+e)}{n}} - 1}{\frac{\ln(V_n + e)}{n}}\cdot\ln(V_n + e)\cdot\ln a$$

$$= \lim_{n\to+\infty}\frac{e^{\left(e^{\frac{\ln(V_n+e)}{n}}-1\right)\ln a} - 1}{\left(e^{\frac{\ln(V_n+e)}{n}} - 1\right)\ln a}\cdot\frac{e^{\frac{\ln(V_n+e)}{n}} - 1}{\frac{\ln(V_n + e)}{n}}\cdot\ln(V_n + e)\cdot\ln a$$

$$= 1 \times 1 \times \ln e\cdot\ln a = \ln a$$

$$\left(\left(\lim_{+\infty} e^{\frac{\ln(V_n+e)}{n}} = e^{\lim_{+\infty}\frac{\ln(V_n+e)}{n}} = e^0 = 1\right)\right);\left(\lim_{x\to 0}\frac{e^n - 1}{n} = 1\right)$$

$\Omega(a) = \ln a$. ***if:*** $x \geq y \geq z$

$$* f(x,y,z) = \ln(ax + by + cz) - (a\ln x + b\ln y + c\ln z)$$

$$* g(x) = f(x,y,z)$$

$$g'(x)=\frac{a}{ax+by+cz}-\frac{a}{x}=a\left(\frac{x-(ax+by+cz)}{x(ax+by+cz)}\right)$$

$$=a\left(\frac{(a+b+c)x-(ax+by+cz)}{x(ax+by+cz)}\right)=\frac{a\big(b(x-y)+c(x-3)\big)}{x(ax+by+cz)}$$

$$\forall x>y>z;g'(x)\geq 0\Rightarrow g(x)\geq g(z)\Leftrightarrow f(x,y,z)\geq f(z,y,z)$$

$$*\ h(y)=f(z,y,z);\ h'(y)=\frac{b}{az+by+cz}-\frac{b}{y}=b\left(\frac{y-(az+by+cz)}{y(az+by+cz)}\right)$$

$$=b\frac{y(1-b)-z(a+c)}{y(az+by+cz)}=\frac{b(y-z)(a+c)}{y(az+by+cz)};\ \forall\, y>z;h'(y)>0$$

$$\Rightarrow h(y)\geq h(z)\Leftrightarrow f(z,y,z)\geq f(z,z,z)\Rightarrow f(x,y,z)\geq f(z,z,z)$$

$$f(z,z,z)=\ln\big((a+b+c)z\big)-\big((a+b+c)\ln z\big)=\ln z-\ln z=0$$

$$\Rightarrow f(x,y,z)\geq 0\Leftrightarrow \ln(ax+by+cz)\geq(a\ln x+b\ln y+c\ln z)$$

$$\Leftrightarrow \ln(ax+by+cz)\geq\ln\left(x^ay^bz^c\right)\Leftrightarrow \Omega(ax+by+cz)\geq\Omega\left(x^ay^bz^c\right)$$

if $x\geq y\geq z$

171. Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1+6+11+16+\cdots+(10k-9)}{2k-1}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Subhajit Chattopadhyay-Bolpur-India, Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

Solution 1 by Subhajit Chattopadhyay-Bolpur-India

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1+6+11+16+\cdots+(10k-9)}{2k-1}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(2k-1)\big(2+10(k-1)\big)}{2\times(2k-1)}=\lim_{n\to\infty}\sum_{k=1}^{n}(5k-4)$$

$$=\lim_{n\to\infty}\frac{5n^2-3n}{2}=\infty$$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$\Omega=\lim_{n\to+\infty}\sum_{h=1}^{n}\frac{1+6+\cdots+(10k-9)}{2k-1}=\lim_{n\to+\infty}\sum_{h=1}^{n}\frac{\sum_{p=0}^{2h-2}(5p+1)}{2k-1}$$

$$=\lim_{n\to+\infty}\sum_{h=1}^{n}\frac{s\cdot\frac{(2h-2)(2h-1)}{2}+(2h+1)}{2h-1}=\lim_{n\to+\infty}\left(\sum_{h=1}^{n}\frac{s}{2}(2h-2)+1\right)$$

$$=\lim_{n\to+\infty}\sum_{h=1}^{n}sk-4=\lim_{n\to+\infty}s\frac{n(n+1)}{2}-4n=$$

$$=\lim_{n\to+\infty}n\left(\frac{s}{2}(n+1)-1\right)=+\infty$$

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1+6+11+\cdots+(10k-9)}{2k-1}=$$

$$1+6+11+\cdots+(10k-9)=\frac{a_1+a_n}{2}n=\frac{1+5n-4}{2}\cdot n=\frac{(5n-3)n}{2}$$

$$a_n=a_1+d(n-1)=1+5n-5=5n-4$$

$$\lim_{n\to\infty}\frac{(5n-3)n}{2n-1}=\lim_{n\to\infty}\frac{5-\frac{3}{n}}{\frac{2}{n}-\frac{1}{n^2}}=\lim_{n\to\infty}\frac{5}{0}=\alpha$$

$$\Omega=\alpha$$

172. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n+3}\sum_{k=2}^{n} \frac{k}{\sqrt[k]{k!}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kays Tomy-Nador-Tunisia, Solution 2 by Rozeta Atanasova-Skopje, Solution 3 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Kays Tomy-Nador-Tunisia

$\frac{1}{n+3}\sum_{k=2}^{n}\frac{k}{\sqrt[k]{k!}} \to_{n\to\infty} e$. ***Let us recall Stirling's formula*** $\ln(n!) \sim n\ln(n) - n$

then we have $\frac{n}{\sqrt[n]{n!}} = \exp\left(\ln\left(\frac{n}{\sqrt[n]{n!}}\right)\right) \sim \exp\left(\ln(n) - \frac{\ln(n!)}{n}\right)$

$\sim \exp(\ln(n) - \ln(n) + 1) \to_{\infty} e$. ***Let us Apply Cesaro lemma***

Now we get. $\frac{1}{n+3}\sum_{k=2}^{n}\frac{k}{\sqrt[k]{k!}} = \underbrace{\frac{n-2}{n+3}}_{\substack{\downarrow\infty \\ 1}} \cdot \frac{\sum_{k=2}^{n}\frac{k}{\sqrt[k]{k!}}}{n-2} \to e$. ***Finally we get***

$$\frac{1}{n+3}\sum_{k=2}^{n}\frac{k}{\sqrt[k]{k!}} \to_{\infty} e$$

Solution 2 by Rozeta Atanasova-Skopje

$$L = \lim_{n\to\infty}\frac{1}{n+3}\sum_{k=2}^{n}\frac{k}{\sqrt[k]{k!}} = \lim_{n\to\infty}\frac{n}{n+3}\cdot\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\frac{1}{\sqrt[k]{\frac{k!}{k^k}}}$$

$$\overset{Cauchy}{=} \lim_{n\to\infty}\frac{1}{\sqrt[n]{\frac{n!}{n^n}}} = \frac{1}{\lim\limits_{n\to\infty} e^{\frac{1}{n}\left(\ln\frac{1}{n}+\ln\frac{2}{n}+\ln\frac{3}{n}+\cdots+\ln\frac{n}{n}\right)}}$$

$$= \frac{1}{e^{\int_0^1 \ln x dx}} = \frac{1}{e^{-1}} = e$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} \overset{Cauchy}{=} \lim_{n\to\infty} \left(\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right)$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n} \right) = \frac{1}{e}$$

$$\lim_{n\to\infty} \frac{1}{n+3} \sum_{k=2}^{n} \frac{k}{\sqrt[k]{k!}} \overset{Cesaro-Stolz}{=} \lim_{n\to\infty} \frac{\sum_{k=2}^{n+1} \frac{k}{\sqrt[k]{k!}} - \sum_{k=2}^{n} \frac{k}{\sqrt[k]{k!}}}{n+4-(n+3)}$$

$$= \lim_{n\to\infty} \frac{n+1}{\sqrt[n+1]{(n+1)!}} = e$$

173. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n^5} \sum_{k=1}^{n} k^4 \arctan\left(\frac{k}{n}\right)^5$$

Proposed by Daniel Sitaru – Romania

Solution by Abdelhak Maoukouf-Casablanca-Morocco

$$\Omega = \lim_{n\to+\infty} \frac{1}{n^5} \sum_{k=1}^{n} k^4 \arctan^5\left(\frac{k}{n}\right) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^4 \arctan^5\left(\frac{k}{n}\right)$$

$$= \int_0^1 x^4 \arctan^5 x \, dx = \left[\frac{x^5}{5} \cdot \arctan^5 x\right]_0^1 - \int_0^1 \frac{x^5}{5} \cdot \frac{5x^4}{x^{10}+1} dx$$

$$= \frac{\pi}{20} - \int_0^1 \frac{x^9}{x^{10}+1}dx = \frac{\pi}{20} - \left[\frac{1}{10}\ln|x^{10}+1|\right]_0^1 = \frac{\pi}{20} - \frac{\ln 2}{10}$$

174. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{m=1}^{n}\left(\left(1+\frac{1}{m}\right)\sum_{p=1}^{m} p!\,(1+p^2)\right)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco,

Solution 2 by Abdallah Almalih-Damascus-Syria

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$$p!\,(1+p^2) = p!\left((p+1)(p+2) - 3(p+1) + 2\right) = [(p+2)! - (p+1)!] - 2[(p+1)! - p!]$$

$$\sum_{p=1}^{m} p!\,(1+p^2) = \sum_{p=1}^{n} [(p+2)! - (p+1)!] - 2[(p+1)! - p!] =$$

$$= [(m+2)! - 2!] - 2[(m+1)! - 1!] = (m+2)! - 2(m+1)!$$

$$\left(1+\frac{1}{m}\right)\left((m+2)! - 2(m+1)!\right) = (m+1)(m+1)! = (m+2)! - (m+1)!$$

$$\sum_{m=1}^{n}\left(1+\frac{1}{m}\right)\left((m+2)! - 2(m+1)!\right) = \sum_{m=1}^{n}(m+2)! - (m+1)! = (n+2)! - 2!$$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{m=1}^{n}\left(\left(1+\frac{1}{m}\right)\sum_{p=1}^{n} p!\,(1+p^2)\right)} = \lim_{n\to\infty}\sqrt[n]{(n+2)! - 2} \approx \lim_{n\to\infty}\sqrt[n]{(n+2)!} =$$

$$= \lim_{n\to\infty}\sqrt[n]{n!}\sqrt[n]{(n+1)(n+2)} \approx \lim_{n\to\infty}\sqrt[n]{n!} \approx \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}} \approx \lim_{n\to\infty}\frac{n}{e} \to \infty$$

Solution 2 by Abdallah Almalih-Damascus-Syria

Find $\lim_{n\to\infty}\sqrt[n]{\sum_{m=1}^{n}\left(1+\frac{1}{m}\right)\sum_{p=1}^{m}\left(p!\,(1+p^2)\right)}$

First let's compute the sum $\sum_{p=1}^{m}\left(p!\,(1+p^2)\right)$

$$p!\,(1+p^2)=p!\,[(p+2)(p+1)-3(p+1)+2]$$
$$=(p+2)!-3(p+1)!+2p!$$
$$=[(p+2)!-(p+1)!]-2[(p+1)!-p!]$$
$$\sum_{p=1}^{m}p!\,(1+p^2)=\sum_{p=1}^{m}[(p+2)!-(p+1)!]-2\sum_{p=1}^{m}[(p+1)!-p!]$$
$$=(m+2)!-2!-2\left((m+1)!-1!\right)=(m+2)!-2-2(m+1)!+2$$
$$=(m+2)!-2(m+1)!=(m+1)!\,[m+2-2]=m(m+1)!$$

Let $a_n=\sum_{m=1}^{n}\left(1+\frac{1}{m}\right)\sum_{p=1}^{m}p!\,(1+p^2)$.

