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401 – 500

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Proposed by

Daniel Sitaru – Romania

Nguyen Viet Hung – Hanoi – Vietnam

Mehmet Șahin – Ankara – Turkey

Ali Can Gullu – Izmir – Turkey

Adil Abdullayev – Baku – Azerbaidian

D.M. Bătinețu Giurgiu – Romania

Neculai Stanciu – Romania

George Apostolopoulos – Messolonghi – Greece

Vasile Jiglau – Romania

Gheorghe Alexe – Romania

Șerban George – Florin – Romania

Nica Nicolae – Romania

Seyran Ibrahimov – Maasilli – Azerbaidian

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Solutions by

Daniel Sitaru – Romania

Kevin Soto Palacios – Huarmey – Peru

Khanh Hung Vu - Ho Chi Minh – Vietnam

Ravi Prakash - New Delhi – India; Geanina Tudose - Romania

Mehmet Șahin – Ankara – Turkey

Soumava Chakraborty – Kolkata – India

Rozeta Atanasova – Skopje

Sanong Hauerai-Nakon Pathom – Thailand

Seyran Ibrahimov – Maasilli – Azerbaidian

Hoang Le Nhat Tung – Hanoi – Vietnam

Soumitra Mandal-Chandar Nagore-India

Myagmarsuren Yadamsuren-Darkhan – Mongolia

George Apostolopoulos-Messolonghi – Greece

Nirapada Pal – Jhargram – India; SK Rejuan - West Bengal – India

Francisco Javier Garcia Capitan – Spain

Boris Colakovic – Belgrade – Serbia; Theodoros Sampas – Greece

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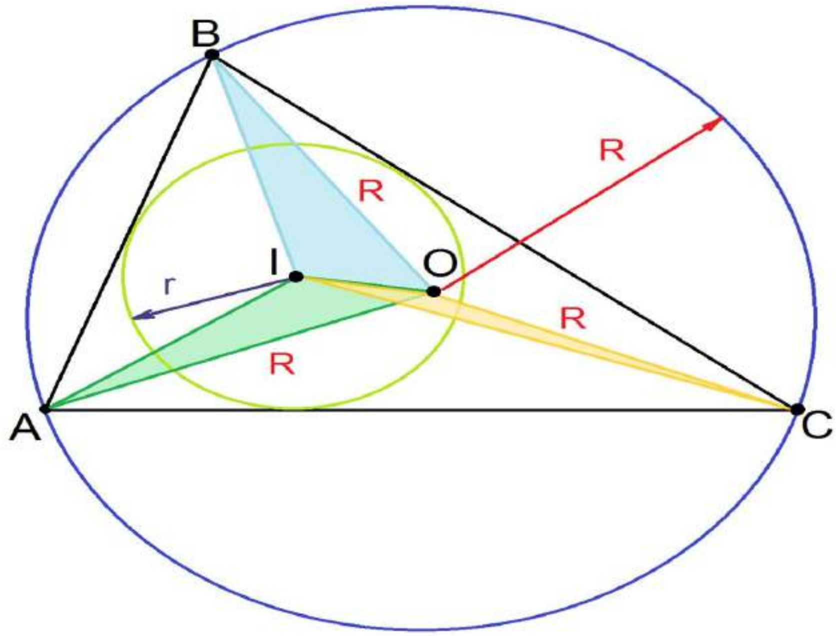
401. Prove that in any triangle ABC,

$$(a+b+c)\cdot\frac{OI}{R}=a\cdot\cos\angle AOI+b\cdot\cos\angle BOI+c\cdot\cos\angle COI$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru



Siendo O – circuncentro e I – incentro. Probar en un triángulo ABC

$$(a+b+c)\cdot\frac{OI}{R}=a\cos(\measuredangle AOI)+b\cos(\measuredangle BOI)+c\cos(\measuredangle COI)$$

Tener en cuenta lo siguiente

$$abc=4pRr, IA^2=bc-4Rr, IB^2=ca-4Rr, IC^2=ab-4Rr$$

$$\Leftrightarrow aIA^2+bIB^2+cIC^2=a(bc-4Rr)+b(ca-4Rr)+c(ab-4Rr)=$$

$$=3abc-4Rr(a+b+c)$$

$$\Leftrightarrow aIA^2+bIB^2+cIC^2=12pRr-9pRr=4pRr=2(a+b+c)Rr$$

$$OI^2=R^2-2Rr, OA=OB=OC=R$$

Aplicando ley de Cosenos en los $\Delta OIA, \Delta OIB, \Delta OIC$ se obtiene

$$\cos(\measuredangle AOI) = \frac{OA^2 + OI^2 - IA^2}{2OA \cdot OI}, \cos(\measuredangle BOI) = \frac{OB^2 + OI^2 - IB^2}{2OB \cdot OI},$$

$$\cos(\measuredangle COI) = \frac{OC^2 + OI^2 - IC^2}{2OC \cdot OI}$$

$$\Leftrightarrow \cos(\measuredangle AOI) = \frac{R^2 + OI^2 - IA^2}{2R \cdot OI}, \cos(\measuredangle BOI) = \frac{R^2 + OI^2 - IB^2}{2R \cdot OI},$$

$$\cos(\measuredangle COI) = \frac{R^2 + OI^2 - IC^2}{2R \cdot OI}$$

Por lo tanto

$$a\cos(\measuredangle AOI) + b\cos(\measuredangle BOI) + c\cos(\measuredangle COI) =$$

$$= \frac{(a+b+c)R^2 + (a+b+c)OI^2 - aIA^2 - bIB^2 - cIC^2}{2R \cdot OI}$$

$$a\cos(\measuredangle AOI) + b\cos(\measuredangle BOI) + c\cos(\measuredangle COI) = \frac{(a+b+c)(R^2+OI^2-2Rr)}{2R \cdot OI} =$$

$$= \frac{(a+b+c) \cdot 2OI^2}{2R \cdot OI} = (a+b+c) \cdot \frac{OI}{R}$$

402. If in $\Delta\ ABC: A_1, A_2 \in (BC), B_1, B_2 \in (AC), C_1, C_2 \in (AB)$,

$$\frac{BA_1}{CA_1} = \frac{CB_1}{AB_1} = \frac{AC_1}{BC_1} = \frac{CA_2}{BA_2} = \frac{AB_2}{CB_2} = \frac{BC_2}{AC_2} \text{ then:}$$

$$AA_1^2 + BB_1^2 + CC_1^2 = AA_2^2 + BB_2^2 + CC_2^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam, Solution 2 by Ravi Prakash-New Delhi-India, Solution 3 by Geanina Tudose-Romania

Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam

If in $\Delta\ ABC, A_1, A_2 \in BC, B_1, B_2 \in AC, C_1, C_2 \in AB,$

$$\frac{BA_1}{CA_1} = \frac{CB_1}{AB_1} = \frac{AC_1}{BC_1} = \frac{CA_1}{BA_1} = \frac{AB_1}{CB_1} = \frac{BC_1}{AC_1}$$

Prove that $AA_1^2 + BB_1^2 + CC_1^2 = AA_2^2 + BB_2^2 + CC_2^2$

We put $\frac{BA_1}{CA_1} = \frac{CB_1}{AB_1} = \frac{AC_1}{BC_1} = \frac{CA_1}{BA_1} = \frac{AB_1}{CB_1} = \frac{BC_1}{AC_1} = k$ ***and*** $BC = a, CA = b, AB = c$

$$\Rightarrow BA_1 = CA_2 = \frac{ak}{k+1}, CB_1 = AB_2 = \frac{bk}{k+1} \text{ and } AC_1 = BC_2 = \frac{ck}{k+1}$$

By Cosin's law of triangle ABA_1 ***and*** ACA_2***, we have:***

$$AA_1^2 - AA_2^2 = AB^2 + BA_1^2 - 2AB \cdot BA_1 \cdot \cos B - \left(AC^2 + CA_2^2 - 2AC \cdot CA_2 \cdot \cos C\right) =$$

$$= AB^2 - AC^2 - 2AB \cdot BA_1 \cdot \cos B + 2AC \cdot A_2C \cdot \cos C$$

$$\Rightarrow AA_1^2 - AA_2^2 = c^2 - b^2 - 2c \cdot \frac{ak}{k+1} \cdot \frac{a^2 + c^2 - b^2}{2ab} + 2b \cdot \frac{ak}{k+1} \cdot \frac{a^2 + b^2 - c^2}{2ab} =$$

$$= c^2 - b^2 + \frac{2(b^2 - c^2) \cdot k}{k+1}$$

Similarly, we have $BB_1^2 - BB_2^2 = a^2 - c^2 + \frac{2(c^2 - a^2) \cdot k}{k+1}$ ***and***

$$CC_1^2 - CC_2^2 = b^2 - a^2 + \frac{2(a^2 - b^2) \cdot k}{k+1}$$

Similarly, we have $BB_1^2 - BB_2^2 = a^2 - c^2 + \frac{2(c^2 - a^2) \cdot k}{k+1}$ ***and***

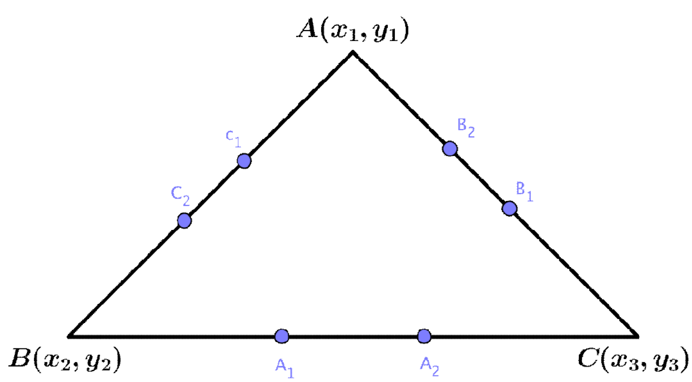
$$CC_1^2 - CC_2^2 = b^2 - a^2 + \frac{2(a^2 - b^2) \cdot k}{k+1}$$

$$\Rightarrow AA_1^2 - AA_2^2 + BB_1^2 - BB_2^2 + CC_1^2 - CC_2^2 = 0 \Rightarrow AA_1^2 + BB_1^2 + CC_1^2 =$$

$$= AA_2^2 + BB_2^2 + CC_2^2$$

(QED)

Solution 2 by Ravi Prakash-New Delhi-India



Let $k = \frac{BA_1}{CA_1} = \frac{CB_1}{AB_1} = \frac{AC_1}{BC_1} = \frac{CA_2}{BA_2} = \frac{AB_2}{CB_2} = \frac{BC_2}{AC_2}$