Then $a_n=\sum_{m=1}^{n}\left(1+\frac{1}{m}\right)(m(m+1)!)$

$$=\sum_{m=1}^{n}\frac{(m+1)}{m}m\,(m+1)!=\sum_{m=1}^{n}(m+1)(m+1)!$$
$$=\sum_{m=1}^{n}[(m+2)!-(m+1)!]=(n+2)!-2!=(n+2)!-2$$

We want to compute $\lim_{n\to\infty}\sqrt[n]{a_n}$

We know that $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$***. So*** $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$

$$=\lim_{n\to\infty}\frac{(n+3)!-2}{(n+2)!-2}=\lim_{n\to\infty}\frac{n+3-\frac{2}{(n+2)!}}{1-\frac{2}{(n+2)!}}=\infty$$

175. Find:

$$\Omega = \prod_{n=0}^{\infty}\left(1 + \left(\frac{1}{e}\right)^{3^n} + \left(\frac{1}{e}\right)^{2\cdot 3^n}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Subhajit Chattopadhyay-Bolpur-India

Solution 2 by Anisoara Dudu-Romania

Solution 1 by Subhajit Chattopadhyay-Bolpur-India

$$\Omega = \prod_{n=0}^{\infty}\left[1 + \left(\frac{1}{e}\right)^{3^n} + \left(\frac{1}{e}\right)^{2\cdot 3^n}\right]. \textit{ Note, } 1 + x_n + x_n^2 = \frac{1-x_n^3}{1-x_n}$$

$$\therefore \Omega = \prod_{n=0}^{\infty}\frac{\left(1-e^{-3^{n+1}}\right)}{\left(1-e^{-3^n}\right)} = \frac{1-e^{-3}}{1-e^{-1}} \times \frac{1-e^{-9}}{1-e^{-3}} \quad \textit{(1)}$$

$$[\because e^{-\infty} = 0] = \frac{e}{e-1}$$

Solution 2 by Anisoara Dudu-Romania

$$\frac{1}{e} = a \textit{ (not) } \Omega = \lim_{n\to 0}\prod_{k=0}^{n}\left[1 + a^{3^k} + (a^2)^{3k}\right];\ 1 + x + x^2 = \frac{x^3-1}{x-1}$$

$$\Omega = \lim_{n\to\infty}\prod_{k=0}^{\infty}\frac{a^{3k+1}-1}{a^{3k-1}} = \lim_{n\to\infty}\frac{a^3-1}{a-1}\cdot\frac{a^9-1}{a^3-1}\cdot\ldots\cdot\frac{a^{3^{n+1}}-1}{a^{3^n}-1}$$

$$= \lim_{n\to\infty}\frac{a^{3^{n+1}}-1}{a-1} = \frac{0-1}{a-1} = \frac{1}{1-a} = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}$$

176. $x_1 = 2, \frac{1}{\sqrt[n+1]{x_{n+1}}-1} = 1 + \frac{1}{\sqrt[n]{x_n}-1}$

Find the closed form and $\lim_{n\to\infty} x_n$

Proposed by Maria Elena Panaitopol-Romania

Solution 1 by Avishek Mitra-Kolkata-India, Solution 2 by Ravi Prakash-New Delhi-India, Solution 3 by Kunihiko Chikaya-Tokyo-Japan, Solution 4 by Abdelhak Maoukouf-Casablanca-Morocco

Solution 1 by Avishek Mitra-Kolkata-India

$$x_1 = 2 \therefore \frac{1}{\sqrt{x_2}-1} = 1 + \frac{1}{2-1} = 2 \quad \therefore x_2 = \left(\frac{3}{2}\right)^2$$

$$\therefore \frac{1}{\sqrt[3]{x_3}-1} = 1 + \frac{1}{\frac{3}{2}-1} = 3 \quad \therefore x_3 = \frac{64}{27}; = \left(\frac{4}{3}\right)^3$$

$$\therefore \frac{1}{\sqrt[4]{x_4}-1} = 1 + \frac{1}{\frac{4}{3}-1} = 4 \quad \therefore x_4 = \frac{625}{256} = \left(\frac{5}{4}\right)^4$$

$$\therefore \textit{Closed form } x_n = \left(1+\frac{1}{n}\right)^n \therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\textit{Let } b_n = \frac{1}{(a_n)^{\frac{1}{n}}-1}, \textit{ then } b_1 = \frac{1}{2-1} = 1. \textit{ Also, } b_{n+1} - b_n = 1 \ \forall n$$

$$\Rightarrow < b_n > \textit{ is an A.P. with } d = 1 \Rightarrow b_n = b_1 + (n-1)(1) = n$$

$$\Rightarrow (a_n)^{\frac{1}{n}} - 1 = \frac{1}{n} \Rightarrow a_n = \left(1+\frac{1}{n}\right)^n \lim_{n\to\infty} a_n = e$$

Solution 3 by Kunihiko Chikaya-Tokyo-Japan

$$x_1 = 2 \quad \frac{1}{\sqrt[n+1]{x_{n+1}}-1} = 1 + \frac{1}{\sqrt[n]{x_n}-1}. \textit{ Find } \lim_{n\to\infty} x_n$$

$$\textit{The sequence } \left\{\frac{1}{\sqrt[n]{x_n}-1}\right\} \textit{ is an arithmetic progression with common}$$

$$\textit{difference 1} \therefore \frac{1}{\sqrt[n]{x_n}-1} = \frac{1}{x_1-1} + 1\cdot(n-1) = n$$

$$\Leftrightarrow \sqrt[n]{x_n} = 1 + \frac{1}{n} \therefore x_n = \left(1 + \frac{1}{n}\right)^n \ (n \geq 1);\ \lim_{n\to\infty} x_n = e$$

Solution 4 by Abdelhak Maoukouf-Casablanca-Morocco

$$\forall k \in \mathbb{N}^* \ \frac{1}{\sqrt[h+1]{x_{h+1}} - 1} = 1 + \frac{1}{\sqrt[h]{x_h} - 1}$$

$$\Rightarrow \sum_{h=1}^{n-1} \frac{1}{\sqrt[h+1]{x_{h+1}} - 1} = \sum_{h=1}^{n-1}\left(1 + \frac{1}{\sqrt[h]{x_h} - 1}\right)$$

$$\Leftrightarrow \frac{1}{\sqrt[n]{x_n} - 1} = (n-1) + \frac{1}{x_1 - 1} \Leftrightarrow x_n = \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} e^{n\ln\left(1+\frac{1}{n}\right)} = \lim_{\frac{1}{n}\to 0} e^{\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}} = e$$

177. **Find:**

$$\Omega = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \sqrt[n]{7^i \cdot 5^{n-i}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Carlos Suarez-Quito-Ecuador, Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 3 by Rozeta Atanasova-Skopje, Solution 4 by Shivam Sharma-New Delhi-India

Solution 1 by Carlos Suarez-Quito-Ecuador

$$\sum_{k=0}^{n} \left(\frac{7}{5}\right)^{\frac{k}{n}} = \frac{\left(\frac{7}{5}\right)^{\frac{1}{n}+1}}{\left(\frac{7}{5}\right)^{\frac{1}{n}} - 1};\ \lim_{n\to\infty} \frac{5\left(-1+\left(\frac{7}{5}\right)^{1+\frac{1}{n}}\right)}{n\left(-1+\left(\frac{7}{5}\right)^{\frac{1}{n}}\right)} = \frac{2}{\log\left(\frac{7}{5}\right)}$$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$\sum_{i=0}^{n} \sqrt[n]{7^i 5^{n-i}} = 5\sum_{i=0}^{n}\left(\sqrt[n]{\frac{7}{5}}\right)^i = 5\frac{\left(\sqrt[n]{\frac{7}{5}}\right)^n - 1}{\left(\sqrt[n]{\frac{7}{5}}\right) - 1} = \frac{2}{\left(\sqrt[n]{\frac{7}{5}}\right) - 1}$$

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\sqrt[n]{7^i 5^{n-i}} = \lim_{n\to\infty}\frac{2}{\ln\frac{7}{5}}\cdot\frac{\frac{\ln\frac{7}{5}}{n}}{e^{\frac{\ln\frac{7}{5}}{n}} - 1} = \frac{2}{\ln 7 - \ln 5}$$

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\sqrt[n]{7^i 5^{n-i}} = \lim_{n\to\infty}\frac{5}{n}\sum_{i=0}^{n}\left(\frac{7}{5}\right)^{\frac{i}{n}} = 5\int_0^1\left(\frac{7}{5}\right)^x dx =$$

$$= 5\int_0^1 e^{x\ln\frac{7}{5}}dx = 5\left[\frac{e^{x\ln\frac{7}{5}}}{\ln\frac{7}{5}}\right]_0^1 = \frac{2}{\ln 7 - \ln 5}$$

Solution 3 by Rozeta Atanasova-Skopje

I. $\Omega = \lim_{n\to\infty}\frac{1}{n}5\sum_{i=0}^{n}\left(\frac{7}{5}\right)^{\frac{i}{n}} = 5\int_0^1\left(\frac{7}{5}\right)^x dx = \frac{2}{\ln\frac{7}{5}}$

II. $\Omega = \lim_{n\to\infty}\frac{1}{n}\cdot 5\sum_{i=0}^{n}\left(\frac{7}{5}\right)^{\frac{i}{n}} = \lim_{n\to\infty}\frac{5}{n}\cdot\frac{\left(\frac{7}{5}\right)^{\frac{1}{n}\cdot n} - 1}{\left(\frac{7}{5}\right)^{\frac{1}{n}} - 1}$

$$= 5\left(\frac{7}{5} - 1\right)\lim_{n\to\infty}\frac{\frac{1}{n}}{\left(\frac{7}{5}\right)^{\frac{1}{n}} - 1} = \frac{2}{\ln\frac{7}{5}}$$

Solution 4 by Shivam Sharma-New Delhi-India

$$\Rightarrow 5\lim_{n\to\infty}\sum_{i=0}^{n}\sqrt[n]{\left(\frac{7}{5}\right)^i}\Rightarrow 5\lim_{n\to\infty}\sum_{i=0}^{n}\left(\frac{7}{5}\right)^{\frac{i}{n}}$$

Applying Reimann sum or limit as a sum, we get, $\frac{i}{n}=x\Rightarrow 5\int_0^1\left(\frac{7}{5}\right)^x dx$

As we know, $\int a^x\,dx=\frac{a^x}{\ln(a)}+c\Rightarrow 5\left[\frac{\left(\frac{7}{5}\right)^x}{\ln\left(\frac{7}{5}\right)}\right]_0^1\Rightarrow 5\left[\frac{\left(\frac{7}{5}-1\right)}{\ln\left(\frac{7}{5}\right)}\right]$ ***(OR)*** $\Omega=\frac{2}{\ln\left(\frac{7}{5}\right)}$

(Answer)

178. $\Omega_n\in[1,\infty),n\geq 1,\lim_{n\to\infty}\Omega_n=\Omega\in\mathbb{R}$

Find:

$$\lim_{n\to\infty}e^{\sqrt[n]{(1+\ln\Omega_1)(1+\ln\Omega_2)\cdot\ldots\cdot(1+\ln\Omega_n)}-1}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Anas Adlany-El Zemamra-Morocco, Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 3 by Abdallah Almalih-Damascus-Syria

Solution 1 by Anas Adlany-El Zemamra-Morocco

Let $a_n=\sqrt[n]{\prod_{k=1}^{n}(1+\ln(\Omega_k))}$***, then***

$$\ln(a_n)=\frac{1}{n}\sum_{k=1}^{n}\ln[1+\ln(\Omega_k)]\to\lim(\ln[1+\ln(\Omega_n)])=\ln(1+\ln(\Omega))$$

Hence; $e^{a_n-1}\to\Omega$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$\lim_{n\to\infty}\Omega_n=\Omega\Leftrightarrow\lim_{n\to\infty}\ln\Omega_n=\ln\Omega$$

$$\lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}(1+\ln\Omega_k)}=\lim_{n\to\infty}e^{\frac{1}{n}\ln\prod_{k=1}^{n}(1+\ln\Omega_k)}=\lim_{n\to\infty}e^{\frac{1}{n}\sum_{k=1}^{n}\ln(1+\ln\Omega_k)}$$

$$\overset{Cesaro}{\cong} \lim_{n\to\infty} e^{\ln(1+\ln\Omega_k)} = 1 + \ln\Omega \Rightarrow \lim_{n\to\infty} e^{\sqrt[n]{\prod_{k=1}^{n}(1+\ln\Omega_k)}-1} = e^{1+\ln\Omega-1} = \Omega$$

Solution 3 by Abdallah Almalih-Damascus-Syria

$\Omega_n \in [1,\infty[, n \geq 1, \lim_{n\to\infty}\Omega_n = \Omega \in \mathbb{R}$ ***find*** $\lim_{n\to\infty} e^{\sqrt[n]{\prod_{k=1}^{n}(1+\ln\Omega k)}-1}$

Sol. Let $a_n = \prod_{k=1}^{n}(1+\ln(\Omega_k))$ ***so*** $a_{n+1} = [1+\ln(\Omega_{n+1})] - a_n$

We know $\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$

so $\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}[1+\ln(\Omega_{n+1})]$

$$\overset{from\ continuance\ of\ \ln\ function}{\cong} 1 + \ln(\Omega)$$

so $\lim_{n\to\infty} e^{\sqrt[n]{a_n}-1} \overset{from\ continuance\ of\ \exp\ function}{\cong} e^{1+\ln(\Omega)-1} = \Omega$

179. Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\int_0^{\infty}\frac{dx}{\left(x^2+\frac{1}{4}\right)^{n+1}}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Shivam Sharma-New Delhi-India, Solution 2 by Ravi Prakash-New Delhi-India, Solution 3 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 4 by Nirapada Pal-Jhargram-India, Solution 5 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Shivam Sharma-New Delhi-India

Let, $I = \int_0^{\infty}\frac{dx}{\left(x^2+\frac{1}{4}\right)^{n+1}}$. ***Let,*** $x = \frac{1}{2}\tan\theta$; $dx = \frac{1}{2}\sec^2\theta\, d\theta$

$$\Rightarrow \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\, d\theta}{\left(\frac{1}{4}\right)^{n+1} \sec^{2n+2}} \Rightarrow \frac{4^{n+1}}{2}\int_0^{\frac{\pi}{2}} \cos^{2n}(\theta)\, d\theta$$

Let, $A = \int_0^{\frac{\pi}{2}} \cos^{2n}(\theta)\, d\theta$*. Let,* $\cos\theta = u, \sin\theta\, d\theta = du$

$\Rightarrow \int_0^1 \frac{u^{2n}}{\sqrt{1-u^2}} du$*. Let,* $u^2 = y \Rightarrow u = (y)^{\frac{1}{2}}; du = \frac{1}{2} y^{\frac{1}{2}-1} dy$

$$\Rightarrow \frac{1}{2}\int_0^1 \frac{y^n}{\sqrt{1-y}} y^{\frac{1}{2}-1} dy$$

$$\Rightarrow \frac{1}{2}\int_0^1 y^{\frac{2n+1}{2}-1} (1-y)^{\frac{1}{2}-1} dy \Rightarrow \frac{1}{2} B\left(\frac{2n+1}{2}, \frac{1}{2}\right)$$

$$\Rightarrow \frac{1}{2}\left[\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n+1)}\right] \Rightarrow \frac{1}{2}\left[\frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{n!}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{\sqrt{\pi}(2n-1)!!\sqrt{\pi}}{n!}\right] \Rightarrow \frac{\pi}{2}\left[\frac{(2n-1)!!}{n!}\right] \Rightarrow \frac{\pi}{2}\left[\frac{(2n)!}{2^n (n!)^2}\right]$$

(OR) $A = \frac{\pi}{2^{n+1}}\left[\frac{(2n)!}{(n!)^2}\right], \forall n > 0$ *(OR)* $I = \frac{4^{n+1}}{2}\left[\frac{\pi}{2^{n+1}}\left(\frac{(2n)!}{(n!)^2}\right)\right]$

(OR) $I = 2\left[\pi\left(\frac{(2n)!}{(n!)^2}\right)\right]$*. Now,* $\sqrt[n]{I} = 2^{\frac{1}{n}} \pi^{\frac{1}{n}} \left(\frac{(2n)!}{(n!)^2}\right)^{\frac{1}{n}}$

As, we know, the Stirling's formula, we get, $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$*, so,*

$$\Omega = \lim_{n\to\infty} (I)^{\frac{1}{n}} \Rightarrow \lim_{n\to\infty} 2^{\frac{1}{n}} \pi^{\frac{1}{n}} \left[\frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{2\pi n}}{\left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^2}\right]^{\frac{1}{n}}$$

Now, applying Ratio Test, we get,

And many terms are cancelling, we get, our limit $\Omega = 4$ (Answer)

Solution 2 by Ravi Prakash-New Delhi-India

Let $I_n = \int_0^\infty \frac{dx}{\left(x^2+\frac{1}{4}\right)^n}$. **Put** $x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\, d\theta$

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{1}{2}\right)\sec^2\theta\, d\theta}{\left(\frac{1}{4}\right)^{n+1}\sec^{2n+2}\theta} = 2^{2n+1}\int_0^{\frac{\pi}{2}}\cos^{2n}\theta\, d\theta$$

$$= 2^{2n+1}\cdot\frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\ldots\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 2^{2n+1}\cdot\frac{(2n)!}{\left(2^n(n!)\right)^2}\cdot\frac{\pi}{2} = \frac{(2n)!\,\pi}{n!\,n!}$$

$$(I_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{I_{n+1}}{I_n} = \lim_{n\to\infty}\frac{(2n+2)!\,\pi}{(n+1)!\,(n+1)!}\cdot\frac{n!\,n!}{(2n)!\,\pi}$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)} = 4$$

Solution 3 by Abdelhak Maoukouf-Casablanca-Morocco

$$I_n = \int_0^\infty \frac{dx}{\left(x^2+\frac{1}{4}\right)^n} = \int_0^\infty \frac{x^2dx}{\left(x^2+\frac{1}{4}\right)^{n+1}} + \int_0^\infty \frac{\frac{1}{4}dx}{\left(x^2+\frac{1}{4}\right)^{n+1}}$$

$$= \left[-\frac{x}{2n\left(x^2+\frac{1}{4}\right)^n}\right] + \int_0^\infty \frac{\frac{1}{4}dx}{2n\left(x^2+\frac{1}{4}\right)^n} + I_{n+1} = \frac{1}{2n}I_n + \frac{I_{n+1}}{4}$$

$$\to \frac{I_{n+1}}{I_n} = 2\cdot\frac{2n-1}{n} \Rightarrow \prod_{k=1}^n\frac{I_{k+1}}{I_k} = \prod_{k=1}^n\frac{2k-1}{k}\times\frac{2k}{k} \Leftrightarrow$$

$$\Leftrightarrow \frac{I_{n+1}}{I_1} = \frac{(2n)!}{(n!)^2}$$

$$I_1 = \int_0^\infty \frac{dx}{\left(x^2+\frac{1}{4}\right)} = [2a\tan 2x]_0^\infty = \pi \to I_{n+1} = \pi\frac{(2n)!}{(n!)^2}$$

$$\Omega=\lim_{n\to\infty}\sqrt[n]{I_{n+1}}=\lim_{n\to\infty}\sqrt[n]{\prod\frac{(2n)!}{2^{2n}(n!)^2}}=\lim_{n\to\infty}\frac{\pi\frac{(2n+2)!}{\left((n+1)!\right)^2}}{\pi\frac{(2n)!}{(n!)^2}}$$

$$=\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2}=4$$

Solution 4 by Nirapada Pal-Jhargram-India

Let $\int_0^\infty\frac{dx}{\left(x^2+\frac{1}{4}\right)^{n+1}}=x_n$

So we have to find $\lim_{n\to\infty}x_n^{\frac{1}{n}}$

But $\lim_{n\to\infty}x_n^{\frac{1}{n}}=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}$ ***(when the limits exists)***

Now, $x_n=\int_0^\infty\frac{dx}{\left(x^2+\frac{1}{4}\right)^{n+1}}=2^{2n}B\left(n,\frac{1}{2}\right)$ ***after putting*** $x=\frac{1}{2}\tan\theta$

So $\Omega=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{2^{2n+2}B\left(n+\frac{3}{2}\right)}{2^{2n}B\left(n,\frac{1}{2}\right)}=\lim_{n\to\infty}2^2\frac{n}{n+\frac{1}{2}}=2^2=4$

Solution 5 by Soumitra Mandal-Chandar Nagore-India

$\Omega=\lim_{n\to\infty}\sqrt[n]{\frac{dx}{\left(x^2+\frac{1}{4}\right)^{n+1}}}$***. Let*** $x=\frac{\tan\theta}{2}\Rightarrow dx=\frac{\sec^2\theta}{2}d\theta$***. When*** $x=0,\theta=0$***;***

when $x\to\infty,\theta=\frac{\pi}{2}$***.*** $\Omega=\lim_{n\to\infty}\sqrt[n]{\int_0^\infty\frac{dx}{\left(x^2+\frac{1}{4}\right)^{n+1}}}=\lim_{n\to\infty}\sqrt[n]{\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sec^2\theta d\theta}{\left(\frac{\sec^2\theta}{4}\right)^{n+1}}}=$

$$=\lim_{n\to\infty}\sqrt[n]{2^{2n+1}\int_0^{\frac{\pi}{2}}\cos^{2n}\theta\,d\theta}$$

$$= \lim_{n\to\infty} \sqrt[n]{2^{2n+1} \cdot \frac{\beta\left(n, \frac{1}{2}\right)}{2}} = \lim_{n\to\infty} \sqrt[n]{2^{2n} \cdot \frac{\Gamma(n)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}} =$$

$$= \lim_{n\to\infty} \sqrt{2^{2n} \cdot \frac{(n-1)!\,\Gamma\left(\frac{1}{2}\right)}{\frac{(2n-1)!}{2^{2n-1}(n-1)!}\sqrt{\pi}}}$$

$$= \lim_{n\to\infty} \sqrt[n]{2^{4n-1} \cdot \frac{\{(n-1)!\}^2}{(2n-1)!}} = \lim_{n\to\infty} \sqrt[n]{2^{4n-1} \cdot \frac{\{(n-1)!\}^2}{(2n-1)(2n-2)\dots 3\cdot 2\cdot 1}}$$

$$= \lim_{n\to\infty} \sqrt[n]{2^{4n-1} \cdot \frac{(n-1)!}{2^{n-1}(2n-1)(2n-3)\cdot\dots\cdot 3\cdot 1}} =$$

$$= \lim_{n\to\infty} \sqrt[n]{2^{3n-1} \cdot \frac{(n-1)(n-2)\cdot\dots\cdot 3\cdot 2\cdot 1}{(2n-1)(2n-3)\cdot\dots\cdot 3\cdot 1}}$$

$$= \lim_{n\to\infty} \sqrt[n]{2^{3n-1} \cdot \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\dots\cdot\left(1-\frac{n-3}{n}\right)\left(1-\frac{n-2}{n}\right)\left(1-\frac{n-1}{n}\right)}{\left(2-\frac{1}{n}\right)\left(2-\frac{3}{n}\right)\dots\left(2-\frac{n-2}{n}\right)\left(2-\frac{n-1}{n}\right)}}$$

$$= \lim_{n\to\infty} 2^{2-\frac{1}{n}} = 4$$ ***(Ans :)***

180. Find:

$$\Omega = \lim_{n\to\infty} n \sqrt[n]{\frac{\left((2n)!!\right)^2}{(2n)!}}$$

Proposed by Daniel Sitaru – Romania

Solution by Abdelhak Maoukouf-Casablanca-Morocco

$$\Omega = \lim_{n\to\infty} n\sqrt[n]{\frac{((2n)!!)^2}{(2n)!}} = \lim_{n\to\infty} n\sqrt[n]{\frac{(2^n(n!))^2}{(2n)!}}$$

$$= \lim_{n\to\infty} n\sqrt[n]{\frac{\left((2^n\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)^2}{\left(\left(\frac{2n}{e}\right)^{2n}\sqrt{4\pi n}\right)}} = \lim_{n\to\infty} n\sqrt[n]{\sqrt{\pi n}} \to \infty$$

181. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\int_0^1 x^{2n}\sqrt{(1-x^2)^{2n+1}}\,dx}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India, Solution 2 by Ravi Prakash-New Delhi-India, Solution 3 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 4 by Geanina Tudose-Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$***, when*** $x = 0, \theta = 0; x = 1, \theta = \frac{\pi}{2}$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\int_0^1 x^{2n}\sqrt{(1-x^2)^{2n+1}}\,dx} = \lim_{n\to\infty} \sqrt[n]{\int_0^{\frac{\pi}{2}} \sin^{2n}\theta\cos^{2n+2}\theta\, d\theta}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{1}{2}\beta(n, n+1)} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{2}\cdot\frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)}} \overset{CAUCHY-D'ALEMBERT}{=}$$

$$= \lim_{n\to\infty}\left(\frac{\Gamma(n+1)\Gamma(n+2)}{\Gamma(2n+3)}\cdot\frac{\Gamma(2n+1)}{\Gamma(n)\Gamma(n+1)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n(n+1)}{(2n+1)(2n+2)}\right) = \frac{1}{4} \quad \textit{(Ans:)}$$

Solution 2 by Ravi Prakash-New Delhi-India

Let $a_n = \int_0^1 x^{2n}(1-x^2)^{\frac{(2n+1)}{2}}dx$. ***Put*** $x^2 = t$, ***so that***

$$a_n = \frac{1}{2}\int_0^1 t^{\frac{2n-1}{2}}(1-t)^{\frac{(2n+1)}{2}}dt$$

$$= \frac{1}{2}\int_0^1 t^{n+\frac{1}{2}-1}(1-t)^{n+\frac{3}{2}-1}dt = \frac{1}{2}\beta\left(n+\frac{1}{2}, n+\frac{3}{2}\right)$$

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\beta\left(n+\frac{3}{2}, n+\frac{5}{2}\right)}{\beta\left(n+\frac{1}{2}, n+\frac{3}{2}\right)}$$

$$= \lim_{n\to\infty}\frac{\sqrt{n+\frac{3}{2}}\sqrt{n+\frac{5}{2}}}{\sqrt{2n+4}}\cdot\frac{\sqrt{2n+2}}{\sqrt{n+\frac{1}{2}}\sqrt{n+\frac{3}{2}}}$$

$$= \lim_{n\to\infty}\frac{\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right)}{(2n+3)(2n+2)} = \frac{1}{4}\lim_{n\to\infty}\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{1}{n}\right)} = \frac{1}{4}$$

Solution 3 by Abdelhak Maoukouf-Casablanca-Morocco

$$I_n=\int_0^1 x^{2n}\sqrt{(1-x^2)^{2n+1}}\overset{x=\sin t}{=}\int_0^{\frac{\pi}{2}}\sin^{2n}t\sqrt{(1-\sin^2 t)^{2n+1}}\cos t\,dt$$

$$=\int_0^{\frac{\pi}{2}}\sin^{2n}t\times\cos^{2n+2}t\,dt=\int_0^{\frac{\pi}{2}}\sin^{2n}t\times\cos^{2n}t\,(1-\sin^2 t)dt$$

$$=\int_0^{\frac{\pi}{2}}\sin^{2n}t\times\cos^{2n}t\,dt-\int_0^{\frac{\pi}{2}}\sin^{2n+2}t\times\cos^{2n}t\,dt$$

$$\overset{t=\frac{\pi}{2}-u}{=}\frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}}\sin^{2n}2t\,dt-\int_0^{\frac{\pi}{2}}\cos^{2n+2}u\times\sin^{2n}u\,du$$

$$\to 2I=\frac{1}{2^n}\int_0^{\frac{\pi}{2}}\sin^{2n}2t\,dt=\frac{1}{2^{n-1}}\int_0^{\frac{\pi}{2}}\sin^{2n}t\,dt=\frac{1}{2^{n-2}}\int_0^{\frac{\pi}{2}}\sin^{2n}t\,dt$$

$$\to I=\frac{1}{2^{n-1}}\int_0^{\frac{\pi}{2}}\sin^{2n}t\,dt=\frac{1}{2^{n-1}}\cdot\frac{\pi}{2}\cdot\frac{(2n)!}{2^{2n}(n!)^2}=\pi\cdot\frac{(2n)!}{2^{4n}(n!)^2}$$

$$\Omega=\lim_{n\to\infty}\sqrt[n]{I_n}=\lim_{n\to\infty}\sqrt[n]{\pi\cdot\frac{(2n)!}{2^{4n}(n!)^2}}=\lim_{n\to\infty}\frac{\pi\cdot\frac{(2n+2)!}{2^{4n+4}\left((n+1)!\right)^2}}{\pi\cdot\frac{(2n)!}{2^{4n}(n!)^2}}$$

$$=\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{16(n+1)^2}=\frac{1}{4}$$