Coordinates of A_1

$$\left(\frac{x_2+kx_3}{k+1}, \frac{y_2+ky_3}{k+1}\right)$$

$$AA_1^2 = \left(x_1 - \frac{x_2-kx_3}{k+1}\right)^2 + \left(y_1 - \frac{y_2+ky_3}{k+1}\right)^2$$

$$= \frac{1}{(k+1)^2}[(x_1-x_2)+k(x_1-x_3)]^2 + \frac{1}{(k+1)^2}[(y_1-y_2)+k(y_1-y_3)]^2$$

$$= \frac{1}{(k+1)^2}\left[\begin{matrix}(x_1-x_2)^2+(y_1-y_2)^2+k^2\{(x_1-x_3)^2+(y_1-y_3)^2\}+\\ +2k(x_1-x_2)(x_1-x_3)+2k(y_1-y_2)(y_1-y_3)\end{matrix}\right]$$

$$= \frac{1}{(k+1)^2}[AB^2+k^2AC^2+2k(x_1-x_2)(x_1-x_3)+2k(y_1-y_2)(y_1-y_3)]$$

$$\therefore AA_1^2 + BB_1^2 + CC_1^2 =$$

$$= \frac{1}{(k+1)^2}\left[\begin{matrix}(k^2+1)(AB^2+BC^2+AC^2)+\\ +2k\left(x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2 - x_1x_2-x_2x_3-x_3x_1-y_1y_2-y_2y_3-y_3y_1\right)\end{matrix}\right]$$

$$= E \text{ (say)}$$

Similarly

$$AA_2^2 + BB_2^2 + CC_2^2 = E$$

$$\therefore AA_1^2 + BB_1^2 + CC_1^2 = AA_2^2 + BB_2^2 + CC_2^2$$

Solution 3 by Geanina Tudose-Romania

$$\text{Let } \frac{BA_1}{CA_1} = \frac{CB_1}{AB_1} = \frac{AC_1}{BC_1} = \frac{CA_2}{BA_2} = \frac{AB_2}{CB_2} = \frac{BC_2}{AC_2} = k$$

$$\text{we have } \overrightarrow{AA_1} = \frac{1}{k+1}\overrightarrow{AB} + \frac{k}{k+1}\overrightarrow{AC}$$

$$AA_1^2 = \overrightarrow{AA_1}\cdot\overrightarrow{AA_1} = \left(\frac{1}{k+1}\overrightarrow{AB} + \frac{k}{k+1}\overrightarrow{AC}\right)\cdot\left(\frac{1}{k+1}\overrightarrow{AB} + \frac{k}{k+1}\overrightarrow{AC}\right)$$

$$= \frac{1}{(k+1)^2}AB^2 + \frac{k^2}{(k+1)^2}AC^2 + \frac{2k}{(k+1)^2}\overrightarrow{AB}\cdot\overrightarrow{AC} = \frac{1}{(k+1)^2}c^2 + \frac{k^2}{(k+1)^2}b^2 + \frac{2k}{(k+1)^2}\cdot bc\cdot\cos A$$

Similarly:

$$BB_1^2 = \frac{1}{(k+1)^2}BC^2 + \frac{k^2}{(k+1)^2}BA^2 + \frac{2k}{(k+1)^2}\overrightarrow{BC}\cdot\overrightarrow{BA} =$$

$$= \frac{1}{(k+1)^2}a^2 + \frac{k^2}{(k+1)^2}c^2 + \frac{2k}{(k+1)^2} \cdot ac \cdot \cos B$$

$$CC_1^2 = \frac{1}{(k+1)^2}b^2 + \frac{k^2}{(k+1)^2}a^2 + \frac{2k}{(k+1)^2}ab\cos C$$

Adding then up we have:

$$S_1 = (a^2+b^2+c^2) \cdot \frac{(k^2+1)}{(k+1)^2} + \frac{2k}{(k+1)^2}(bc\cos A +$$

$$+ac \cdot \cos B + ab \cdot \cos C)$$

Similarly $\overrightarrow{AA_2} = \frac{1}{k+1}\overrightarrow{AC} + \frac{k}{k+1}\overrightarrow{AB} \Rightarrow AA_2^2 = \frac{1}{(k+1)^2}AC^2 + \frac{k^2}{(k+1)^2}AB^2 +$

$$+\frac{2k}{(k+1)^2} \cdot AC \cdot AB \cdot \cos A$$

$$BB_2^2 = \frac{1}{(k+1)^2}AB^2 + \frac{k^2}{(k+1)^2}BC^2 + \frac{2k}{k+1}AB \cdot BC \cdot \cos B$$

$$CC_2^2 = \frac{1}{(k+1)^2}CB^2 + \frac{k^2}{(k+1)^2}CA^2 + \frac{2k}{k+1}CB \cdot CA - \cos C$$

Hence $S_2 = (a^2+b^2+c^2) \cdot \frac{(k^2+1)}{(k+1)^2} + \frac{2k}{k+1}(bc \cdot \cos A + ac \cdot \cos B + ab\cos C)$

Therefore $S_1 = S_2$

403. In $\Delta\ ABC$

$$\max(A,B,C) = 135^\circ \Leftrightarrow \frac{s+r}{R+r} = \sqrt{2}$$

Proposed by Mehmet Șahin – Ankara – Turkey

Solution by Daniel Sitaru – Romania

$$m(\sphericalangle A) = 135^\circ \to R = \frac{a}{2\sin A} = \frac{a\sqrt{2}}{2}; S = \frac{1}{2}bc\sin 135^\circ = \frac{\sqrt{2}bc}{4},$$

$$a^2 = b^2 + c^2 + \sqrt{2}bc$$

$$\frac{s+r}{R+r} = \sqrt{2} \leftrightarrow s = \sqrt{2}R + r(\sqrt{2}-1) \leftrightarrow s = a + r(\sqrt{2}-1)$$

$$a+b+c = 2a + 2r(\sqrt{2}-1) \leftrightarrow b+c-a = (\sqrt{2}-1)\cdot\frac{\sqrt{2}bc}{2s}$$

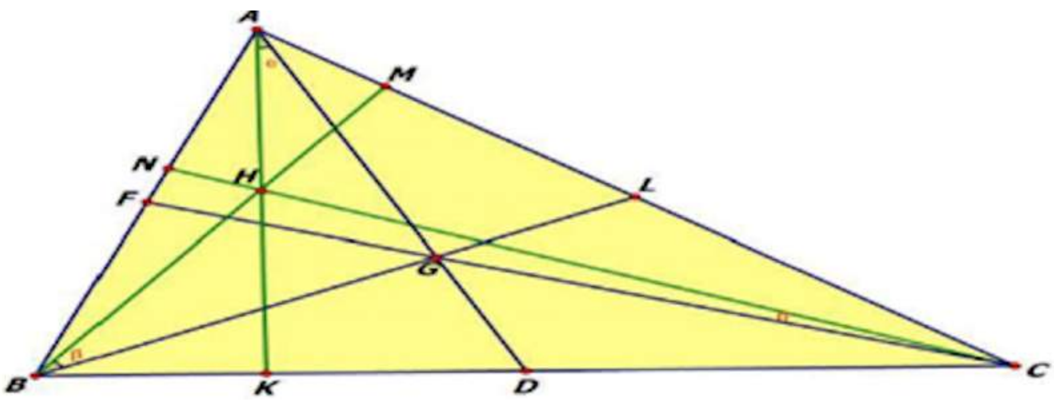
$$(a+b+c)(b+c-a) = (2-\sqrt{2})bc \leftrightarrow (b+c)^2 - a^2 = (2-\sqrt{2})bc$$

$$(b+c)^2 - b^2 - c^2 - \sqrt{2}bc = (2-\sqrt{2})bc \leftrightarrow (2-\sqrt{2})bc = (2-\sqrt{2})bc$$

404. If in $\Delta\ ABC$ acute, $a > b > c, H$ – orthocenter, G - centroid

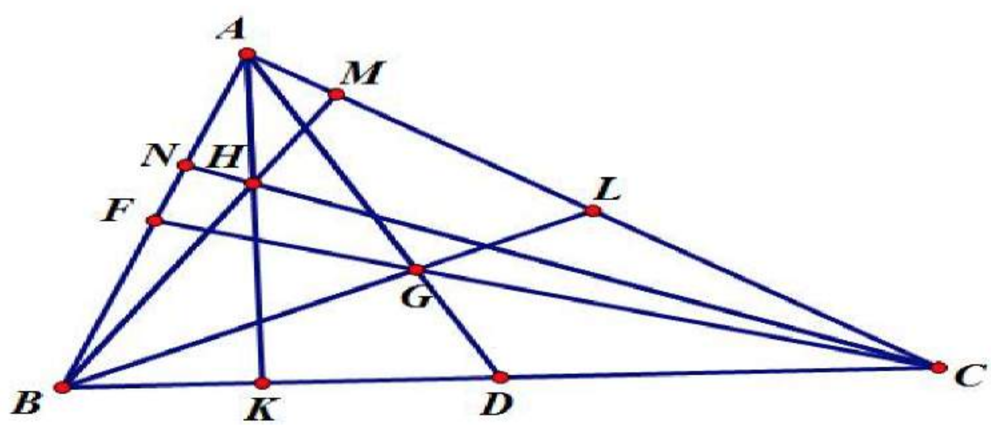
$m(\sphericalangle HAG) = \alpha, m(\sphericalangle HBG) = \beta, m(\sphericalangle HCG) = \theta$ then:

$$\tan\beta = \tan\alpha + \tan\theta$$



Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam



If in $\Delta\ ABC$ – acute, $a > b > c$, H - orthocenter, G - centroid,

$\angle HAG = \alpha, \angle HBG = \beta, \angle HCG = \gamma$ ***then*** $\tan\beta = \tan\alpha + \tan\gamma$

We have $h_b = \frac{2S}{b} \Rightarrow h_b^2 = \frac{4S^2}{b^2} = \frac{4\cdot\frac{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}{16}}{b^2} =$

$$= \frac{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)}{4b^2}$$

We have $\tan\beta = \frac{ML}{BM} = \frac{\sqrt{m_a^2 - h_b^2}}{h)b} = \frac{\sqrt{\frac{2a^2+2c^2-b^2}{4} - \frac{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}{4b^2}}}{\frac{2S}{b}} =$

$$= \frac{\sqrt{\frac{(2a^2 + 2c^2 - b^2)\cdot b^2 - 2(a^2b^2 + b^2c^2 + c^2a^2) + (a^4 + b^4 + c^4)}{4b^2}}}{\frac{2S}{b}}$$

$$\Rightarrow \tan\beta = \frac{\sqrt{\frac{a^4+c^4-2a^2c^2}{4b^2}}}{\frac{2S}{b}} = \frac{\frac{a^2-c^2}{2b}}{\frac{2S}{b}} = \frac{a^2-c^2}{4S} \ \textit{(1) (Since } a > c\textit{)}$$