Solution 4 by Geanina Tudose-Romania

Consider $\int_0^1 x^{2n}\sqrt{(1-x^2)^{3n+1}}\,dx$. ***Let*** $x=\sin x$, $x=0\Rightarrow\sin 0=0$

$$dx = \cos\alpha \ dx \quad x = 1 \Rightarrow \alpha = \frac{\pi}{2}$$

$$= \int_0^{\frac{\pi}{2}} \sin^{2n} x \cdot \sqrt{(1-\sin^3\alpha)^{3n+1}} \cos\alpha \, dx = \int_0^{\frac{\pi}{2}} \sin^{2n}\alpha \cdot \cos^{2n+1}\alpha \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{2n}\alpha \cos^{2n+3}\alpha \, d\alpha = \int_0^{\frac{\pi}{2}} (\sin^2\alpha \cos^2\alpha)^n \cdot \cos^2\alpha \, d\alpha$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\sin^3 2\alpha}{2^3}\right)^n \cdot \frac{\cos 2\alpha + 1}{2} d\alpha$$

$$= \underbrace{\frac{1}{2^{2n+1}}, \int_0^{\frac{\pi}{2}} \sin^{2n} 2\alpha \cdot \cos 2\alpha \, d\alpha}_{I_1} + \underbrace{\frac{1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} \sin^{2n} 2\alpha \, d\alpha}_{I_2}$$

$$I_1 = \frac{1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} \sin^{2n} 2\alpha \cdot \cos 2\alpha \, d\alpha$$

$$= \frac{1}{2^{2n+1}} \cdot \frac{1}{(2n+1)\cdot 2} \sin^{2n+1} 2\alpha \Big|_0^{\frac{\pi}{2}} = 0$$

$$I_2 = \frac{1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} \sin^{2n} 2\alpha \, d\alpha = \frac{1}{2^{2n+2}} \int_0^{\pi} \sin^{2n} x \, dx; \ x = 2\alpha \Rightarrow d\alpha = 2d\alpha$$

$$= \frac{1}{2^{2n+2}} \left(\int_0^{\frac{\pi}{2}} \sin^{2n}\alpha \, d\alpha \underbrace{\int_{\frac{\pi}{2}}^{\pi} \sin^{2n} x \, dx}_{y = x - \frac{\pi}{2}} \right) = \frac{1}{2^{2n+2}} \left(\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx + \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx \right)$$

But $I_{2n} = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}$ ***. Hence*** $I = I_2 = \frac{1}{2^{n+\alpha}} \ 8 \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \pi$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{1}{2^{2n+2}} \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \pi}.$$ ***Using Cauchy D'Alembert*** $\lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!}{(2n)!!}} =$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+1)!!}{(2n+2)!!} \cdot \frac{(2n)!!}{(2n-1)!!} = \lim_{n\to\infty} \frac{2n+1}{2n+2} = 1$$

Thus $\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{1}{2^{2n+2}}} \cdot \sqrt[n]{\frac{(2n-1)!!}{(2n)!!}} \cdot \pi^{\frac{1}{n}} = \frac{1}{4}$

182. Find:

$$\Omega = \lim_{n\to\infty} \frac{13 \cdot 25 \cdot 37 \cdot \ldots \cdot (12n-11)}{7 \cdot 19 \cdot 31 \cdot \ldots \cdot (12n-5)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 2 by Shivam Sharma-New Delhi-India

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$U_n = \prod_{k=1}^{n} \left(\frac{12k-11}{12k-5}\right)$ ***we have*** $0 < U_n < 1$***; So we put*** $\lim U_n = l$

$$U_n = \frac{12n-11}{12-5} U_{n-1} \Leftrightarrow (12(n+1)-11)U_n - (12n-11)U_{n-1} = 6U_n$$

$$\Rightarrow \sum_{k=2}^{n} \left((12(k+1)-11)U_k - (12k-11)U_{k-1}\right) = \sum_{k=2}^{n} 6U_k$$

$$\Leftrightarrow (12(n+1)-11)U_n + 5U_1 = 6\sum_{k=1}^{n} U_k$$

$$\Leftrightarrow 12U_n + \frac{U_n + 5U_1}{n} = 6 \underbrace{\frac{1}{n}\sum_{k=1}^{n} U_k}_{Cesaro's\ Lemma}$$

$$\overset{n\to\infty}{\Longrightarrow} 12l + 0 = 6l \Leftrightarrow l = 0 \rightarrow \lim U_n = 0$$

Solution 2 by Shivam Sharma-New Delhi-India

$$\Omega=\lim_{n\to\infty}\left[\frac{13\cdot25\cdot37\cdot\ldots\cdot(12n-11)}{7\cdot9\cdot31\cdot\ldots\cdot(12n-5)}\right]\Rightarrow\lim_{n\to\infty}\left[\frac{\prod_{k=2}^{n}(12k-11)}{\prod_{k=1}^{n}(12k-5)}\right]$$

$$\Rightarrow\lim_{n\to\infty}\left[\frac{\dfrac{12^n\Gamma\left(n+\frac{1}{12}\right)}{\Gamma\left(\frac{1}{12}\right)}}{\dfrac{12^n\Gamma\left(n+\frac{7}{12}\right)}{\Gamma\left(\frac{7}{12}\right)}}\right]\Rightarrow\lim_{n\to\infty}\left[\frac{\Gamma\left(1+\frac{1}{12}\right)\Gamma\left(\frac{7}{n}\right)}{\Gamma\left(n+\frac{7}{12}\right)\Gamma\left(\frac{1}{12}\right)}\right]$$

Now, applying Ratio test, we get, $\Omega=0$

(Answer)

183. Find:

$$\Omega=\lim_{x\to0}\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\tan^2\frac{x}{2^k}\right)\right)$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution 1 by Daniel Sitaru-Romania, Solution 2 by Mehmet Sahin-Ankara-Turkey

Solution 1 by Daniel Sitaru-Romania

$$\Omega=\lim_{x\to0}\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\tan^2\frac{x}{2^k}\right)\right)=\lim_{x\to0}\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{\cos^2\frac{x}{2^k}-\sin^2\frac{x}{2^k}}{\cos^2\frac{x}{2^k}}\right)\right)$$

$$=\lim_{x\to0}\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{\cos\frac{x}{2^{k-1}}}{\cos^2\frac{x}{2^k}}\right)\right)=\lim_{x\to0}\left(\lim_{n\to\infty}\frac{\cos x}{\cos\frac{x}{2^n}}\cdot\frac{1}{\prod_{k=1}^{n}\cos\frac{x}{2^k}}\right)=$$

$$=\lim_{x\to0}\left(\lim_{n\to\infty}\frac{\cos x}{\cos\frac{x}{2^n}}\cdot\frac{1}{\prod_{k=1}^{n}\cos\frac{x}{2^k}}\right)=$$

$$= \lim_{x\to 0}\left(\lim_{n\to\infty}\frac{\cos x}{\cos\frac{x}{2^n}}\cdot\frac{2^n\sin\frac{x}{2^n}}{2^n\sin\frac{x}{2^n}\prod_{k=1}^{n}\cos\frac{x}{2^k}}\right)=$$

$$= \lim_{x\to 0}\left(\lim_{n\to\infty}\frac{\cos x}{\cos\frac{x}{2^n}}\cdot\frac{\sin\frac{x}{2^n}}{\frac{2}{2^n}}\cdot\frac{x}{\sin x}\right)=1$$

Solution 2 by Mehmet Sahin-Ankara-Turkey

$$1-\tan^2 x = 1-\frac{\sin^2 x}{\cos^2 x}=\frac{\cos^2 x-\sin^2 x}{\cos^2 x}=\frac{\cos 2x}{\cos^2 x}$$

$x \to \frac{x}{2},\frac{x}{2^2},\frac{x}{2^4},\dots,\frac{x}{2^n}$ ***write. In this case***

$$\left(1-\tan^2\frac{x}{2}\right)\left(1-\tan^2\frac{x}{4}\right)\left(1-\tan^2\frac{x}{8}\right)\dots\left(1-\tan^2\frac{x}{2^n}\right)$$

$$=\frac{\cos 2\cdot\frac{x}{2}}{\cos^2\frac{x}{2}}\cdot\frac{\cos 2\cdot\frac{x}{4}}{\cos^2\frac{x}{4}}\cdot\frac{\cos 2\cdot\frac{x}{8}}{\cos^2\frac{x}{8}}\dots\frac{\cos 2\cdot\frac{x}{2^n}}{\cos^2\frac{x}{2^n}}=$$

$$=\frac{\cos x\cdot\cos\frac{x}{2}\cdot\cos\frac{x}{4}\dots\cos\frac{x}{2^{n-1}}}{\cos^2\frac{x}{2}\cdot\cos^2\frac{x}{4}\cdot\cos^2\frac{x}{8}\dots\cos^2\frac{x}{2^n}}$$

$$=\frac{\cos x}{\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}\dots\cos\frac{x}{2^{n-1}}\cdot\cos^2\frac{x}{2^n}}$$

$$=\frac{\cos x\cdot\sin\frac{x}{2}\cdot\sin\frac{x}{4}\cdot\sin\frac{x}{8}\dots\sin\frac{x}{2^{n-1}}\cdot\sin\frac{x}{2^n}}{\left(\sin\frac{x}{2}\cdot\cos\frac{x}{2}\right)\cdot\left(\sin\frac{x}{4}\cdot\cos\frac{x}{4}\right)\cdot\left(\sin\frac{x}{8}\cdot\cos\frac{x}{8}\right)\dots\left(\sin\frac{x}{2^{n-1}}\cdot\cos\frac{x}{2^{n-1}}\right)\cdot\left(\sin\frac{x}{2^n}\cdot\cos\frac{x}{2^n}\right)\cdot\cos\frac{x}{2^n}}$$

$$=\frac{2^n\cdot\cos x\cdot\sin\frac{x}{2}\cdot\sin\frac{x}{4}\cdot\sin\frac{x}{8}\dots\sin\frac{x}{2^{n-1}}\cdot\sin\frac{x}{2^n}}{\sin x\cdot\sin\frac{x}{2}\cdot\sin\frac{x}{4}\cdot\sin\frac{x}{5}\dots\sin\frac{x}{2^{n-2}}\cdot\sin\frac{x}{2^{n-1}}\cdot\left(\cos\frac{x}{2^n}\right)}$$

$$=2^n\cdot\frac{\cos x}{\sin x}\cdot\frac{\sin\frac{x}{2^n}}{\cos\frac{x}{2^n}}=2^n\cdot\cot x\cdot\frac{\sin\frac{x}{2^n}}{\cos\frac{x}{2^n}}=\cot x\cdot\frac{\sin\frac{x}{2^n}}{\frac{x}{2^n}}\cdot x\cdot\frac{1}{\cos\frac{x}{2^n}}$$

$$= x\cdot\cot x\cdot\frac{\sin\frac{x}{2^n}}{\frac{x}{2^n}}\cdot\frac{1}{\cos\frac{x}{2^n}}$$

$$\lim_{n\to\infty}(x\cdot\cot x)\left(\frac{\sin\frac{x}{2^n}}{\frac{x}{2^n}}\cdot\frac{1}{\cos\frac{x}{2^n}}\right)=x\cdot\cot x\ \therefore$$

$$\lim_{x\to 0} x\cdot\cot x=\left(\lim_{x\to 0}\frac{x}{\sin x}\right)\cdot\left(\lim_{x\to 0}\cos x\right)=1.$$

as desired $\therefore$

184. **Solve over the set of real numbers the following system of equations written on base – 42 numeral system:**

$$a_a^2+a_3^2+a_4^2+\cdots+a_{2018}^2=4\cdot(97a_1-1)$$
$$a_1^2+a_3^2+a_4^2+\cdots+a_{2018}^2=4\cdot(97a_2-1)$$
$$a_1^2+a_2^2+a_4^2+\cdots+a_{2018}^2=4\cdot(97a_3-1)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_1^2+a_2^2+a_3^2+\cdots+a_{2017}^2=4\cdot(97a_{2018}-1)$$

Proposed by Koczinger Éva and Kovács Béla – Romania

Solution by Koczinger Éva and Kovács Béla – Romania

$$2017_{42}=2\cdot 42^3+42+7=148225=385^2=(9\cdot 42+7)^2=(97_{42})^2$$

Taking into account: $2017_{42}=(97_{42})^2$***, adding the equations;***

$$\sum_{k=1}^{2018}\left(2017a_k^2-4\cdot 97a_k+4\right)=0\Leftrightarrow\sum_{k=1}^{2018}\left(97^2\cdot a_k^2-4\cdot 97a_k+4\right)=0\Leftrightarrow$$

$$\Leftrightarrow\sum_{k=1}^{2018}(97a_k-2)^2=0\Leftrightarrow a_1=a_2=\cdots=a_{2018}=\frac{2}{97}$$

So, the solution to the equation system is: $\left(\frac{2}{97},\frac{2}{97},\frac{2}{97},\dots,\frac{2}{97}\right)$.

Or, rewriting the equations into decimal numeral system:

$$a_2^2 + a_3^2 + a_4^2 + \cdots + a_{148226}^2 = 4\cdot(385a_1 - 1)$$
$$a_1^2 + a_3^2 + a_4^2 + \cdots + a_{148226}^2 = 4\cdot(385a_2 - 1)$$
$$a_1^2 + a_2^2 + a_4^2 + \cdots + a_{148226}^2 = 4\cdot(385a_3 - 1)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\,..$$
$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{148225}^2 = 4\cdot(385a_{148226} - 1)$$

Adding then, we get:

$$\sum_{k=1}^{148226}\left(148225a_k^2 - 4\cdot 385a_k + 4\right) = 0 \Leftrightarrow \sum_{k=1}^{148226}\left(385^2\cdot a_k^2 - 4\cdot 385a_k + 4\right) = 0 \Leftrightarrow$$
$$\sum_{k=1}^{148226}(385a_k - 2)^2 = 0 \Leftrightarrow a_1 = a_2 = \cdots = a_{148226} = \frac{2}{385}$$

So, the solution to the equation system is: $\left(\frac{2}{385},\frac{2}{385},\frac{2}{385},\dots,\frac{2}{385}\right)$.

185. Solve for real numbers:

$$\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1 \\ x^2 + y^2 = \left(\dfrac{x^2}{5} + \dfrac{y^2}{4}\right)^2 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian, Solution 2 by Shahlar Maharramov-Jebrail-Azerbaidian, Solution 3 by Uche Eliezer Okeke-Anambra-Nigeria, Solution 4 by Boris Colakovic-Belgrade-Serbia, Solution 5 by Kunihiko Chikaya-Tokyo-Japan

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

Solve for real numbers: $\frac{x^2}{25}+\frac{y^2}{16}=1$

$$x^2+y^2=\left(\frac{x^2}{5}+\frac{y^2}{4}\right)^2 \Rightarrow x^2=a \quad y^2=b;\ 16a+25b=400$$

$$a+b=\frac{a^2}{25}+\frac{b^2}{16}+\frac{ab}{10}=\frac{16a^2+25b^2+40ab}{400}=\frac{(4a+5b)^2}{400}$$

$$400(a+b)=(4a+5b)^2;\ (16a+25b)(a+b)=(4a+5b)^2$$

$$16a^2+16ab+25ab+25b^2=16a^2+25b^2+40ab$$

$$ab=0\Rightarrow$$

$\Rightarrow a=0 \quad x=0\Rightarrow 25b=400 \quad b=16 \quad y=\pm 4$ ***answer*** $(0;4)$ ***and*** $(0;\,-4)$

$\Rightarrow b=0 \quad y=0\Rightarrow 16a=400 \quad a=25 \quad x=\pm 5$ ***answer*** $(5;0)$ ***and*** $(-5;-0)$

Solution 2 by Shahlar Maharramov-Jebrail-Azerbaidian

$x=5\sin t,\ y=4\cos t$***, put these in second equation*** $\Rightarrow$

$$\Rightarrow 25\sin^2 t+16\cos^2 t=(5\sin^2 t+4\cos^2 t)^2\Rightarrow$$

$$\Rightarrow 25\sin^2 t\underbrace{(1-\sin^2 t)}_{\cos^2 t}+16\cos^2 t\underbrace{(1-\cos^2 t)}_{\sin^2 t}=40\sin^2 t\cos^2 t$$

$$\Rightarrow \sin^2 t\cos^2 t=0\Rightarrow\frac{1}{2}\sin^2 2t=0\Rightarrow\sin 2t=0\Rightarrow 2t=\pi k\Rightarrow t=\frac{\pi}{2}k$$

$k=0,1,2,3$ ***after these it repeated***

1) $k=0,x=5\sin 0=0,y=4$

2) $k=1,x=5\sin\frac{\pi}{2}=5,y=0$

3) $k=2,x=0,y=-4$

4) $k=3,\ x=-5,y=0$

Solution 3 by Uche Eliezer Okeke-Anambra-Nigeria

$$\frac{x^2}{25}+\frac{y^2}{16}=1 \quad (1)$$

$$x^2+y^2=\left(\frac{x^2}{5}+\frac{y^2}{4}\right)^2 \quad (2)$$

Set $x=5\sin\theta$ **(3),** $y=\cos\theta$ **(4)**

Transformation of (2) gives

$$\Leftrightarrow (5\sin\theta)^2+(4\cos\theta)^2=\left(\frac{(5\sin\theta)^2}{5}+\frac{(4\cos\theta)^2}{4}\right)^2$$

$$\Leftrightarrow 25\sin^2\theta+16\cos^2\theta=(5\sin^2\theta+4\cos^2\theta)^2$$

$$\Leftrightarrow 25(1-\cos^2\theta)+16\cos^2\theta=(5(1-\cos^2\theta)+4\cos^2\theta)^2$$

$$\Leftrightarrow \cos^2\theta-\cos^2\theta=0\Leftrightarrow \left.\begin{matrix}\cos\theta=0\Rightarrow y=4\cos\theta=4(0)=0\\ \cos\theta=\pm1\Rightarrow y=\pm4\cos\theta=\pm4(1)=\pm4\end{matrix}\right\}$$

$$\Rightarrow \left.\begin{matrix}\cos^2\theta=0\Leftrightarrow \sin^2\theta=\sqrt{1-0^2}=0\Leftrightarrow \sin\theta=\pm1\Leftrightarrow x=\pm5\sin\theta=\pm5(1)=\pm5\\ \cos\theta=1\Leftrightarrow \cos\theta=\sqrt{1-1^2}=0\Leftrightarrow x=5\sin\theta=5(0)=0\end{matrix}\right\}$$

Solution $(x,y)=(\pm5,0)(0,\pm4)$

Solution 4 by Boris Colakovic-Belgrade-Serbia

$$\textbf{Substitutions } \frac{x}{5}=u;\frac{y}{4}=v\Rightarrow\Rightarrow\begin{cases}u^2+v^2=1 & (1)\\ 25u^2+16v^2=(5u^2+4v^2)^2 & (2)\end{cases}$$

From (2) $\Rightarrow 25(1-v^2)+16v^2=(5-v^2)^2\Leftrightarrow 25-9v^2=25-10v^2+v^4\Leftrightarrow$

$$\Leftrightarrow v^2(v^2-1)=0\Leftrightarrow v=0 \vee v=\pm1.$$

For $v=0\Rightarrow u=\pm1$ **For** $v=\pm1\Rightarrow u=0$**. Solutions are** $(\pm5,0);(0,\pm4)$

Solution 5 by Kunihiko Chikaya-Tokyo-Japan

$$\begin{cases}\dfrac{a}{25}+\dfrac{b}{16}=1 & (*)\\ a+b=\left(\dfrac{a}{5}+\dfrac{b}{4}\right)^2\end{cases}$$

$$a = x^2 \geq 0;\ b = y^2 \geq 0;\ \textit{Cauchy – Schwarz}$$

$$\left(\frac{\sqrt{a}}{5}\sqrt{a} + \frac{\sqrt{b}}{4}\sqrt{b}\right)^2 \leq \left\{\left(\frac{\sqrt{a}}{5}\right)^2 + \left(\frac{\sqrt{b}}{4}\right)^2\right\}\left\{\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2\right\}$$

$$\therefore \left(\frac{a}{5} + \frac{b}{4}\right)^2 \leq 1 \cdot (a + b).\ \textit{Equality:} \begin{pmatrix}\frac{\sqrt{a}}{5}\\ \frac{\sqrt{b}}{4}\end{pmatrix} = \begin{pmatrix}\sqrt{a}\\ \sqrt{b}\end{pmatrix}\ \&\ (*) \Leftrightarrow ab = 0\ \&\ (*) \Leftrightarrow$$

$$\textit{Ans}\ \begin{matrix}(x, y) = (\pm 5, 0), (0, \pm 4)\\ (a, b) = (25, 0), (0, 16)\end{matrix}$$

186. Find $A, B, C \in (0, \pi), A + B + C = \pi$ such that:

$$\begin{cases}\cos A\,|\cos B| + \cos B\,|\cos A| = 1 + \cos 2C\\ \cos B\,|\cos C| + \cos C\,|\cos B| = 1 + \cos 2A\\ \cos C\,|\cos A| + \cos A\,|\cos C| = 1 + \cos 2B\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Boris Colakovic-Belgrade-Serbia, Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian, Solution 3 by Ravi Prakash-New Delhi-India

Solution 1 by Boris Colakovic-Belgrade-Serbia

$$\begin{cases}\cos A\,|\cos B| + \cos B\,|\cos A| = 1 + \cos 2C \ldots (1)\\ \cos B\,|\cos C| + \cos C\,|\cos B| = 1 + \cos 2A \ldots (2)\\ \cos C\,|\cos A| + \cos A\,|\cos C| = 1 + \cos 2B \ldots (3)\end{cases}$$

From (1) $\Rightarrow \cos A\,|\cos B| + \cos B\,|\cos A| = 2\cos^2 C > 0 \Rightarrow \cos A > 0, \cos B > 0$

From (2) $\Rightarrow \cos B\,|\cos C| + \cos C\,|\cos B| = 2\cos^2 A > 0 \Rightarrow \cos B > 0, \cos C > 0$

From (3) $\Rightarrow \cos C\,|\cos A| + \cos A\,|\cos C| = 2\cos^2 B > 0 \Rightarrow \cos A > 0, \cos C > 0$

$$\Rightarrow A, B, C \in \left(0, \frac{\pi}{2}\right)$$

$$\begin{cases}\cos A \cos B = \cos^2 C \ldots (4)\\ \cos B \cos C = \cos^2 A \ldots (5)\\ \cos A \cos C = \cos^2 B \ldots (6)\end{cases}$$

$$(5) - (4) \Rightarrow \cos A\,(\cos A - \cos B) = \cos C\,(\cos B - \cos C) \ldots (7)$$

$$(6)-(5)\Rightarrow \cos B(\cos B-\cos C)=\cos A(\cos C-\cos A)\ldots(8)$$

$$(6)-(4)\Rightarrow \cos B(\cos B-\cos A)=\cos C(\cos A-\cos C)\ldots(9)$$

$$(7)+(8)\Rightarrow(\cos B-\cos C)(\cos B+\cos C)=$$

$$=\cos A(\cos C-\cos B)\Rightarrow(\cos B-\cos C)\underbrace{(\cos A+\cos B+\cos C)}_{>0}=0$$

$$\Rightarrow \cos B=\cos C\Rightarrow B=C$$

From (9) $\Rightarrow \cos C(\cos C-\cos A)=\cos C(\cos A-\cos C)\Rightarrow 2(\cos C-\cos A)\cos C=0\Rightarrow$

$\cos C=\cos A\Rightarrow A=C\quad \cos C\neq 0$ ***because*** $c\in\left(0,\frac{\pi}{2}\right)$

$$A=B=C=\frac{\pi}{3}$$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

If $\cos A=a,\ \cos B=b,\cos C=c$ ***then*** $a,b,c\neq 0$

$$\begin{cases} a|b|+b|a|=2c^2\\ b|c|+c|b|=2a^2\\ c|a|+a|c|=2b^2\end{cases}$$

1) $a,b,c>0\to ab>0,\ bc>0,\ ac>0$ ***must***

2) $a,b,c<0\ \underbrace{\Rightarrow}_{not\ true\ because}\ a,b,c\neq 0$

$$\begin{cases} ab=c^2\\ bc=a^2\Rightarrow\\ ac=b^2\end{cases}\quad \begin{matrix} 1)\ a=b=c\ \ true\\ 2)\ \begin{matrix} a>b>c\\ a^2>b^2\\ bc>ac\to b>0\ not\ true\end{matrix}\end{matrix}$$

answer $a=b=c,\cos A=\cos B=\cos C=\frac{1}{2};\ A=B=C=\frac{\pi}{3}$

Solution 3 by Ravi Prakash-New Delhi-India

$$\cos A|\cos B|+\cos B|\cos A|=1+\cos 2C\quad (1)$$

$$\cos B|\cos C|+\cos C|\cos B|=1+\cos 2A\quad (2)$$

$$\cos C|\cos A|+\cos A|\cos C|=1+\cos 2B\quad (3)$$

Assume C ***is obtuse, then*** $|\cos C|=-\cos C$***, (2) becomes*** $0=2\cos^2 A$

Not possible. If $C = \frac{\pi}{2}$*, then (2) become* $0 = 2\cos^2 A$*. Not possible.*

Thus, A, B, C *must be all acute angles. From (1)*

$$2\cos A\cos B = 2\cos^2 C \Rightarrow 2\cos^2 C = \cos(A+B) + \cos(A-B)$$

$$\Rightarrow 2\cos^2 C + \cos C \le 1 \Rightarrow (2\cos C - 1)(\cos C + 1) \le 0$$

$$\Rightarrow 0 < \cos C \le \frac{1}{2} \Rightarrow \frac{\pi}{3} \le C < \frac{\pi}{2}.$$ *Similarly, from (2), (3)*

$\frac{\pi}{3} \le A, B < \frac{\pi}{2}$*. As* $A + B + C = \pi_\phi$ *and* $\frac{\pi}{3} \le A, B, C < \frac{\pi}{2}$,

we get $A = B = C = \frac{\pi}{3}$

187. Solve for real numbers:

$$\begin{cases} 27^x + 2 = 3^{y+1} \\ 27^y + 2 = 3^{z+1} \\ 27^z + 2 = 3^{x+1} \end{cases}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaidian

Solution 1 by Boris Colakovic-Belgrade-Serbia, Solution 2 by Rustem Zeynalov-Baku-Azerbaidian

Solution 1 by Boris Colakovic-Belgrade-Serbia

$$\begin{cases} 3^{3x} + 2 = 3^{y+1} \\ 3^{3y} + 2 = 3^{z+1} \\ 3^{3z} + 2 = 3^{x+1} \end{cases}$$ *. Substitutions* $3^x = a; 3^y = b, 3^z = c;$ $$\begin{cases} a^3 + 2 = 3b \\ b^3 + 2 = 3c \\ c^3 + 2 = 3a \end{cases}$$

$$a^3 + 1 + 1 \overset{AM-GM}{\geq} 3\sqrt[3]{a^3} = 3a$$

$$b^3 + 1 + 1 \overset{AM-GM}{\geq} 3\sqrt[3]{b^3} = 3b$$

$$c^3 + 1 + 1 \overset{AM-GM}{\geq} 3\sqrt[3]{c^3} = 3a$$

Equality holds for $a = 1; b = 1; c = 1$

$$3^x = 1 \Rightarrow x = 0, 3^x = 1 \Rightarrow y = 0, 3^z = 1 \Rightarrow z = 0$$

Solution 2 by Rustem Zeynalov-Baku-Azerbaidian

$$\begin{cases} 27^x + 2 = 3^{y+1} \\ 27^y + 2 = 3^{z+1} \\ 27^z + 2 = 3^{x+1} \end{cases}\text{. Symetric } x = y = z; \; 27^x + 2 = 3^{x+1}; \; 3^x = a$$

$$a^3 + 2 = 3a; \; a^3 - 3a + 2 = 0; \; a^3 - 1 - 3a + 3 = 0$$

$$(a - 1)(a^2 + a + 1) - 3(a - 1) = 0; \; (a - 1)(a^2 + a - 2) = 0$$

$$a - 1 = 0, a^2 + a - 2 = 0; a_1 = 1; a_2 = -2; \; a_3 = 1$$

$$3^x = -2 \; \emptyset; \; 3^x = 1; \; x = 0; \; x = y = z = 0$$

188. Solve for real numbers:

$$[\tan x] \cdot (\cot x - [\cot x]) = (\tan x - [\tan x]) \cdot [\cot x]$$

$[*]$ – **great integer function**

Proposed by Rovsen Pirgulyev-Sumgait-Azerbaidian

Solution by Ravi Prakash-New Delhi-India

$$[\tan x](\cot x - [\cot x]) = (\tan x - [\tan x])[\cot x] \quad (1)$$

For $0 < x < \frac{\pi}{4}, 0 < \tan x < 1, [\tan x] = 0$ ***and*** $[\cot x] \geq 1$

Now (1) becomes $0 = (\tan x)[\cot x] \neq 0$

$\therefore$ ***(1) has no solution for*** $0 < x < \frac{\pi}{4}$.