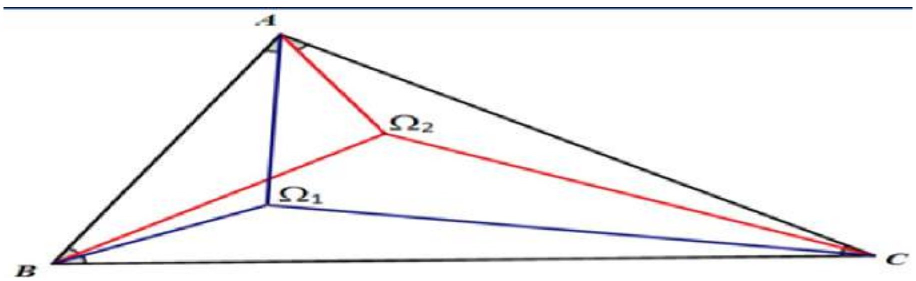
Similarly, we have $\tan\alpha = \frac{b^2-c^2}{4S}$ ***and*** $\tan\gamma = \frac{a^2-b^2}{4S}$

$$\Rightarrow \tan\alpha + \tan\gamma = \frac{a^2-c^2}{4S} \quad \textit{(2)}$$

(1) and (2) $\Rightarrow \tan\beta = \tan\alpha + \tan\gamma$

405. If Ω_1, Ω_2 - Brocard's point in $\Delta\ ABC$ then:

$$A\Omega_1 \cdot B\Omega_1 \cdot C\Omega_1 = A\Omega_2 \cdot B\Omega_2 \cdot C\Omega_2$$



Proposed by Ali Can Gullu-Izmir-Turkey

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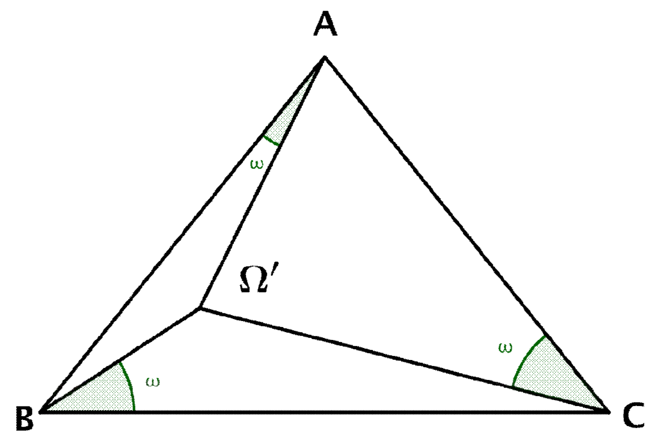
Solution by Mehmet Sahin-Ankara-Turkey

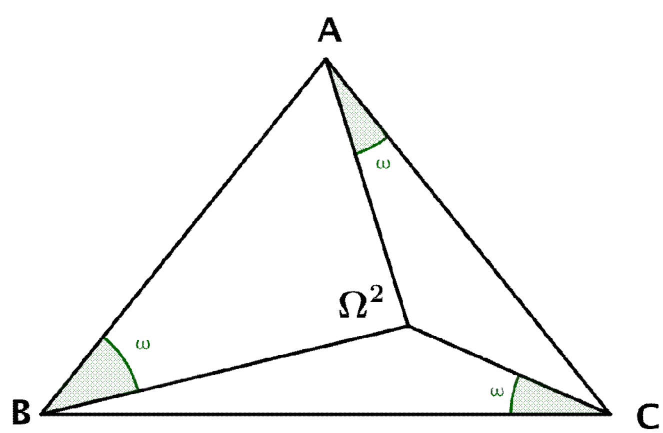


$$\frac{B\Omega'}{\sin\omega} = 2R_c \Rightarrow B\Omega' = 2R_c \sin\omega$$

$$\frac{C\Omega'}{\sin\omega} = 2R_a \Rightarrow C\Omega' = 2R_a \cdot \sin\omega$$

$$\frac{A\Omega'}{\sin\omega} = 2R_b \Rightarrow A\Omega' = 2R_b \cdot \sin\omega$$

$$A\Omega' \cdot B\Omega' \cdot C\Omega' = 8R_aR_bR_c \cdot \sin^3\omega = 8R^3 \cdot \sin^3\omega = (2R\sin\omega)^3 \quad \textbf{\textit{(1)}}$$



$$\frac{B\Omega^{(2)}}{\sin\omega} = 2R'_a \Rightarrow B\Omega^{(2)} = 2R'_a \cdot \sin\omega$$

$$\frac{C\Omega^{(2)}}{\sin\omega} = 2R'_a \Rightarrow C\Omega^{(2)} = 2R'_a \cdot \sin\omega$$

$$\frac{A\Omega^{(2)}}{\sin\omega} = 2R'_c \Rightarrow A\Omega^{(2)} = 2R'_c \cdot \sin\omega$$

$$A\Omega^{(2)} \cdot B\Omega^{(2)} \cdot C\Omega^{(2)} = 8R'_a \cdot R'_b \cdot R'_c \cdot \sin^3\omega = 8 \cdot R^3 \cdot \sin^3\omega$$

$$= (2R\sin\omega)^3 \quad \textbf{(2)}$$

From (1) and (2)

$$A\Omega' \cdot B\Omega' \cdot C\Omega' = A\Omega^{(2)} \cdot B\Omega^{(2)} \cdot C\Omega^{(2)}$$

as desired

406. In $\Delta\ ABC - N$ - ninepoint center

$$12r^2 \le AN^2 + BN^2 + CN^2 \le 3R^2$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Solution 2 by Mehmet Șahin-Ankara-Turkey

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo N − nine point center. Probar en un $\Delta\ ABC$

$$12r^2 \le NA^2 + NB^2 + NC^2 \le 3R^2;$$

Teorema Leibniz

Para cualquier punto P en el plano de triángulo ABC teniendo centroide G, se cumple

$$9PG^2 + a^2 + b^2 + c^2 = 3(PA^2 + PB^2 + PC^2);\ \textbf{Sea } P = N\textbf{, donde}$$

$$NG = \frac{1}{6}OH = \frac{1}{6}\sqrt{9R^2 - (a^2 + b^2 + c^2)} \ge 0 \Leftrightarrow 9R^2 \ge a^2 + b^2 + c^2$$

$$\Rightarrow 9NG^2 + a^2 + b^2 + c^2 = 3(NA^2 + NB^2 + NC^2)$$

$$\Rightarrow 3(NA^2 + NB^2 + NC^2) = 9NG^2 + a^2 + b^2 + c^2 \ge a^2 + b^2 + c^2 \ge$$

$$\geq ab + bc + ca \geq 18Rr \geq 36r^2$$

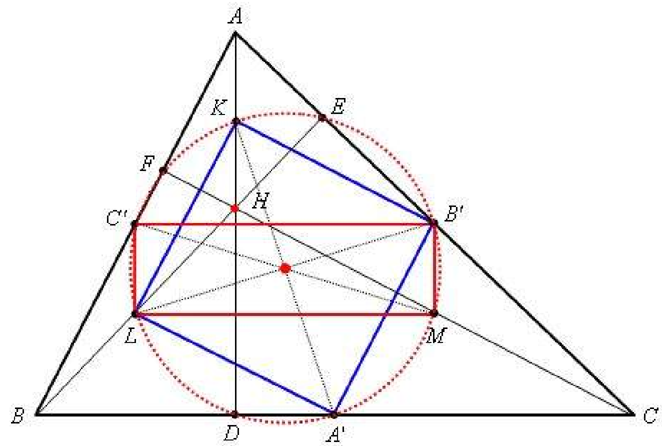
$$\Rightarrow NA^2 + NB^2 + NC^2 \geq 12r^2;$$ ***Por último***

$$3(NA^2 + NB^2 + NC^2) = 9 \cdot \frac{1}{36}\left(9R^2 - (a^2 + b^2 + c^2)\right) + a^2 + b^2 + c^2$$

$$\Rightarrow 3(NA^2 + NB^2 + NC^2) = \frac{9R^2 + 3(a^2 + b^2 + c^2)}{4} \leq \frac{9R^2 + 27R^2}{4} = 9R^2$$

$$\Rightarrow NA^2 + NB^2 + NC^2 \leq 3R^2$$ ***(LQQD)***

Solution 2 by Mehmet Șahin-Ankara-Turkey



H: Orthocenter, O: circumcenter

In triangle O and H are isogonal conjugate points.

$$|AH| = 2R \cdot \cos A, |OH| = R$$

In triangle AHO, $[AN]$ is a median, where N is ninepoint of ABC

$$|AN|^2 = \frac{|AH|^2 + |AO|^2}{2} - \frac{|OH|^2}{4}$$

$$|BN|^2 = \frac{|BH|^2 + |BO|^2}{2} - \frac{|OH|^2}{4}$$

$$|CN|^2 = \frac{|CH|^2 + |CO|^2}{2} - \frac{|OH|^2}{4}$$

The following equality well known:

$$|OH|^2 = R^2 - \frac{(a^2 + b^2 + c^2)}{9}$$

+

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$$\underbrace{|AN|^2+|BN|^2+|CN|^2}_{T}=\frac{|AH|^2+|BH|^2+|CH|^2+|AO|^2+|BO|^2+|CO|^2}{2}-\frac{3}{4}|OH|^2$$

$$T=\frac{(2R\cos A)^2+(2R\cos B)^2+(2R\cos C)^2+3R^2}{2}-\frac{3}{4}\left(R^2-\frac{a^2+b^2+c^2}{9}\right)$$

$$T=2R^2\cdot(\cos^2 A+\cos^2 B+\cos^2 C)+\frac{3R^2}{2}-\frac{3R^2}{4}+\frac{a^2+b^2+c^2}{12}$$

$$T=2R^2[3-(\sin^2 A+\sin^2 B+\sin^2 C)]+\frac{3R^2}{4}+\frac{a^2+b^2+c^2}{12}$$

$$T=2R^2\left(3-\frac{a^2+b^2+c^2}{4R^2}\right)+\frac{3R^2}{4}+\frac{a^2+b^2+c^2}{12}$$

$$T=\frac{27R^2}{4}-\frac{a^2+b^2+c^2}{2}+\frac{a^2+b^2+c^2}{12}$$

$$T=\frac{27R^3}{4}-\frac{5}{12}\cdot(a^2+b^2+c^2)\quad (*)$$

$$36r^2\le a^2+b^2+c^2\le 9R^2\Rightarrow R^2\ge\frac{a^2+b^2+c^2}{9}\Rightarrow$$

$$\Rightarrow T\ge\frac{27}{4}\left(\frac{a^2+b^2+c^2}{9}\right)-\frac{S}{12}(a^2+b^2+c^2)$$

$$\Rightarrow T\ge\frac{a^2+b^2+c^2}{3}\ge\frac{36r^2}{3}=12r^2$$

$$T\le\frac{27R^2}{4}-\frac{S}{12}\cdot 36r^2\Rightarrow T\le 3R^2\le\frac{27R^2}{4}-15r^2\Leftrightarrow 2r\le R$$

407. In $\Delta\ ABC$:

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\ge 1+\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}+\frac{\cos^2\frac{B}{2}}{\cos^2\frac{A}{2}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

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Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un $\Delta\ ABC$

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq 1+\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}+\frac{\cos^2\frac{B}{2}}{\cos^2\frac{A}{2}}$$

Recordar las siguientes identidades en un $\Delta\ ABC$

$$r_a=p\tan\frac{A}{2},\qquad r_b=p\tan\frac{B}{2},\qquad r_c=p\tan\frac{C}{2},$$

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

$$\Leftrightarrow \tan^2\frac{A}{2}+1=\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right),$$

$$\tan^2\frac{B}{2}+1=\left(\tan\frac{B}{2}+\tan\frac{A}{2}\right)\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