For $x = \frac{\pi}{4}$, ***(1) becomes*** $1(1 - 1) = (1 - 1)(1)$ ***which is clearly holds.***

For $\frac{\pi}{4} < x < \frac{\pi}{2}, [\tan x] \geq 1$ ***and*** $[\cot x] = 0$. ***Now (1) becomes***

$[\tan x] \cot x = 0$. ***i.e.*** $0 = [\tan x] \cot x \neq 0$

$\therefore$ ***(1) has no solution for*** $\frac{\pi}{4} < x < \frac{\pi}{2}$. ***Next, let*** $-\frac{\pi}{4} < x < 0$,

$[\tan x] = -1, [\cot x] \leq -2$. ***Write (1) as***

$$(-1)(\cot x - [\cot x]) = (\tan x + 1)[\cot x] \Rightarrow -\cot x = (\tan x)[\cot x] \quad (2)$$

Let $[\cot x] = k$, then $k \le -2$ and $\cot x \le k$ LHS of (2) $\ge -k$

and RHS of (2) $< -k$. Thus, (1) has not solution for $-\frac{\pi}{4} < x < 0$.

For $x = -\frac{\pi}{4}$, (1) is satisfied.

Similarly, (1) has no solution for $-\frac{\pi}{2} < x < -\frac{\pi}{4}$

As $\tan x$ and $\cot x$ are periodic with period π, we get solution set to be $(2k+1)\frac{\pi}{4}$ where k is an integer.

189. Prove that:

$$\sin^2\frac{7\pi}{18}\cdot\sin\frac{5\pi}{18}-\sin^2\frac{\pi}{18}\cdot\sin\frac{7\pi}{18}+\sin^2\frac{5\pi}{18}\cdot\sin\frac{\pi}{18}=\frac{3}{4}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Evaluar la siguiente expresión

$$E=\sin^2\frac{7\pi}{18}\cdot\sin\frac{5\pi}{18}-\sin^2\frac{\pi}{18}\cdot\sin\frac{7\pi}{18}+\sin^2\frac{5\pi}{18}\cdot\sin\frac{\pi}{18}$$

Tener en cuenta las siguientes identidades

$$2\sin^2 x = 1-\cos 2x,\ \sin x+\sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),$$

$$\sin x-\sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\sin 10^\circ+\sin 50^\circ = 2\sin 30^\circ\cos 20^\circ = \sin 70^\circ \Leftrightarrow \sin 10^\circ+\sin 50^\circ-\sin 70^\circ = 0$$

Lo pedido es equivalente

$$4E = 4\sin^2 70^\circ\cdot\sin 50^\circ-4\sin^2 10^\circ\cdot\sin 70^\circ+4\sin^2 50^\circ\cdot\sin 10^\circ$$

$$4E = 2(1-\cos 140^\circ)\sin 50^\circ-2(1-\cos 20^\circ)\sin 70^\circ+2(1-\cos 100^\circ)\sin 10^\circ$$

$$4E = 2(1+\cos 40^\circ)\sin 50^\circ-2(1-\cos 20^\circ)\sin 70^\circ+2(1+\cos 80^\circ)\sin 10^\circ$$

$$E = 2(\sin 10^\circ+\sin 50^\circ-\sin 70^\circ)+2\sin 50^\circ\cos 40^\circ+2\sin 70^\circ\cos 20^\circ+2\sin 10^\circ\cos 80^\circ$$

$$4E = 2(\sin 10^\circ + \sin 50^\circ - \sin 70^\circ) + (\sin 90^\circ + \sin 10^\circ) + (\sin 90^\circ + \sin 50^\circ) +$$
$$+(\sin 90^\circ - \sin 70^\circ)$$
$$4E = 3(\sin 10^\circ + \sin 50^\circ - \sin 70^\circ) + 3\sin 90^\circ$$
$$4E = 3\sin 90^\circ = 3 \Leftrightarrow E = \frac{3}{4}$$

190. $\sum_{k=1}^{\infty}\left(\frac{1}{(4n-2)^2-1}\right) > 2\sum_{k=1}^{\infty}\left(\frac{1}{(4n-1)^2-1}\right)$.

Proposed by Uche Eliezer Okeke-Anambra-Nigeria

Solution by Ravi Prakash-New Delhi-India

$$S_1 = \sum_{k=1}^{\infty}\left(\frac{1}{(4n-2)^2-1}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{4n-3} - \frac{1}{4n-1}\right]$$
$$= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) = \frac{1}{2}\int_0^1 (1 - x^2 + x^4 - x^6 + \cdots) = \frac{1}{2}\int_0^1 \frac{dx}{1+x^2}$$
$$S_2 = 2\sum_{n=1}^{\infty}\frac{1}{(4n-1)^2-1} = \sum_{n=1}^{\infty}\left(\frac{1}{4n-2} - \frac{1}{4n}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$
$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) = \frac{1}{2}\int_0^1 (1 - x + x^2 - x^3 + \cdots)\,dx = \frac{1}{2}\int_0^1 \frac{dx}{1+x}$$
$$\therefore S_1 - S_2 = \frac{1}{2}\int_0^1 \left(\frac{1}{1+x^2} - \frac{1}{1+x}\right)dx = \frac{1}{2}\int_0^1 \frac{x(1-x)}{(1+x^2)(1+x)}dx > 0$$
$$\Rightarrow S_1 > S_2$$

191. Find:

$$\Omega = \int_a^b \tan(\arccos(\sin(\arctan x)))\,dx,\ 0 < a < b < \frac{\pi}{2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 2 by Anas Adlany-El Zemamra-Morocco, Solution 3 by Rozeta Atanasova-Skopje

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$$\Omega = \int_a^b \tan(\arccos(\sin(\arctan(x))))\,dx$$

$$= \int_a^b \tan\left(\frac{\pi}{2} - \arcsin(\sin(\arctan(x)))\right)dx \because \arctan x \in \left]0; \frac{\pi}{2}\right[$$

$$= \int_a^b \tan\left(\frac{\pi}{2} - \arctan(x)\right)dx = \int_a^b \tan\left(\arctan\left(\frac{1}{x}\right)\right)dx = \int_a^b \frac{1}{x}dx = \ln\left(\frac{b}{a}\right)$$

Solution 2 by Anas Adlany-El Zemamra-Morocco

We prove that $\tan(\arccos(\sin(\arctan(x)))) = \frac{1}{x}$.

First, let $t = \arctan(x)$ ***then***

$$\tan(\arccos(\sin(t))) = \tan\left(\arccos\left(\cos\left(\frac{\pi}{2} - t\right)\right)\right) = \tan\left(\frac{\pi}{2} - t\right) = \frac{1}{\tan(t)} = \frac{1}{x}$$

Hence $\Omega = \int_a^b \tan(\arccos(\sin(\arctan(x))))\,dx = \int_a^b \frac{dx}{x} = \ln\left(\frac{b}{a}\right)$

Solution 3 by Rozeta Atanasova-Skopje

$$\arccos(\sin x) = \frac{\pi}{2} - x \Rightarrow$$

$$\Omega = \int_a^b \tan(\arccos(\sin(\arctan x)))\,dx = \int_a^b \tan\left(\frac{\pi}{2} - \arctan x\right)dx =$$

$$= \int_a^b \cot(\arctan x)\,dx = \int_a^b \frac{dx}{x} = \ln\frac{b}{a}$$

192. If $m, n, p \geq 2$

$$\Omega(n) = 4^{n-1}\int_{\frac{1}{2}}^{\frac{3}{2}}\left(1+\frac{1}{\sqrt{2-x}}\right)\left(1+\frac{1}{\sqrt[n]{x}}\right)dx$$

then:

$$\Omega(n)\Omega(m)\Omega(p) \geq 64\sqrt[3]{mnp}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$\text{Let } I_n = \int_{\frac{1}{2}}^{\frac{3}{2}}\left(1+\frac{1}{x^{\frac{1}{n}}}\right)\left(1+\frac{1}{(2-x)^{\frac{1}{n}}}\right)dx = \int_{\frac{1}{2}}^{\frac{3}{2}}\left[1+\frac{1}{x^{\frac{1}{n}}}+\frac{1}{(2-x)^{\frac{1}{n}}}+\frac{1}{[x(2-x)]^{\frac{1}{n}}}\right]dx$$

$$\text{Also } \frac{1}{2^{\frac{1}{n}}}+\frac{1}{(2-x)^{\frac{1}{n}}} \geq \frac{2}{[x(2-x)]^{\frac{1}{2n}}} \text{ and } x(2-x) = 1-(1-x)^2 \leq 1$$

$$\therefore \frac{1}{x^{\frac{1}{n}}}+\frac{1}{(2-x)^{\frac{1}{n}}} \geq 2 \text{ and } \frac{1}{[x(2-x)]^{\frac{1}{n}}} \geq 1. \text{ Thus,}$$

$$I_n \geq \int_{\frac{1}{2}}^{\frac{3}{2}} u\,dx = 4 \Rightarrow \Omega(n) \geq 4^{n-1}u = 4^n$$

$$\text{Now, } \Omega(n)\Omega(m)\Omega(p) \geq 4^{m+n+p} \geq 4^{3(mnp)^{\frac{1}{3}}} = 64^{(mnp)^{\frac{1}{3}}}$$

193. If $a \in \mathbb{R}, f:[a, a+2] \to \mathbb{R}, f \in C^2([a, a+2]), 6 \leq f''(x) \leq 12$, then:

$$1+f(a+1) \leq \frac{1}{2}\int_a^{a+2} f(x)dx \leq 2+f(a+1)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

Let $\varphi(x) = f(x) - 3x^2$ ***for all*** $x \in [a, a+2]$, $\varphi''(x) = f''(x) - 6 \geq 0$

hence φ ***is convex. By Hermite Hadamard inequality***

$$\varphi(a+1) \leq \frac{1}{2}\int_a^{a+2} \varphi(x)\,dx \Rightarrow f(a+1) - 3(a+1)^2 \leq \frac{1}{2}\int_a^{a+2} f(x)\,dx - \frac{3}{2}\int_a^{a+2} x^2\,dx$$

$\Rightarrow 1 + f(a+1) \leq \frac{1}{2}\int_a^{a+2} f(x)\,dx$***. Let*** $G(x) = 6x^2 - f(x)$ ***for all***

$x \in [a, a+2]$. $G''(x) = 12 - f''(x) \geq 0$ ***for all*** $x \in [a, a+2]$ ***hence*** G ***is***

convex. By Hermite Hadamard inequality

$$\frac{1}{2}\int_a^{a+2} G(x)\,dx \geq G(a+1) \Rightarrow \frac{6}{2}\int_a^{a+2} x^2 dx - \frac{1}{2}\int_a^{a+2} f(x)\,dx \geq 6(a+1)^2 - f(a+1)$$

$$\Rightarrow 2 + f(a+1) \geq \frac{1}{2}\int_a^{a+2} f(x)\,dx \therefore 2 + f(a+1) \geq \frac{1}{2}\int_a^{a+2} f(x)\,dx \geq 1 + f(a+1)$$

194. If $a, b > 0$ then:

$$2\int_0^{\sqrt{ab}} e^x \ln(x+1)\,dx \leq \int_0^a e^x \ln(x+1)\,dx + \int_0^b e^x \ln(x+1)\,dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 2 by Anunoy Chakraborty-India, Solution 3 by Geanina Tudose-Romania, Solution 4 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$\forall x > 0$: $f(x) = e^x \ln(x+1)$ & $F(x)$ ***his primitive function.***

So, $F''(x) = f'(x) = \left(\frac{1}{x+1} + \ln(x+1)\right)e^x > 0$.

So, F ***is a convex function*** $\Rightarrow F\left(\frac{a+b}{2}\right) \leq \frac{F(a)+F(b)}{2}$

$$I=2\int_0^{\sqrt{ab}} f(x)\,dx \overset{AM-GM}{\leq} \int_0^{\frac{a+b}{2}} f(x)\,dx \quad \therefore f(x)\geq 0$$

$$=\int_0^a f(x)\,dx+\int_a^{\frac{a+b}{2}} f(x)\,dx+\int_0^b f(x)\,dx+\int_b^{\frac{a+b}{2}} f(x)\,dx$$

$$=\int_0^a f(x)\,dx+\int_0^b f(x)\,dx+2F\left(\frac{a+b}{2}\right)-F(a)-F(b)$$

$$\leq\int_0^a f(x)\,dx+\int_0^b f(x)\,dx$$

$$\rightarrow 2\int_0^{\sqrt{ab}} e^x\ln(x+1)\leq\int_0^a e^x\ln(x+1)\,dx+\int_0^b e^x\ln(x+1)\,dx$$

Solution 2 by Anunoy Chakraborty-India

Let $f(n)=\int_0^n e^n(1+\ln x)dn\,;f'(n)=e^n(1+\ln n)$

$$f''(n)=\frac{e^n}{n}+e^n(1+\ln n)>0; f''(n)>0\,;\forall\, n>0$$

By Jensen's inequality, $f(a)+f(b)\geq 2f\left(\frac{a+b}{2}\right)$

$$\therefore\int_0^a e^n\,(1+\ln n)dn+\int_0^b e^n(1+\ln n)dn\geq 2\int_0^{\frac{(a+b)}{2}} e^n(1+\ln n)\,dn$$

$2\int_0^{\frac{a+b}{2}} e^n(1+\ln n)dn\geq 2\int_0^{\sqrt{ab}} e^n(1+\ln n)\,dn$ ***[By AM-GM Inequality]***

$$\therefore \int_0^a e^n(1+\ln n)\,dn + \int_0^b e^n(1+\ln n)\,dn \geq 2\int_0^{\sqrt{ab}} e^n(1+\ln n)\,dn$$

Solution 3 by Geanina Tudose-Romania

Let $f:[0,+\infty)\to\mathbb{R}, f(x) = e^x\ln(x+1)$ ***continuous and let*** F ***be an antiderivative. Since*** $f(x)\geq 0, (\forall)\ x\in[0,+\infty) \Rightarrow F$ ***is strictly increasing***

$$f'(x) = e^x\ln(x+1) + \frac{e^x}{x+1} = e^x\left(\ln(x+1)+\frac{1}{x+1}\right) > 0 \Rightarrow$$

F ***a convexe function***

$$2\int_0^{\sqrt{ab}} e^x\ln(x+1)\,dx \leq \int_0^a e^x\ln(x+1)\,dx + \int_0^b e^x\ln(x+1)\,dx$$

$$\Leftrightarrow 2F(\sqrt{ab}) - 2F(0) \leq F(a) - F(0) + F(b) - F(0) \Leftrightarrow F(\sqrt{ab}) \leq \frac{F(a)+F(b)}{2}$$

$$\left.\begin{array}{c} Since\ \sqrt{ab}\leq \dfrac{a+b}{2} \\ F\ increasing \end{array}\right\} \Rightarrow F(\sqrt{ab}) \leq F\left(\frac{a+b}{2}\right) \overset{Jensen}{\leq} \frac{F(a)+F(b)}{2}$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

Let $f(t) = \int_0^t e^x\ln(x+1)\,dx$ ***for all*** $t\geq 0$

$f''(t) = e^t\left(\ln(t+1)+\frac{1}{t+1}\right) > 0$ ***for all*** $t\geq 0$***. Hence*** f ***is convex***

$\therefore$ ***for*** $a,b\geq 0, f(a)+f(b)\geq 2f\left(\frac{a+b}{2}\right)$

$$\int_0^a e^x\ln(1+x)\,dx + \int_0^b e^x\ln(1+x)\,dx \geq 2\int_0^{\frac{a+b}{2}} e^x\ln(1+x)\,dx \geq$$

$$\geq 2\int_0^{\sqrt{ab}} e^x\ln(1+x)\,dx$$

195. Find:

$$\Omega = \lim_{x\to 0}\frac{\int_0^x \frac{\sqrt[3]{1+2x}\cdot\sqrt[5]{1+3x}-1}{x}dx}{\int_0^x \frac{\sqrt[3]{1+3x}\cdot\sqrt[5]{1+2x}-1}{x}dx}$$

Proposed by Daniel Sitaru – Romania

Solutions 1,2 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 3 by Serban George Florin-Romania, Solution 4 by Lazaros Zachariadis-Thessaloniki-Greece

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$$f(t) = \frac{\sqrt[3]{1+2t}\cdot\sqrt[5]{1+3t}-1}{t} = \frac{\sqrt[3]{1+2t}\cdot\sqrt[5]{1+3t}-\sqrt[5]{1+3t}+\sqrt[5]{1+3t}-1}{t} =$$

$$= \frac{\sqrt[5]{1+3t}\left(\sqrt[3]{1+2t}-1\right)+\left(\sqrt[5]{1+3t}-1\right)}{t} = \frac{\sqrt[5]{1+3t}\frac{(1+2t-1)}{\sum_{k=0}^{2}\left(\sqrt[3]{1+2t}\right)^k}+\frac{(1+3t-1)}{\sum_{k=0}^{4}\left(\sqrt[5]{1+3t}\right)^k}}{t} =$$

$$= \frac{2\sqrt[5]{1+3t}}{\sum_{k=0}^{2}\left(\sqrt[3]{1+2t}\right)^k} + \frac{3}{\sum_{k=0}^{4}\left(\sqrt[5]{1+3t}\right)^k}$$

$$g(t) = \frac{\sqrt[3]{1+3t}\cdot\sqrt[5]{1+2t}-1}{t} = \frac{3\sqrt[5]{1+2t}}{\sum_{k=0}^{2}\left(\sqrt[3]{1+3t}\right)^k} + \frac{2}{\sum_{k=0}^{4}\left(\sqrt[5]{1+2t}\right)^k} \because \textit{By the same way}$$

$$\Omega = \lim_{x\to 0}\frac{\int_0^x f(t)dt}{g(t)dt} \overset{Hospital}{=} \lim_{x\to 0}\frac{\int_0^x\left(\frac{2\sqrt[5]{1+3t}}{\sum_{k=0}^{2}\left(\sqrt[3]{1+2t}\right)^k}+\frac{3}{\sum_{k=0}^{4}\left(\sqrt[5]{1+3t}\right)^k}\right)dt}{\int_0^x\left(\frac{3\sqrt[5]{1+2t}}{\sum_{k=0}^{2}\left(\sqrt[3]{1+3t}\right)^k}+\frac{2}{\sum_{k=0}^{4}\left(\sqrt[5]{1+2t}\right)^k}\right)dt}$$

$$\overset{Hospital}{=} \lim_{x\to 0}\frac{\left(\frac{2\sqrt[5]{1+3x}}{\sum_{k=0}^{2}\left(\sqrt[3]{1+2x}\right)^k}+\frac{3}{\sum_{k=0}^{4}\left(\sqrt[5]{1+3x}\right)^k}\right)}{\left(\frac{3\sqrt[5]{1+2x}}{\sum_{k=0}^{2}\left(\sqrt[3]{1+3x}\right)^k}+\frac{2}{\sum_{k=0}^{4}(1+2x)^k}\right)} = \frac{\left(\frac{2}{\sum_{k=0}^{2}1}+\frac{3}{\sum_{k=0}^{4}1}\right)}{\left(\frac{3}{\sum_{k=0}^{2}1}+\frac{2}{\sum_{k=0}^{4}1}\right)} = \frac{\frac{2}{3}+\frac{3}{5}}{\frac{3}{3}+\frac{2}{5}} = \frac{19}{21}$$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$\Omega=\lim_{x\to 0}\frac{\int_0^x \frac{\sqrt[3]{1+2t}\cdot\sqrt[5]{1+3t}-1}{t}dt}{\int_0^x \frac{\sqrt[3]{1+3t}\cdot\sqrt[5]{1+2t}-1}{t}dt}\overset{Hospital}{=}\lim_{x\to 0}\frac{\frac{\sqrt[3]{1+2x}\cdot\sqrt[5]{1+3x}-1}{x}}{\frac{\sqrt[3]{1+3x}\cdot\sqrt[5]{1+2x}-1}{x}}$$

$$=\lim_{x\to 0}\frac{\sqrt[3]{1+2x}\cdot\sqrt[5]{1+3x}-1}{\sqrt[3]{1+3x}\cdot\sqrt[5]{1+2x}-1}=\lim_{x\to 0}\frac{\exp\left(\ln\left(\sqrt[3]{1+2x}\cdot\sqrt[5]{1+3x}\right)\right)-1}{\exp\left(\ln\left(\sqrt[3]{1+3x}\cdot\sqrt[5]{1+2x}\right)\right)-1}$$

$$=\lim_{x\to 0}\frac{\exp\left(\frac{\ln(1+2x)}{3}+\frac{\ln(1+3x)}{5}\right)-1}{\exp\left(\frac{\ln(1+3x)}{3}+\frac{\ln(1+2x)}{5}\right)-1}$$

$$=\lim_{x\to 0}\underbrace{\frac{\frac{\exp\left(\frac{\ln(1+2x)}{3}+\frac{\ln(1+3x)}{5}\right)-1}{\frac{\ln(1+2x)}{3}+\frac{\ln(1+3x)}{5}}}{\frac{\exp\left(\frac{\ln(1+3x)}{3}+\frac{\ln(1+2x)}{5}\right)-1}{\frac{\ln(1+3x)}{3}+\frac{\ln(1+2x)}{5}}}}_{\to 1}\cdot\frac{\frac{\ln(1+2x)}{3}+\frac{\ln(1+3x)}{5}}{\frac{\ln(1+3x)}{3}+\frac{\ln(1+2x)}{5}}$$

$$=\lim_{x\to 0}\frac{\frac{\ln(1+2x)}{3}+\frac{\ln(1+3x)}{5}}{\frac{\ln(1+3x)}{3}+\frac{\ln(1+2x)}{5}}\underset{x\to 0}{\overset{\ln(1+ax)\sim ax}{=}}\lim_{x\to 0}\frac{\frac{2x}{3}+\frac{3x}{5}}{\frac{3x}{3}+\frac{2x}{5}}=\frac{\frac{2}{3}+\frac{3}{5}}{\frac{3}{3}+\frac{2}{5}}=\frac{19}{21}$$

Solution 3 by Serban George Florin-Romania

$$\Omega\overset{\frac{0}{0}L.H.}{=}\lim_{x\to 0}\frac{\sqrt[3]{1+2x}\cdot\sqrt[5]{1+3x}-1}{x}\cdot\frac{x}{\sqrt[3]{1+3x}\sqrt[5]{1+2x}-1}=$$

$$\Omega\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{\sqrt[3]{1+2x}\cdot\sqrt[5]{1+3x}-1}{\sqrt[3]{1+3x}\cdot\sqrt[5]{1+2x}-1}$$

$$\Omega \overset{L.H.}{=} \lim_{x\to 0} \frac{\frac{1}{3}(1+2x)^{-\frac{2}{3}}\cdot 2\cdot \sqrt[5]{1+3x}+\frac{1}{5}(1+3x)^{-\frac{4}{5}}\cdot 3\sqrt[3]{1+2x}}{\frac{1}{3}(1+3x)^{-\frac{2}{3}}\cdot 3\cdot \sqrt[5]{1+2x}+\frac{1}{5}(1+2x)^{-\frac{4}{5}}\cdot 2\cdot \sqrt[3]{1+3x}}$$

$$\Omega = \frac{\frac{2}{3}+\frac{3}{5}}{1+\frac{2}{5}} = \frac{19}{15}\cdot\frac{5}{7} = \frac{19}{21}$$

Solution 4 by Lazaros Zachariadis-Thessaloniki-Greece

$$\lim_{x\to 0}\frac{\int_0^x \frac{\sqrt[3]{1+2t}\cdot\sqrt[5]{1+3t}-1}{t}dt}{\int_0^x \frac{\sqrt[3]{1+3t}\cdot\sqrt[5]{1+2t}-1}{t}dt} \overset{\left(\frac{0}{0}\right)}{\underset{DLH}{=}} \lim_{x\to 0}\frac{\sqrt[3]{1+2x}\cdot\sqrt[5]{1+3x}-1}{\sqrt[3]{1+3x}\cdot\sqrt[5]{1+2x}-1}$$

$$\underset{when\ x\to 0}{\overset{e^x\approx x+1}{=}} \lim_{x\to 0}\frac{\sqrt[3]{e^{2x}}\cdot\sqrt[5]{e^{3x}}-1}{\sqrt[3]{e^{3x}}\cdot\sqrt[5]{e^{2x}}-1} = \lim_{x\to 0}\frac{e^{\frac{2x}{3}}\cdot e^{\frac{3x}{5}}-1}{e^{x}\cdot e^{\frac{2x}{3}}-1}$$

$$= \lim_{x\to 0}\frac{e^{\frac{19x}{15}}-1}{e^{\frac{21x}{15}}-1} = \lim_{x\to 0}\frac{\frac{19x}{15}}{\frac{21x}{15}} = \frac{19}{21}$$

196. Find:

$$\Omega = \lim_{n\to\infty}\frac{\sqrt{1!+\sqrt{2!+\sqrt{3!+\sqrt{4!\dots\sqrt{n!}}}}}}{n}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

We first show that $n! < 2^{2^n}$ $\forall n \ge 1$. ***For*** $n = 1, n! = 1! = 1$ ***and***