Siendo $x=\tan\frac{A}{2}>0, y=\tan\frac{B}{2}>0, z=\tan\frac{C}{2}>0$

La desigualdad propuesta es equivalente

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq 1+\frac{y^2+1}{x^2+1}+\frac{x^2+1}{y^2+1}\Leftrightarrow$$

$$\Leftrightarrow \frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq 1+\frac{(y+x)(y+z)}{(x+z)(x+y)}+\frac{(x+z)(x+y)}{(y+x)(y+z)}$$

$$\Leftrightarrow \frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq 1+\frac{y+z}{x+z}+\frac{x+z}{y+x}$$

$$\Leftrightarrow \frac{x}{y}(y+z)+\frac{y}{z}(y+z)+\frac{z}{x}(y+x)\geq y+z+\frac{(y+z)^2}{x+z}+x+z$$

$$\Leftrightarrow x+\frac{xz}{y}+\frac{y^2}{z}+y+\frac{zy}{x}+\frac{z^2}{x}\geq 2z+x+y+\frac{(y+z)^2}{x+z}$$

$$\Leftrightarrow \frac{xz}{y}+\frac{yz}{x}+\frac{y^2}{z}+\frac{z^2}{x}\geq 2z+\frac{(y+z)^2}{x+z}$$

Aplicando la desigualdad de MA $\geq$ MG y Cauchy

$$\frac{xz}{y}+\frac{yz}{x}\geq 2z \quad \textbf{(A)}$$

$$\frac{y^2}{z}+\frac{z^2}{x}\geq\frac{(y+z)^2}{x+z} \quad \textbf{(B)}$$

Sumando (A)+(B)

$$\Rightarrow \frac{xz}{y}+\frac{yz}{x}+\frac{y^2}{z}+\frac{z^2}{x}\geq 2z+\frac{(y+z)^2}{x+z} \quad \textbf{(LQQD)}$$

408. In any $\Delta\ ABC$ with $\prod(a^2-bc)\neq 0$:

$$\frac{9(a^2+b^2+c^2-ab-bc-ca)}{\sum(a^2-bc)^2}<\sum\frac{1}{a^2+bc}<\frac{1}{4Rr}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en totdo triángulo ABC, de tal manera que

$$(a^2-bc)(b^2-ca)(c^2-ab)\neq 0$$

$$\frac{9(a^2+b^2+c^2-ab-bc-ca)}{(a^2-bc)^2+(b^2-ca)^2+(c^2-ab)^2}<\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}<\frac{1}{4Rr}$$

De la condición se puede afirmar que

$$a^2\neq bc, \quad b^2\neq ca, \quad c^2\neq ab \Leftrightarrow a\neq b\neq c$$

Además

$$a^2+b^2+c^2-ab-bc-ca=\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2>0$$

Como $a\neq b\neq c$; Aplicando la desigualdad de Cauchy

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}>\frac{9}{a^2+bc+b^2+ca+c^2+ab}$$

Por último

$$\frac{9}{a^2+b^2+c^2+ab+bc+ca}=\frac{9(a^2+b^2+c^2-ab-bc-ca)}{(a^2+b^2+c^2+ab+bc+ca)(a^2+b^2+c^2-ab-bc-ca)}=$$

$$= \frac{9\left(\sum a^2 - ab\right)}{\sum (a^2 - bc)^2} \quad \text{(LQQD)}$$

donde → $(a^2 + b^2 + c^2 - ab - bc - ca)(a^2 + b^2 + c^2 + ab + bc + ca) =$

$$= (a^2 + b^2 + c^2)^2 - (ab + bc + ca)^2 = \sum (a^2 - bc)^2$$

Como a, b, c son lados de un $\Delta\, ABC$ se cumple

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}$$

Es necesario probar

$$\frac{1}{a^2 + bc} + \frac{1}{b^2 + ca} + \frac{1}{c^2 + ab} < \frac{1}{2ab} + \frac{1}{2bc} + \frac{1}{2ca}$$

Aplicando MA > MG

$$\frac{1}{a^2 + bc} + \frac{1}{b^2 + ca} + \frac{1}{c^2 + ab} < \frac{1}{2a\sqrt{bc}} + \frac{1}{2b\sqrt{ca}} + \frac{1}{2c\sqrt{ab}} <$$

$$< \frac{1}{4}\left(\frac{1}{ab} + \frac{1}{ac}\right) + \frac{1}{4}\left(\frac{1}{bc} + \frac{1}{ba}\right) + \frac{1}{4}\left(\frac{1}{ca} + \frac{1}{cb}\right) = \sum \frac{1}{2ab} = \frac{1}{4Rr}$$

409. In ΔABC:

$$\left(\frac{m_a}{ar_a} + \frac{m_b}{br_b} + \frac{m_c}{cr_c}\right)\left(\frac{ar_a}{m_a} + \frac{br_b}{m_b} + \frac{cr_c}{m_c}\right) \geq \frac{s\sqrt{3}}{r}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{m_a} \geq \frac{b}{m_b} \Leftrightarrow a^2(4m_b^2) \geq b^2(4m_a^2)$$

$$\Leftrightarrow a^2(2c^2 + 2a^2 - b^2) \geq b^2(2b^2 + 2c^2 - a^2)$$

$$\Leftrightarrow 2c^2(a^2 - b^2) + 2(a^2 + b^2)(a^2 - b^2) \geq 0$$

$$\Leftrightarrow (a^2 - b^2)(a^2 + b^2 + c^2) \geq 0 \overset{(1)}{\Leftrightarrow} a \geq b$$

Similarly, $\frac{b}{m_b} \geq \frac{c}{m_c} \Leftrightarrow b \geq c$ ***(2)***

WLOG, we may assume $a \geq b \geq c$

Then, (1), (2) $\Rightarrow \frac{a}{m_a} \geq \frac{b}{m_b} \geq \frac{c}{m_c}$***. Also,*** $r_a \geq r_b \geq r_c$

$$\therefore \frac{a}{m_a} \cdot r_a + \frac{b}{m_b} \cdot r_b + \frac{c}{m_c} \cdot r_c$$

$$\overset{Chebyshev}{\geq} \frac{1}{3}\left(\frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c}\right)(r_a + r_b + r_c)$$

$$= \left(\frac{4R + r}{3}\right)\left(\frac{a^2}{am_a} + \frac{b^2}{bm_b} + \frac{c^2}{cm_c}\right)$$

$$\underset{(i)}{\overset{Bergstrom}{\geq}} \left(\frac{4R + r}{3}\right) \cdot \frac{(a + b + c)^2}{(am_a + bm_b + cm_c)}$$

Again: $\because \frac{a}{m_a} \geq \frac{b}{m_b} \geq \frac{c}{m_c}, \therefore \frac{m_a}{a} \leq \frac{m_b}{b} \leq \frac{m_c}{c}$

And $\because r_a \geq r_b \geq r_c, \therefore \frac{1}{r_a} \leq \frac{1}{r_b} \leq \frac{1}{r_c}$

$$\therefore \frac{m_a}{a} \cdot \frac{1}{r_a} + \frac{m_b}{b} \cdot \frac{1}{r_b} + \frac{m_c}{c} \cdot \frac{1}{r_c}$$

$$\overset{Chebyshev}{\geq} \frac{1}{3}\left(\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c}\right)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right)$$

$$= \frac{1}{3r}\left(\frac{m_a^2}{am_a} + \frac{m_b^2}{bm_b} + \frac{m_c^2}{cm_C}\right) \underset{(ii)}{\overset{Bergstrom}{\geq}} \frac{1}{3r} \cdot \frac{(m_a + m_b + m_c)^2}{(am_a + bm_b + cm_c)}$$

(i)×(ii) $\Rightarrow LHS \overset{(iii)}{\geq} \left(\frac{4R+r}{9r}\right)\frac{(a+b+c)^2(m_a+m_b+m_c)^2}{(am_a+bm_b+cm_c)^2}$

Now, $\because a \geq b \geq c, \therefore m_a \leq m_b \leq m_c$

$$\therefore am_a + bm_b + cm_c \overset{Chebyshev}{\leq} \frac{1}{3}(a + b + c)(m_a + m_b + m_c)$$

$$\Rightarrow (am_a + bm_b + cm_c)^2 \leq \frac{(a + b + c)^2(m_a + m_b + m_c)^2}{9}$$

$$\Rightarrow \frac{1}{(am_a+bm_b+cm_c)^2} \geq \frac{9}{(a+b+c)^2(m_a+m_b+m_c)^2} \quad \textit{(iv)}$$

(iii), (iv) $\Rightarrow$ ***LHS***

$$\geq \left(\frac{4R+r}{9r}\right)(a+b+c)^2(m_a+m_b+m_c)^2 \cdot \frac{9}{(a+b+c)^2(m_a+m_b+m_c)^2}$$

$$= \frac{4R+r}{r} \geq \frac{s\sqrt{3}}{r} \quad \textit{(Trucht) (Proved)}$$

410. In $\Delta\ ABC, I$ - incentre:

$$\sum \sqrt{a} \cdot IA^2 \geq \frac{\sqrt{abc}(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Solution 2 by Khanh Hung Vu-Ho Chi Minh-Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

KLAMKIN INERTIAL MOMENT

Siendo a, b, c los lados de un triángulo ABC y PA, PB, PC son las distancias de un punto P en en plano ABC

Se cumple para todos los números R "x, y, z" se tiene lo siguiente:

$(x+y+z)(xPA^2+yPB^2+zPC^2) \geq yza^2+zxb^2+xyc^2$ ***... (A continuación lo demostraremos)***

La manera clásica es de la siguiente forma:

$$(xPA^{\rightarrow}+yPB^{\rightarrow}+zPC^{\rightarrow})^2 \geq 0$$

$\Rightarrow x^2PA^2+y^2PB^2+z^2PC^2+2xyPA^{\rightarrow}PB^{\rightarrow}+2yzPB^{\rightarrow}PC^{\rightarrow}+2zxPA^{\rightarrow}PC^{\rightarrow} \geq 0$ ***... (B)***

Desde que: $2PA^{\rightarrow}PB^{\rightarrow} = PA^2+PB^2-c^2, 2PB^{\rightarrow}PC^{\rightarrow} = PB^2+PC^2-a^2,$

$$2PA^{\rightarrow}PC^{\rightarrow} = PA^2+PC^2-b^2$$

Tenemos en ... (B)

$$\Rightarrow x^2PA^2 + y^2PB^2 + z^2PC^2 + xy(PA^2 + PB^2 - c^2) + yz(PB^2 + PC^2 - a^2) +$$
$$+zx(PA^2 + PC^2 - b^2) \geq 0$$
$$\Rightarrow (x^2PA^2 + xyPA^2 + xzPA^2) + (y^2PB^2 + yxPB^2 + yzPB^2) + (z^2PC^2 + zxPC^2 + zyPC^2) \geq$$
$$\geq yza^2 + zxb^2 + xyc^2$$
$$\Rightarrow (x + y + z)(xPA^2 + yPB^2 + zPC^2) \geq yza^2 + zxb^2 + xyc^2 \quad \textit{(LQQD)}$$

Sea $P = I$ (Incentro), $x = \sqrt{a} > 0, y = \sqrt{b} > 0, z = \sqrt{c} > 0$

$$\Leftrightarrow (\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a}IA^2 + \sqrt{b}IB^2 + \sqrt{c}IC^2) \geq a^2\sqrt{bc} + b^2\sqrt{ca} + c^2\sqrt{ab}$$
$$\Leftrightarrow \sqrt{a}IA^2 + \sqrt{b}IB^2 + \sqrt{c}IC^2 \geq \frac{\sqrt{abc}(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \quad \textit{(LQQD)}$$

Solution 2 by Khanh Hung Vu-Ho Chi Minh-Vietnam

In $\Delta\, ABC$, prove that $\sqrt{a} \cdot IA^2 + \sqrt{b} \cdot IB^2 + \sqrt{c} \cdot IC^2 \geq \frac{\sqrt{abc}(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$

We have $\forall\, a, b, c > 0$, we have only one point S satisfy:

$$\sqrt{a} \cdot \overrightarrow{SA} + \sqrt{b} \cdot \overrightarrow{SB} + \sqrt{c} \cdot \overrightarrow{SC} = \vec{0}$$

Applying Jacobi's theorem for I is the incenter of triangle ABC, we have:

$$\sqrt{a} \cdot IA^2 + \sqrt{b} \cdot IB^2 + \sqrt{c} \cdot IC^2 = (\sqrt{a} + \sqrt{b} + \sqrt{c}) \cdot IS^2 +$$
$$+ \frac{1}{\sqrt{a} + \sqrt{b} + \sqrt{c}}(\sqrt{a} \cdot AB^2 + \sqrt{bc} \cdot BC^2 + \sqrt{ca} \cdot CA^2)$$
$$\Rightarrow \sqrt{a} \cdot IA^2 + \sqrt{b} \cdot IB^2 + \sqrt{c} \cdot IC^2 = (\sqrt{a} + \sqrt{b} + \sqrt{c}) \cdot IS^2 +$$
$$+ \frac{1}{\sqrt{a} + \sqrt{b} + \sqrt{c}}(\sqrt{ab} \cdot c^2 + \sqrt{bc} \cdot a^2 + \sqrt{ca} \cdot b^2)$$
$$\Rightarrow \sqrt{a} \cdot IA^2 + \sqrt{b} \cdot IB^2 + \sqrt{c} \cdot IC^2 \geq \frac{1}{\sqrt{a} + \sqrt{b} + \sqrt{c}}(\sqrt{ab} \cdot c^2 + \sqrt{bc} \cdot a^2 + \sqrt{ca} \cdot b^2)$$
$$\Rightarrow \sqrt{a} \cdot IA^2 + \sqrt{b} \cdot IB^2 + \sqrt{c} \cdot IC^2 \geq \frac{\sqrt{abc}(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \quad \textit{(QED)}$$

The equality occurs when $S \equiv I \Rightarrow \frac{\sqrt{a}}{a} = \frac{\sqrt{b}}{b} = \frac{\sqrt{c}}{c} \Rightarrow a = b = c$

411. In $\Delta\ ABC$:

$$\sum\frac{(a^3+b^3)(a^5+b^5)}{a^2b^2(a+b)^2}\geq 4\sqrt{3}S$$

Proposed by D.M. Bătinețu Giurgiu, Neculai Stanciu – Romania

Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by Rozeta Atanasova-Skopje, Solution 3 by Sanong Hauerai-Nakon Pathom-Thailand

Solution 4 by Seyran Ibrahimov-Maasilli-Azerbaidian

Solution 1 by Ravi Prakash-New Delhi-India

For $a,b>0$, consider

$$2(a^3+b^3)(a^5+b^5)-a^2b^2(a+b)^2(a^2+b^2)$$

$$=2(a^8+a^3b^5+a^5b^3+b^8)-a^2b^2(a^2+b^2+2ab)(a^2+b^2)$$

$$=2(a^8+a^3b^5+a^5b^3+b^8)-a^2b^2(a^4+b^4+2a^2b^2+2a^3b+2ab^3)$$

$$=2(a^8+a^3b^5+a^5b^3+b^8)-a^6b^2-a^2b^6-2a^4b^4-2a^5b^3-2a^3b^5$$

$$=(a^8+b^8-2a^4b^4)+a^6(a^2-b^2)-b^6(a^2-b^2)$$

$$=(a^4-b^4)^2+(a^6-b^6)(a^2-b^2)\geq 0$$

$$\Rightarrow\frac{(a^3+b^3)(a^5+b^5)}{a^2b^2(a+b)^2}\geq\frac{1}{2}(a^2+b^2)$$

$$\Rightarrow\sum\frac{(a^3+b^3)(a^5+b^5)}{a^2b^2(a+b)^2}\geq\frac{1}{2}[(a^2+b^2)+(b^2+c^2)+(c^2+a^2)]$$

$$\Rightarrow\sum\frac{(a^3+b^3)(a^5+b^5)}{a^2b^2(a+b)^2}\geq E\ \textbf{where}\ E=a^2+b^2+c^2$$

$$\Rightarrow E-4\sqrt{3}S=a^2+b^2+c^2-4\sqrt{3}\left\{\frac{1}{2}ab\sin c\right\}$$

$$=a^2+b^2+a^2+b^2-2ab\cos c-2\sqrt{3}ab\sin c$$

$$=2(a^2+b^2)-4ab\sin\left(\frac{\pi}{3}+c\right)$$

$$\geq 2(a^2 + b^2 - 2ab) = 2(a-b)^2 \geq 0 \Rightarrow E \geq 4\sqrt{3}S$$

$$\textit{Thus, } \sum \frac{(a^3+b^3)(a^5+b^5)}{a^2b^2(a+b)^2} \geq 4\sqrt{3}S$$

Solution 2 by Rozeta Atanasova-Skopje

$$\sqrt[3]{\frac{a^3+b^3}{2}} \overset{(M_3 \geq M_1)}{\geq} \frac{a+b}{2} \Rightarrow a^3 + b^3 \geq \frac{(a+b)^3}{2^2} \quad \dots (1)$$

$$\sqrt[5]{\frac{a^5+b^5}{2}} \overset{(M_5 \geq M_1)}{\geq} \frac{a+b}{2} \Rightarrow a^5 + b^5 \geq \frac{(a+b)^5}{2^4} \dots (2)$$

$$\frac{1}{a^2b^2} = \frac{1}{\left(\sqrt{ab}\right)^4} \overset{AM-GM}{\geq} \frac{2^4}{(a+b)^4} \dots (3)$$

From (1), (2) and (3) $\Rightarrow$

$$LHS \geq \sum \left(\frac{a+b}{2}\right)^2 \geq ab + ac + bc = 2S\left(\frac{1}{\sin C} + \frac{1}{\sin B} + \frac{1}{\sin A}\right)$$

$$\overset{Jensen}{\geq} 2S \cdot 3 \cdot \frac{1}{\sin \frac{\pi}{3}} = 4\sqrt{3}S = RHS$$

Solution 3 by Sanong Hauerai-Nakon Pathom-Thailand

$$\frac{(a^3+b^3)(a^5+b^5)}{a^2b^2(a+b)^2} + \frac{(b^3+c^3)(b^5+c^5)}{b^2c^2(b+c)^2} + \frac{(c^3+a^3)(c^5+a^5)}{c^2a^2(c+a)^2}$$

$$\geq \frac{(a^4+b^4)^2}{a^2b^2(a+b)^2} + \frac{(b^4+c^4)^2}{b^2c^2(b+c)^2} + \frac{(c^4+a^4)^2}{c^2a^2(c+a)^2}$$

$$\geq \frac{2a^2b^2(a^4+b^4)}{a^2b^2(a+b)^2} + \frac{2b^2c^2(b^4+c^4)}{b^2c^2(b+c)^2} + \frac{2c^2a^2(c^4+a^4)}{c^2a^2(c+a)^2}$$

$$= \frac{2(a^2+b^2)^2}{2(a+b)^2} + \frac{2(b^2+c^2)^2}{2(b+c)^2} + \frac{2(c^2+a^2)^2}{2(c+a)^2}$$

$$\geq \frac{(a+b)^2(a^2+b^2)}{2(a+b)^2} + \frac{(b+c)^2(b^2+c^2)}{2(b+c)^2} + \frac{(c+a)^2(c^2+a^2)}{2(b+c)^2}$$

$$= a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} \geq 4\sqrt{3}S$$

Solution 4 by Seyran Ibrahimov-Maasilli-Azerbaidian

Ionescu – Weizenböck Lemma 1: $a^2+b^2+c^2 \geq 4\sqrt{3}S$

Lemma 2: $a^3+b^3 \geq \frac{1}{2}(a+b)(a^2+b^2)$

Lemma 2: $a^5+b^5 \geq \frac{1}{16}(a+b)^5$

$$LHS \geq \sum \frac{1}{32} \cdot \frac{(a+b)^4(a^2+b^2)}{a^2b^2} \overset{AM-GM}{\geq} \sum \frac{1}{2} \cdot (a^2+b^2) \geq 4\sqrt{3}S$$

Proved

412. In $\Delta\, ABC$:

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 R^2 \geq \sqrt{abc}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru,

Solution 2 by Khanh Hung Vu-Ho Chi Minh-Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

KLAMKIN INERTIAL MOMENT

Siendo a, b, c los lados de un triángulo ABC y PA, PB, PC son las distancias de un punto P en en plano ABC

Se cumple para todos los números R " , y, z " se tiene lo siguiente:

$(x+y+z)(xPA^2+yPB^2+zPC^2) \geq yza^2+zxb^2+xyc^2$ ***.. (A continuación lo demostraremos)***

La manera clásica es de la siguiente forma:

$$(xPA^{\rightarrow}+yPB^{\rightarrow}+zPC^{\rightarrow})^2 \geq 0$$

$\Rightarrow x^2PA^2+y^2PB^2+z^2PC^2+2xyPA^{\rightarrow}PB^{\rightarrow}+2yzPB^{\rightarrow}PC^{\rightarrow}+2zxPA^{\rightarrow}PC^{\rightarrow} \geq 0$ ***… (B)***

Desde que: $2PA^{\rightarrow}PB^{\rightarrow} = PA^2+PB^2-c^2$,

$2PB^{\rightarrow}PC^{\rightarrow} = PB^2+PC^2-a^2$, $2PA^{\rightarrow}PC^{\rightarrow} = PA^2+PC^2-b^2$

Tenemos en … (B)

$$\Rightarrow x^2PA^2 + y^2PB^2 + z^2PC^2 + xy(PA^2 + PB^2 - c^2) + yz(PB^2 + PC^2 - a^2) + zx(PA^2 + PC^2 - b^2) \ge 0$$

$$\Rightarrow (x^2PA^2 + xyPA^2 + xzPA^2) + (y^2PB^2 + yxPB^2 + yzPB^2) + (z^2PC^2 + zxPC^2 + zyPC^2) \ge$$

$$\ge yza^2 + zxb^2 + xyc^2$$

$$\Rightarrow (x + y + z)(xPA^2 + yPB^2 + zPC^2) \ge yza^2 + zxb^2 + xyc^2 \quad \dots \textit{(LQQD)}$$

Siendo $P = 0$ (Circuncentro) $\Leftrightarrow OA = OB = OC = R$,

$$x = \sqrt{a} > 0, y = \sqrt{b} > 0, z = \sqrt{c} > 0$$

$$\Leftrightarrow (\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a}OB^2 + \sqrt{b}OB^2 + \sqrt{c}OC^2) \ge a^2\sqrt{bc} + b^2\sqrt{ca} + c^2\sqrt{ab}$$

$$\Leftrightarrow (\sqrt{a} + \sqrt{b} + \sqrt{c})^2R^2 \ge \sqrt{abc}(a\sqrt{a} + b\sqrt{b} + c\sqrt{c})$$

(LQQD)

Solution 2 by Khanh Hung Vu-Ho Chi Minh-Vietnam

In ΔABC, prove that $(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 . R^2 \ge \sqrt{abc}(a\sqrt{a} + b\sqrt{b} + c\sqrt{c})$

We have $\forall a, b, c > 0$**, we have only one point S satisfy:**

$$\sqrt{a} \cdot \overrightarrow{SA} + \sqrt{b} \cdot \overrightarrow{SB} + \sqrt{c} \cdot \overrightarrow{SC} = \vec{0}$$

Applying Jacobi's theorem for O is the circumcenter of triangle ABC, we have:

$$\sqrt{a} \cdot OA^2 + \sqrt{b} \cdot OB^2 + \sqrt{c} \cdot OC^2 = (\sqrt{a} + \sqrt{b} + \sqrt{c}) \cdot OS^2 +$$

$$+ \frac{1}{\sqrt{a} + \sqrt{b} + \sqrt{c}}(\sqrt{ab} \cdot AB^2 + \sqrt{bc} \cdot BC^2 + \sqrt{ca} \cdot CA^2)$$

$$\Rightarrow \sqrt{a} \cdot OA^2 + \sqrt{b} \cdot OB^2 + \sqrt{c} \cdot OC^2 = (\sqrt{a} + \sqrt{b} + \sqrt{c}) \cdot OS^2 +$$

$$+ \frac{1}{\sqrt{a} + \sqrt{b} + \sqrt{c}}(\sqrt{ab} \cdot c^2 + \sqrt{bc} \cdot a^2 + \sqrt{ca} \cdot b^2)$$

$$\Rightarrow (\sqrt{a} + \sqrt{b} + \sqrt{c})^2 \cdot R^2 = (\sqrt{a} + \sqrt{b} + \sqrt{c})^2 \cdot OS^2 + \sqrt{abc}(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}) \ge$$

$$\ge \sqrt{abc}(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}) \textbf{ (QED)}$$

The equality occurs when $S \equiv O \Rightarrow a = b = c$

413. In $\Delta\, ABC, I -$ incentre, $R_a, R_b, R_c -$ circumradii in $\Delta\, BIC, \Delta\, CIA, \Delta\, AIB$

$$3 \le \frac{R_a^2 + R_b^2 + R_c^2}{R^2} \le \frac{r_a^2 + r_b^2 + r_c^2}{r_a r_b + r_b r_c + r_c r_a} + 2$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru, Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam, Solution 3 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $I -$ Incentro, R_a, R_b, R_c, circunradio en los triángulos BIC, CIA, AIB. Probar en un triángulo ABC

$$3 \le \frac{R_a^2 + R_b^2 + R_c^2}{R^2} \le \frac{r_a^2 + r_b^2 + r_c^2}{r_a r_b + r_b r_c + r_c r_a} + 2$$

Tener en cuenta las siguientes identidades

$$R_a = 2R\sin\frac{A}{2}, R_b = 2R\sin\frac{B}{2}, R_c = 2R\sin\frac{C}{2}$$

$$r_a r_b + r_b r_c + r_c r_a = p^2, r_a = p\tan\frac{A}{2}, r_b = p\tan\frac{B}{2}, r_c = p\tan\frac{C}{2}$$

Además

$$m^2 + n^2 + p^2 + 2mnp = 1, \text{ donde } m = \sin\frac{A}{2}, n = \sin\frac{B}{2}, p = \sin\frac{C}{2}$$

Como $m, n, p > 0$; Aplicando $MA \ge MG$

$$1 = m^2 + n^2 + p^2 + 2mnp \ge 4\sqrt[3]{2m^3n^3p^3} \Leftrightarrow mnp \le \frac{1}{8}$$

Lo cual implica

$$\frac{R_a^2 + R_b^2 + R_c^2}{R^2} = 4(m^2 + n^2 + p^2) = 4(1 - 2mnp) \ge 4\left(1 - \frac{1}{4}\right) = 3$$

En el RHS es equivalente

$$4\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)\le \tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}+2$$

$$\Leftrightarrow 4\left(1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\le \tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}+2$$

$$\Leftrightarrow \tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}+8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ge 2$$

LEMMA

Siendo $x,y,z\ge 0$, se cumple la siguiente desigualdad

$$\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{8xyz}{(x+y)(y+z)(z+x)}\ge 2 \quad (A)$$

Realizamos los siguientes cambios de variables

$$x=\tan\frac{A}{2}>0,\ y=\tan\frac{B}{2}>0,\ z=\tan\frac{C}{2}>0$$

Reemplazando en (A) se obtiene

$$\Rightarrow \tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}+8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ge 2$$

(LQQD)

Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam

Prove that: $3\le \frac{R_a^2+R_b^2+R_c^2}{R^2}\le \frac{r_a^2+r_b^2+r_c^2}{r_ar_b+r_br_c+r_cr_a}+2$

We prove: $3\le \frac{R_a^2+R_b^2+R_c^2}{R^2}$ ***(1)***

Let $BC=a; CA=b; AB=c; S=$ ***area of triangle***

$$IB=\sqrt{\frac{ac(a+c-b)}{a+b+c}};\ IC=\sqrt{\frac{ab(a+b-c)}{a+b+c}};\ S_{BIC}=\frac{a\cdot r}{2}$$

$$\Rightarrow R_a=\frac{IB\cdot IC\cdot BC}{4S_{BIC}}=\frac{\sqrt{\frac{ac(a+c-b)}{a+b+c}}\cdot\sqrt{\frac{ab(a+b-c)}{a+b+c}}\cdot a}{2ar}$$

$$\Leftrightarrow R_a^2 = \frac{\frac{a^2bc(a+c-b)(a+b-c)}{(a+b+c)^2}}{4r^2} = \frac{\frac{a^2bc(a+c-b)(a+b-c)}{(a+b+c)^2}}{\frac{16S^2}{(a+b+c)^2}}$$

$$\Leftrightarrow R_a^2 = \frac{a^2bc(a+c-b)(a+b-c)}{\prod(b+c-a)\cdot(\sum a)}$$

$$\Rightarrow \sum \frac{R_a^2}{R^2} = \sum \frac{\frac{a^2bc(a+c-b)(a+b-c)}{\prod(b+c-a)\cdot(\sum a)}}{\frac{(abc)^2}{(\sum a)\prod(b+c-a)}} = \sum \frac{a(a+c-b)(a+b-c)}{abc}$$

Therefore: $3 \le \frac{R_a^2+R_b^2+R_c^2}{R^2}$

$$\Leftrightarrow 3 \le \frac{\sum a(a+c-b)(a+b-c)}{abc} \Leftrightarrow \sum a(a^2-(b-c)^2) \ge 3abc$$

$$\Leftrightarrow \sum a^3 + 3abc \ge \sum ab\,(a+b)$$

$\Leftrightarrow \sum a\,(a-b)(a-c) \ge 0$ ***(True because Schur)*** $\Rightarrow$ ***(1) True***

We prove: $\frac{\sum R_a^2}{R^2} \le \frac{\sum r_a^2}{\sum r_a r_b} + 2$ ***(3)***

$$\Leftrightarrow \frac{\sum a(b+a-c)(a+c-b)}{abc} \le \frac{(\sum r_a)^2}{\sum r_a r_b}$$

$$\Leftrightarrow \frac{\sum a\,(b+a-c)(a+c-b)}{abc} \le \frac{\left(\sum \frac{2s}{b+c-a}\right)^2}{\sum \frac{2s}{c+a-b}\cdot\frac{2s}{b+c-a}}$$

$\Leftrightarrow \sum \frac{a(b+a-c)(a+c-b)}{abc} \le \frac{[\sum(b+c-a)(c+a-b)]^2}{(\sum a)\prod(b+c-a)}$ ***(2)***

Let $\begin{cases} b+c-a=2x \\ c+a-b=2y \\ a+b-c=2z \end{cases} \Leftrightarrow \begin{cases} a=y+z \\ b=z+x \\ c=x+y \end{cases}$

(2) $\Leftrightarrow \sum \frac{(y+z)\cdot 4yz}{\prod(x+y)} \le \frac{(\sum xy)^2}{xyz\sum x}$

$$\Leftrightarrow \frac{(\sum xy)^2}{xyz\sum x} - 3 \geq \frac{4\sum yz(y+z)}{\prod(x+y)} - 3$$

$$\Leftrightarrow \sum (x-y)^2 \left(z^2 \cdot \prod(x+y) - 2xyz^2\left(\sum x\right)\right) \geq 0$$

$$\begin{cases} s_c = z^2\prod(x+y) - 2xyz^2\left(\sum x\right) \\ s_a = x^2\prod(x+y) - 2x^2xyz\left(\sum x\right) \\ s_b = y^2 \cdot \prod(x+y) - 2xy^2z\left(\sum x\right) \end{cases}$$

Suppose: $x \geq y \geq z > 0$

$$S_b = y^2\left[\prod(x+y) - 2xz\left(\sum x\right)\right]$$

$$S_b = y^2 \cdot [x^2(y-z) + x(y^2 - z^2) + y^2z + yz^2] > 0$$

Similar: $S_a > 0 \Rightarrow S_b + S_a > 0$

$$S_b + S_c = \prod(x+y)\,(y^2 + z^2) - 2xyz\left(\sum x\right)(y+z)$$

$$\geq \prod(x+y) \cdot 2yz - 2xyz\left(\sum x\right) \cdot (y+z)$$

$$= (y+z) \cdot 2yz \cdot yz = 2y^2z^2(y+z) > 0$$

Therefore: $S_b > 0, S_b + S_c > 0; S_b + S_a > 0$ ***By SOS*** $\Rightarrow$ ***QED***

Then (1), (2), (3) $\Rightarrow 3 \leq \frac{\sum R_a^2}{R^2} \leq \frac{\sum r_a^2}{\sum r_a r_b} + 2$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$R_a = 2R\sin\frac{A}{2}, R_b = 2R\sin\frac{B}{2}, R_c = 2R\sin\frac{C}{2}, \sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}$$

$$\sin\frac{B}{2} = \sqrt{\frac{(p-c)(p-a)}{ca}} \text{ and } \sin\frac{C}{2} = \sqrt{\frac{(p-a)(p-b)}{ab}}$$

$$\therefore \frac{1}{R^2}\sum_{cyc} R_a^2 = 4\sum_{cyc}\frac{(p-a)(p-b)}{ab} = \frac{4}{abc}\sum_{cyc} a(p-a)(p-b)$$

$$= \frac{1}{Rrp}(2p^3 - 2p(p^2 + r^2 + 4Rr) + 12Rrp) = \frac{2(2Rr - r^2)}{Rr}$$

$$\geq 3\ [\because R \geq 2r]$$

$$2 + \frac{r_a^2 + r_b^2 + r_c^2}{r_a r_b + r_b r_c + r_c r_a} \geq \frac{1}{R^2}\sum_{cyc} R_a^2 = \frac{2(2Rr - r^2)}{Rr}$$

$$\Leftrightarrow \frac{\left(\sum_{cyc}\frac{\Delta}{p-a}\right)^2}{\sum_{cyc}\frac{\Delta^2}{(p-a)(p-b)}} \geq \frac{2(2Rr - r^2)}{Rr} = \frac{2(2R - r)}{R}$$

$$\Leftrightarrow \frac{1}{p(p-a)(p-b)(p-c)}\left(\sum_{cyc}(p-a)(p-b)\right)^2 \geq \frac{2(2R - r)}{R}$$

$$\Leftrightarrow \frac{R(r+4R)^2}{2(2R-r)} \geq p^2, \text{ which is true}$$

414. In $\Delta\ ABC$:

$$\frac{a^2\sin^2 A}{\sin B\sin C} + \frac{b^2\sin^2 B}{\sin C\sin A} + \frac{c^2\sin^2 C}{\sin A\sin B} \geq 36r^2$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru, Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam, Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$\frac{a^2\sin^2 A}{\sin B\sin C} + \frac{b^2\sin^2 B}{\sin C\sin A} + \frac{c^2\sin^2 C}{\sin A\sin B} \geq 36r^2$$

La desigualdad propuesta es equivalente

$$\Leftrightarrow \frac{a^2\cdot\frac{a^2}{4R^2}}{\frac{bc}{4R^2}}+\frac{b^2\cdot\frac{b^2}{4R^2}}{\frac{ca}{4R^2}}+\frac{c^2\cdot\frac{c^2}{4R^2}}{\frac{ab}{4R^2}}\ge 36r^2$$

$$\frac{a^4}{bc}+\frac{b^4}{ca}+\frac{c^4}{ab}\ge 36r^2$$

Aplicando la desigualdad de Cauchy y $MA\ge MG$

$$\left(\frac{a^4}{bc}+\frac{b^4}{ca}+\frac{c^4}{ab}\right)\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)\ge\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)^2\ge 9$$

$$\Rightarrow\left(\frac{a^4}{bc}+\frac{b^4}{ca}+\frac{c^4}{ab}\right)\left(\frac{1}{2Rr}\right)\ge 9\Leftrightarrow\frac{a^4}{bc}+\frac{b^4}{ca}+\frac{c^4}{ab}\ge 18Rr\ge 36r^2$$

(Válido por la desigualdad de Euler)

Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam

In Δ ABC. Prove that: $\frac{a^2\sin^2 A}{\sin B\sin C}+\frac{b^2\sin^2 B}{\sin C\sin A}+\frac{c^2\sin^2 C}{\sin A\sin B}\ge 36r^2$

$$\Leftrightarrow\sum\frac{a^2\left(\frac{a}{2R}\right)^2}{\frac{b}{2R}\cdot\frac{c}{2R}}\ge 36\cdot\left(\frac{2S}{a+b+c}\right)^2\quad (S=\text{area of } ABC)$$

$$\Leftrightarrow\sum\frac{a^4}{bc}\ge 9\cdot\frac{16S^2}{(a+b+c)^2}=9\cdot\frac{(a+b+c)\prod(b+c-a)}{(a+b+c)^2}$$

$$\Leftrightarrow\sum\frac{a^4}{bc}\ge\frac{9\prod(b+c-a)}{a+b+c}\quad (1)$$

By AM-GM: $\prod(b+c-a)\le\frac{(\sum(b+c-a))^3}{27}=\frac{(\sum a)^3}{27}$

$$\Rightarrow\frac{9\prod(b+c-a)}{\sum a}\le\frac{9(\sum a)^3}{27\sum a}=\frac{(\sum a)^2}{3}\le\sum a^2\quad (2)$$

$$\sum\frac{a^4}{bc}\ge\frac{\left(\sum a^2\right)^2}{\sum bc}\ge\frac{\left(\sum a^2\right)^2}{\sum a^2}=\sum a^2\quad (3)$$

(2), (3) $\Rightarrow\sum\frac{a^4}{bc}\ge\frac{9\prod(b+c-a)}{\sum a}\Rightarrow$ ***(1) true ⇒ QED***

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{a^2 \cdot \sin^2 A}{\sin B \cdot \sin C} \cdot \sum (\sin B \cdot \sin C) \overset{CBS}{\geq}$$

$$\geq \left(\sum a \cdot \sin A\right)^2 = \frac{1}{4R^2} \cdot \left(\sum a^2\right)^2 = \frac{1}{4R^2} \cdot (p^2 - 4Rr - r^2)^2 \geq$$

$$\overset{GERRETSEN}{\geq} \frac{1}{R^2} \cdot (16Rr - 5r^2 - 4Rr - r^2)^2 = \frac{36r^2}{R^2} \cdot (2R - r)^2$$

$$\frac{36r^2 \cdot (2R-r)^2}{R^2} \geq 36r^2 \cdot \sum (\sin A \cdot \sin B) \quad \textbf{(ASSURE)}$$

$$\frac{(2R-r)^2}{R^2} \geq \sum \sin A \cdot \sin B = \frac{1}{4R^2} \cdot (ab + bc + ca)$$

$$4 \cdot (2R - r)^2 \geq ab + bc + ca = p^2 + 4Rr + r^2$$

$$16R^2 - 20Rr + 3r^2 \geq p^2$$

$$16R^2 - 20Rr + 3r^2 \overset{Euler}{\geq} 4R^2 + 24Rr - 20Rr + 3r^2 =$$

$$= 4R^2 + 4Rr + 3r^2 \geq p^2 \quad \textbf{(GERRETSEN)}$$

415. If in $\Delta\ ABC$, I - incentre, R_a, R_b, R_c – circumradii of $\Delta\ BIC, \Delta\ CIA, \Delta\ AIB$

$$2 \leq \frac{R_a}{r_a} + \frac{R_b}{r_b} + \frac{R_c}{r_c} \leq \frac{R}{r}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru, Solution 2 by George Apostolopoulos-Messolonghi-Greece, Solution 3 by Soumava Chakraborty-Kolkata-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un $\Delta\ ABC$, siendo I – Incentro, R_a, R_b, R_c los circunradios de los triángulos $\Delta\ BIC, \Delta\ CIA, \Delta\ AIB$

Probar que

$$2 \leq \frac{R_a}{r_a} + \frac{R_b}{r_b} + \frac{R_c}{r_c} \leq \frac{R}{r}$$

Siendo I – incentro, tener en cuenta las siguientes identidades

$$R_a = 2R\sin\frac{A}{2}, \quad R_b = 2R\sin\frac{B}{2}, \quad R_c = 2R\sin\frac{C}{2}$$

Además en un $\Delta\, ABC$ se cumple las siguientes identidades y desigualdades

$$r_a = p\tan\frac{A}{2}, \quad r_b = p\tan\frac{B}{2}, \quad r_c = p\tan\frac{C}{2}$$

$$p \geq 3\sqrt{3}r, \frac{p}{R} = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} =$$

$$= \sin A + \sin B + \sin C \leq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2}$$

El RHS es equivalente

$$\frac{R_a}{r_a} + \frac{R_b}{r_b} + \frac{R_c}{r_c} = \frac{2R}{p}\left(\frac{\sin\frac{A}{2}}{\tan\frac{B}{2}} + \frac{\sin\frac{B}{2}}{\tan\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\tan\frac{C}{2}}\right) = \frac{2R}{p}\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) \leq$$

$$\leq \frac{2R}{3\sqrt{3}r}\cdot\frac{3\sqrt{3}}{2} = \frac{R}{r}$$

El LHS es equivalente

$$\frac{R_a}{r_a} + \frac{R_b}{r_b} + \frac{R_c}{r_c} = \frac{2R}{p}\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) \geq \frac{2R}{p}\cdot 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{2R}{p}\cdot\frac{p}{R} = 2$$

(LQQD)

Solution 2 by George Apostolopoulos-Messolonghi-Greece

It is well-known that $R_aR_bR_c = 2R^2r$,

$$R_a = 2R\sin\frac{A}{2}, R_b = 2R\sin\frac{B}{2}, R_c = 2R\sin\frac{C}{2}$$

$$\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2} = \frac{r}{4R}, \text{ and } \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2}$$

Also $R_a = s\cdot\tan\frac{A}{2},\dots$ ***and*** $r_a\cdot r_b\cdot r_c = r\cdot s^2$ $\left(s=\frac{a+b+c}{2}\right)$

So, by AM – GM Inequality, we get

$$\frac{R_a}{r_a}+\frac{R_b}{r_b}+\frac{R_c}{r_c}\geq 3\sqrt[3]{\frac{R_a\cdot R_b\cdot R_c}{r_a\cdot r_b\cdot r_c}} = 3\sqrt[3]{\frac{2R^2r}{r\cdot s^2}} = 3\sqrt[3]{\frac{2R^2}{s^2}}\geq$$

$$3\sqrt[3]{\frac{2R^2}{\left(\frac{3\sqrt{3}}{2}R\right)^2}} = 3\sqrt[3]{\frac{8}{27}} = 2$$

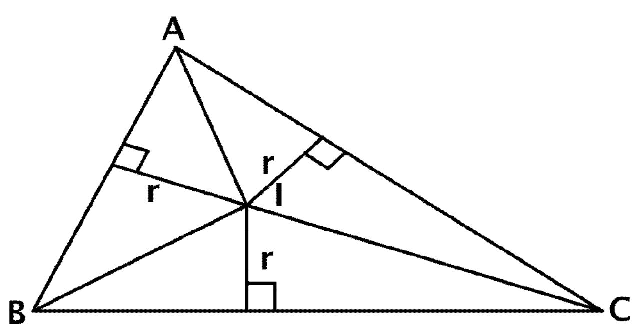
Also, we have $\frac{R_a}{r_a} = \frac{2R\sin\frac{A}{2}}{s\cdot\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}} = \frac{2R}{s}\cdot\cos\frac{A}{2}\leq\frac{2R}{3\sqrt{3}r}\cos\frac{A}{2}$

So $\frac{R_a}{r_a}+\frac{R_b}{r_b}+\frac{R_c}{r_c}\leq\frac{2R}{3\sqrt{3}r}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\leq\frac{2R}{3\sqrt{3}r}\cdot\frac{3\sqrt{3}}{2} = \frac{R}{2}$

Namely $2\leq\frac{R_a}{r_a}+\frac{R_b}{r_b}+\frac{R_c}{r_c}\leq\frac{R}{r}$

Equality holds when the triangle ABC ***is equilateral.***

Solution 3 by Soumava Chakraborty-Kolkata-India



$$2\overset{(a)}{\leq}\frac{R_a}{r_a}+\frac{R_b}{r_b}+\frac{R_c}{r_c}\overset{(b)}{\leq}\frac{R}{r}$$

$$AI=\frac{r}{\sin\frac{A}{2}};\quad BI=\frac{r}{\sin\frac{B}{2}};\quad CI=\frac{r}{\sin\frac{C}{2}}$$

$\because$ ***circumradius of Δ*** $= \frac{\text{product of sides}}{\text{area}}$

$$\therefore R_a = \frac{BI \cdot CI \cdot BC}{4\, area\, (\Delta BIC)} = \frac{\frac{r}{\sin\frac{B}{2}} \cdot \frac{r}{\sin\frac{C}{2}} \cdot a}{4\frac{1}{2} area}$$

$$= \frac{r^2 a}{2 area \sin\frac{B}{2}\sin\frac{C}{2}} = \frac{r}{2\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\therefore \frac{R_a}{r_a} = \frac{r}{2\sin\frac{B}{2}\sin\frac{C}{2} s\tan\frac{A}{2}} = \frac{r\cos\frac{A}{2}}{2s\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}$$

Similarly, $\frac{R_b}{r_b} = \frac{r\cos\frac{A}{2}}{2s\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$ ***and,*** $\frac{R_c}{r_c} = \frac{r\cos\frac{C}{2}}{2s\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$

$$\therefore \sum\frac{R_a}{a} = \frac{r}{2s\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left(\sum\cos\frac{A}{2}\right)$$

$$= \frac{r}{2s}\sqrt{\frac{bc}{(s-b)(s-c)}}\cdot\sqrt{\frac{ca}{(s-c)(s-a)}}\cdot\sqrt{\frac{ab}{(s-a)(s-b)}}\cdot\sum\cos\frac{A}{2}$$

$$= \frac{r}{2s}\cdot\frac{abc}{(s-a)(s-b)(s-c)}\cdot\sum\cos\frac{A}{2}$$

$$= \frac{r}{2s}\cdot\frac{4Rrs^2}{r^2s^2}\cdot\sum\cos\frac{A}{2} \overset{(1)}{\le} \frac{2R}{s}\left(\sum\cos\frac{A}{2}\right)$$

$$\overset{A-G}{\ge} \frac{2R}{s}\cdot 3\sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$= \frac{6R}{s}\sqrt[3]{\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ca}}\cdot\sqrt{\frac{s(s-c)}{ab}}}$$

$$=\frac{6R}{s}\sqrt[3]{\frac{s}{4Rrs}\cdot rs}=\frac{6R}{s}\sqrt[3]{\frac{s}{4R}}\overset{?}{\geq}2\Leftrightarrow\frac{216R^3}{s^3}\cdot\frac{s}{4R}\overset{?}{\geq}8\Leftrightarrow 27R^2\overset{?}{\geq}4s^2$$

$$\Leftrightarrow 2s\overset{?}{\leq}3\sqrt{3}R\Leftrightarrow s\overset{?}{\leq}\frac{3\sqrt{3}R}{2}$$ → ***true, by Mitrinovic ⇒ (a) is true***

$$(1)\Rightarrow\sum\frac{R_a}{r_a}=\frac{2R}{s}\left(\sum\cos\frac{A}{2}\right)\overset{C-B-S}{\leq}\frac{2R}{s}\sqrt{3}\sqrt{\sum\cos^2\frac{A}{2}}$$

$$=\frac{2R}{s}\sqrt{\frac{3}{2}}\sqrt{1+\cos A+1+\cos B+1+\cos C}$$

$$=\frac{2R}{s}\cdot\sqrt{\frac{3}{2}}\sqrt{3+1+\frac{r}{R}}=\frac{2R}{s}\cdot\sqrt{\frac{3}{2}}\sqrt{\frac{4R+r}{R}}\overset{?}{\leq}\frac{R}{r}$$

$$\Leftrightarrow\frac{4}{s^2}\cdot\frac{3}{2}\cdot\frac{4R+r}{R}\overset{?}{\leq}\frac{1}{r^2}$$

$$\Leftrightarrow s^2R\overset{?}{\geq}6r^2(4R+r)\quad\textbf{(2)}$$

Gerretsen ⇒ LHS of (2) $\geq R(16Rr-5r^2)\overset{?}{\geq}6r^2(4R+r)$

$$\Leftrightarrow R(16R-5r)\overset{?}{\geq}6r(4R+r)$$

$$\Leftrightarrow 16R^2-29Rr-6r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)(16R+3r)\overset{?}{\geq}0$$

→ ***true by Euler ⇒ (2) is true ⇒ (b) is true***

416. If $t\in(0,\pi)$ then in $\Delta\ ABC$:

$$\frac{a^2\sqrt{a}}{\sqrt{b}\sin^2 t+\sqrt{c}\cos^2 t}+\frac{b^2\sqrt{b}}{\sqrt{c}\sin^2 t+\sqrt{a}\cos^2 t}+\frac{c^2\sqrt{c}}{\sqrt{a}\sin^2 t+\sqrt{b}\cos^2 t}\geq 4\sqrt{3}S$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam, Solution 2 by Nirapada Pal-Jhargram-India

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

If $t \in (0,\pi)$***, then in*** $\Delta\ ABC$: $\sum \frac{a^2\sqrt{a}}{\sqrt{b}\sin^2 t+\sqrt{c}\cdot\cos^2 t} \geq 4\sqrt{3}S$

By Cauchy – Schwarz:

$$\sum \frac{a^2\sqrt{a}}{\sqrt{b}\sin^2 t+\sqrt{c}\cdot\cos^2 t} = \sum \frac{a^3}{\sqrt{ab}\cdot\sin^2 t+\sqrt{ac}\cdot\cos^2 t} \geq$$

$$\geq \frac{\left(\sum\sqrt{a^3}\right)^2}{\sum\sqrt{ab}(\sin^2 t+\cos^2 t)} = \frac{\left(\sum\sqrt{a^3}\right)^2}{\sum\sqrt{ab}}$$

We prove: $\frac{\left(\sum\sqrt{a^3}\right)^2}{\sum\sqrt{ab}} \geq 4\sqrt{3}\cdot S$

$$\Leftrightarrow \left(\sum\sqrt{a}\right)^2 \geq \left(\sum\sqrt{ab}\right)\cdot\sqrt{3\left(\sum a\right)\prod(b+c-a)} \quad \textit{(1)}$$

By Hölder: $\left(\sum\sqrt{a^3}\right)\left(\sum\sqrt{a^3}\right)\cdot(1+1+1) \geq \left(\sum a\right)^3$

$$\Rightarrow \left(\sum\sqrt{a^3}\right)^2 \geq \frac{\left(\sum a\right)^3}{3} \geq \frac{\left(\sum\sqrt{ab}\right)\cdot\left(\sum a\right)^2}{3} \geq \frac{\left(\sum\sqrt{ab}\right)\cdot 3\sum ab}{3}$$

$$\Rightarrow \left(\sum\sqrt{a^3}\right)^2 \geq \left(\sum\sqrt{ab}\right)\cdot\sqrt{3abc\left(\sum a\right)} \geq \left(\sum\sqrt{ab}\right)\sqrt{3\left(\sum a\right)\prod(b+c-a)}$$

$\Rightarrow$ ***(1) true*** $\Rightarrow \sum \frac{a^2\sqrt{a}}{\sqrt{b}\sin^2 t+\sqrt{c}\cos^2 t} \geq 4\sqrt{3}\cdot S$

Solution 2 by Nirapada Pal-Jhargram-India

$$\sum \frac{a^2\sqrt{a}}{\sqrt{b}\sin^2 t+\sqrt{c}\cos^2 t} = \sum \frac{\left(a^{\frac{3}{2}}\right)^2}{\sqrt{ab}\sin^2 t+\sqrt{ac}\cos^2 t}$$

$$\overset{Bergstorm}{\geq} \frac{\left(\sum a^{\frac{3}{2}}\right)^2}{\sum(\sqrt{ab}\sin^2 t+\sqrt{ac}\cos^2 t)} \overset{AM-GM}{\geq} \frac{(3\sqrt{abc})^2}{\sum\sqrt{ab}} \geq \frac{9abc}{\sum a} = \frac{9\cdot 4RS}{2s} \geq$$

$\geq \frac{36RS}{2\cdot\frac{3\sqrt{3}R}{2}} = 4\sqrt{3}S$***. As*** $\sum a^2 \geq \sum ab \overset{Curry}{\geq} 4\sqrt{3}S$

417. In $\Delta\, ABC$:

$$\frac{16}{3}\le\left(\sec\frac{A}{2}\cdot\sec\frac{B}{2}\right)^2+\left(\sec\frac{B}{2}\cdot\sec\frac{C}{2}\right)^2+\left(\sec\frac{C}{2}\cdot\sec\frac{A}{2}\right)^2\le\frac{4}{3}\left(\frac{R}{r}\right)^2$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Daniel Sitaru – Romania , Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

Solution 1 by Daniel Sitaru – Romania

$$\sum\sec^2\frac{A}{2}\sec^2\frac{B}{2}=\frac{8R(4R+r)}{s^2}$$

$$\frac{16}{3}\le\frac{8R(4R+r)}{s^2}\leftrightarrow 2s^2\le 3R(4R+r)$$

$$2s^2\overset{GERRETSEN}{\le} 2(4R^2+4Rr+3r^2)\le 12R^2+3Rr\leftrightarrow$$

$$(R-2r)(4R+3r)\ge 0\quad \textbf{(1)}$$

$$\frac{8R(4R+r)}{s^2}\le\frac{4R^2}{3r^2}\leftrightarrow s^2\ge 24r^2+\frac{6r^2}{R}$$

$$s^2\overset{GERRETSEN}{\ge} 16Rr-5r^2\ge 24r^2+\frac{6r^3}{R}\leftrightarrow(R-2r)(16R+3r)\ge 0\quad \textbf{(2)}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{16}{3}\le\sum\sec^2\frac{A}{2}\cdot\sec^2\frac{B}{2}\le\frac{4}{3}\cdot\left(\frac{R}{2}\right)^2$$

$$\sum\sec^2\frac{A}{2}\cdot\sec^2\frac{B}{2}=\sum\left(\frac{1}{\cos\frac{A}{2}\cdot\cos\frac{B}{2}}\right)^2=$$

$$=\sum\frac{1}{\left[\frac{1}{2}\cdot\left(\cos\frac{A+B}{2}+\cos\frac{A-B}{2}\right)\right]^2}=$$

$$\left(\cos\frac{A-B}{2}=\frac{a+b}{c}\cdot\sin\frac{C}{2}\right)=4\cdot\sum\frac{1}{\left(\sin\frac{C}{2}+\cos\frac{A-B}{2}\right)^2}$$

$$=4\cdot\sum\frac{1}{\sin^2\frac{C}{2}\cdot