$2^{2^1} = 2^2 = 4 \therefore 1! < 2^{2^1}$. ***Assume*** $k! < 2^{2^k}$, ***for some*** $k \in \mathbb{N}$. ***Now,***

$$k+1<2^{2^k}\Rightarrow k!\,(k+1)<2^{2^k}\cdot 2^{2^k}=2^{2^{k+1}}\Rightarrow (k+1)!<2^{2^{k+1}}$$

$$\therefore\ n!<2^{2^n}\quad \forall n\geq 1$$

Now, for $n\geq 1, n!<2^{2^n}\Rightarrow \sqrt{n!}<\left(2^{2^n}\right)^{\frac{1}{2}}=2^{(2^n)\left(\frac{1}{2}\right)}=2^{2^{n-1}}\Rightarrow$

$$\Rightarrow (n-1)!+\sqrt{n!}<2^{2^{n-1}}+2^{2^{n-1}}=(2)2^{2^{n-1}}\Rightarrow$$

$$\Rightarrow \sqrt{(n-1)!+\sqrt{n!}}<\sqrt{2}\cdot 2^{2^{n-2}}<2\left(2^{2^{n-2}}\right)\Rightarrow$$

$$\Rightarrow (n-2)!+\sqrt{(n-1)!+\sqrt{n!}}<3\left(2^{2^{n-2}}\right)\Rightarrow$$

$$\Rightarrow \sqrt{(n-2)!+\sqrt{(n-1)!+\sqrt{n!}}}<\sqrt{3}\left(2^{2^{n-3}}\right)<(3)\left(2^{2^{n-3}}\right)$$

Continuing in this way we get

$$\sqrt{1!+\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}}}<\left(\sqrt{n}\right)2^{2^{n-n}}=2\sqrt{n}$$

$$\Rightarrow 0<\frac{\sqrt{1!+\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}}}}{n}<\frac{2\sqrt{n}}{n}=\frac{2}{\sqrt{n}}$$

As $\lim_{n\to\infty}\frac{2}{\sqrt{n}}=0$***, we get*** $\Omega=\lim_{n\to\infty}\frac{\sqrt{1!+\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}}}}{n}=0$

197. $\Omega=\lim_{n\to\infty}\frac{\sqrt{2\sqrt{3\sqrt{4\sqrt{5\ldots\sqrt{n}}}}}}{n}$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$\text{For } n \geq 2, \text{ let } a_n = \sqrt{2\sqrt{3\sqrt{4\sqrt{5\ldots\sqrt{n}}}}} \Rightarrow a_n^2 = 2\sqrt{3\sqrt{4\sqrt{5\ldots\sqrt{n}}}} \Rightarrow$$

$$\Rightarrow (a_n^2)^2 = 2^2\cdot 3\sqrt{4\sqrt{5\ldots\sqrt{n}}} \Rightarrow \left(a_n^{2^2}\right)^2 = 2^2\cdot 3^2\cdot 4\sqrt{5\ldots\sqrt{n}}$$

$$\Rightarrow \left(a_n^{2^3}\right)^2 = 2^{2^3}\cdot 3^{2^2}\cdot 4^2\cdot 5\sqrt{6\ldots\sqrt{n}}$$

… … … … … … … … … … … … … … … … … … ..

$$a_n^{2^{n-1}} = 2^{2^{n-2}}\cdot 3^{2^{n-3}}\cdot 4^{2^{n-4}}\ldots(n-1)^2 n$$

$$\Rightarrow \left(\frac{a_n}{n}\right)^{2^{n-1}} = \left(\frac{2}{n}\right)^{2^{n-2}}\left(\frac{3}{n}\right)^{2^{n-3}}\ldots\left(\frac{n-1}{n}\right)^2\cdot\frac{n}{n}\cdot\frac{1}{n} \leq \left(\frac{2}{n}\right)^{2^{n-2}}$$

$$\Rightarrow \frac{a_n}{n} \leq \sqrt{\frac{2}{n}} \Rightarrow 0 < \frac{a_n}{n} \leq \sqrt{\frac{2}{n}}\ \forall n \geq 2. \text{ As } \lim_{n\to\infty}\sqrt{\frac{2}{n}} = 0, \text{ we get}$$

$$\lim_{n\to\infty}\frac{a_n}{n} = 0$$

198. Find:

$$\Gamma = \lim_{n\to\infty}\left(\Omega(n) - \Omega(n+1)\right), \Omega(n) = \int_1^e \frac{dx}{x(1+x^3)^n}, n \in \mathbb{N}^*$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$\Omega(n) = \int_1^e \frac{dx}{x(1+x^3)^n} = \frac{1}{3}\int_1^e \frac{3x^2}{x^3(1+x^3)^n}dx$$

Put $1+x^3=t, 3x^2dx=dt \therefore \Omega(n)=\frac{1}{3}\int_2^{1+e^3}\frac{dt}{(t-1)t^n}$. *For* $n\geq 2$

$$\Omega(n)-\Omega(n-1)=\frac{1}{3}\int_2^{1+e^3}\frac{1}{t-1}\left(\frac{1}{t^n}-\frac{1}{t^{n-1}}\right)dt=\frac{1}{3}\int_2^{1+e^3}\frac{1}{t-1}\cdot\frac{(1-t)}{t^n}dt=$$

$$=-\frac{1}{3}\int_2^{1+e^3}t^{-n}\,dt=-\frac{1}{3(-n+1)}[t^{-n+1}]_2^{1+e^3}=\frac{1}{3(n-1)}\left[\frac{1}{(1+e^3)^{n-1}}-\frac{1}{2^{n-1}}\right]$$

$$\therefore \Gamma=\lim_{n\to\infty}[\Omega(n)-\Omega(n-1)]=0$$

For $n\geq 2, 1\leq x\leq e\Rightarrow 2\leq 1+x^3\leq 1+e^3\Rightarrow 2^n\leq(1+x^3)^n\leq(1+e^3)^n$

$$\Rightarrow 2^n\leq x(1+x^3)^n\leq e(1+e^3)^n\Rightarrow\frac{1}{e(1+e^3)^n}\leq\frac{1}{x(1+x^3)^n}\leq\frac{1}{2^n}\Rightarrow$$

$\Rightarrow\frac{e-1}{e(1+e^3)n}\leq\int_1^e\frac{dx}{x(1+x^3)^n}\leq\frac{e-1}{2^n}$. *As* $\lim_{n\to\infty}\frac{e-1}{e(1+e^3)^n}=0=\lim_{n\to\infty}\frac{e-1}{2^n}$

$$\therefore\lim_{n\to\infty}\int_1^e\frac{1}{x(1+x^3)^n}dx=0\Rightarrow\lim_{n\to\infty}\Omega(n)=0\Rightarrow\lim_{n\to\infty}\big(\Omega(n)-\Omega(n-1)\big)=0$$

199. Find:

$$\Omega=\lim_{n\to\infty}\frac{\left(\sqrt[n]{n!}+n\right)^n}{(2n)!}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 2 by Ravi Prakash-New Delhi-India, Solution 3 by Geanina Tudose-Romania, Solution 4 by Su Tanaya-India

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco,

$$\Omega = \lim_{n\to\infty}\frac{\left(\sqrt[n]{n!}+n\right)^n}{(2n)!} \overset{Stirlling}{=} \lim_{n\to\infty}\frac{\left(\sqrt[n]{\left(\frac{n}{e}\right)^n}+n\right)^n}{\left(\frac{2n}{e}\right)^{2n}} = \lim_{n\to\infty}\frac{\left(n(e^{-1}+1)\right)^n}{\left(\frac{2n}{e}\right)^{2n}}$$

$$= \lim_{n\to\infty}\left(\frac{n(1+e^{-1})}{4n^2}e^2\right)^n = \lim_{n\to\infty}\left(\frac{(e^2+e)}{4n}\right)^n = \lim_{n\to\infty}\left(\frac{(e^2+e)}{4n}\right)^n$$

$$= \lim_{n\to\infty} e^{n\ln\left(\frac{(e^2+e)}{4n}\right)} = e^{-\infty} = 0$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{For } n \geq 1, n! \leq n^n \Rightarrow (n!)^{\frac{1}{n}} \leq n \Rightarrow \left(n+(n!)^{\frac{1}{n}}\right)^n \leq (2n)^n \Rightarrow$$

$$\Rightarrow 0 < \frac{\left(n+(n!)^{\frac{1}{n}}\right)^n}{(2n)!} \leq \frac{(2n)^n}{(2n)!} = b_n \ \text{(say)} \quad (1)$$

$$\text{Now, } \frac{b_n}{b_{n+1}} = \frac{(2n)^n}{(2n!)}\cdot\frac{(2n+2)!}{(2n+2)^{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot(2n+1) \to \infty \text{ as } n\to\infty$$

$$\text{Let } R > 1, \text{ then } \exists\, m \in \mathbb{N} \text{ s.t.}$$

$$\frac{b_n}{b_{n+1}} > R \ \ \forall n \geq m \Rightarrow \frac{b_m}{b_{m+1}}\cdot\frac{b_{m+1}}{b_{m+2}}\cdot\ldots\cdot\frac{b_{n-1}}{b_n} > R^{n-m} \ \ \forall n > m$$

$$\Rightarrow 0 < b_n < \frac{R^m b_m}{R^n} \quad \forall n > m$$

$$\text{As, } \frac{1}{R^n} \to 0 \text{ as } n\to\infty, \text{ we get } b_n \to 0 \text{ as } n\to\infty. \text{ From (1), we get}$$

$$\frac{\left(n+(n!)^{\frac{1}{n}}\right)^n}{(2n!)} \to 0 \text{ as } n\to\infty$$

Solution 3 by Geanina Tudose-Romania

$$\Omega_n = \lim_{n\to\infty}\frac{\left(\sqrt[n]{n!}+n\right)^n}{(2n)!} \text{ we have } \sqrt[n]{n!} \leq n \text{ since } n! \leq n^n.$$

$$\textit{Hence } \Omega_n \le \frac{(n+n)^n}{(2n)!} = \frac{(2n)^n}{(2n)!} = a_n$$

$$\frac{a_{n+1}}{a_n} = \frac{\left(2(n+1)\right)^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(2n)^n} = \frac{(2n+2)^{n+1}}{(2n+1)(2n+2)} \cdot \frac{1}{(2n)^n} =$$

$$= \frac{1}{2n+1} \cdot \left(\frac{2n+2}{2n}\right)^n = \frac{1}{2n+1} \cdot \left(1 + \frac{1}{n}\right)^n \to 0$$

$$\textit{Since } \frac{a_{n+1}}{a_n} \to 0 < 1 \Rightarrow \left.\begin{matrix} \lim_{n\to\infty} a_n = 0 \\ 0 \le \Omega_n \le a_n \end{matrix}\right\} \Rightarrow \lim \Omega_n = 0$$

Solution 4 by Su Tanaya-India

$$0 < \frac{\left(\sqrt[n]{n!} + n\right)^n}{(2n)!} = \left(\frac{\sqrt[n]{n!}}{\sqrt[n]{(2n)!}} + \frac{n}{\sqrt[n]{(2n)!}}\right)^n < \left(\frac{1}{\left((n+1)\dots(2n)\right)^{\frac{1}{n}}} + \frac{1}{(n!)^{\frac{1}{n}}}\right)^n$$

$$< \left(\frac{2}{(n!)^{\frac{1}{n}}}\right)^n = \frac{2^n}{n!} \because \lim_{n\to\infty} \frac{2^n}{n!} = 0, \lim_{n\to\infty} \frac{\left(\sqrt[n]{n!}+n\right)^n}{(2n)!} = 0$$

By Sandwich theorem

200. Find

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \arctan\left(\frac{9}{9 + (3k+5)(3k+8)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India, Solution 2 by Serban George Florin-Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \tan^{-1}\left(\frac{9}{9 + (3k+5)(3k+8)}\right)$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\tan^{-1}\left(k+\frac{8}{3}\right)-\tan^{-1}\left(k+\frac{5}{3}\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan^{-1}\left(n+\frac{8}{3}\right)-\tan^{-1}\frac{8}{3}\right)=\frac{\pi}{2}-\tan^{-1}\frac{8}{3}$$

Solution 2 by Serban George Florin-Romania

$$\arctan x-\arctan y=\arctan\frac{x-y}{1+xy}$$

$$\arctan\frac{9}{9+(3k+5)(3k+8)}=\arctan\frac{1}{1+\frac{3k+5}{3}\cdot\frac{3k+8}{3}}=$$

$$=\arctan\frac{\frac{3k+8}{3}-\frac{3k+5}{3}}{1+\frac{3k+5}{3}\cdot\frac{3k+8}{3}}=\arctan\frac{3k+8}{3}-\arctan\frac{3k+5}{3}$$

$$\sum_{k=1}^{n}\arctan\left(\frac{9}{9+(3k+5)(3k+8)}\right)=\sum_{k=1}^{n}\left(\arctan\frac{3k+8}{3}-\arctan\left(\frac{3k+5}{3}\right)\right)$$

$$=\arctan\frac{11}{3}-\arctan\frac{8}{3}+\arctan\frac{14}{3}-\arctan\frac{11}{3}+\cdots+\arctan\left(\frac{3n+8}{3}\right)-$$

$$-\arctan\left(\frac{3n+5}{3}\right)=\arctan\left(\frac{3n+8}{3}\right)-\arctan\frac{8}{3}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\arctan\left(\frac{9}{9+(3k+5)(3k+8)}\right)=$$

$$=\Omega=\lim_{n\to\infty}\left(\arctan\left(\frac{3n+8}{3}\right)-\arctan\frac{8}{3}\right)=\frac{\pi}{2}-\arctan\frac{8}{3}$$

Its nice to be important but more important its to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru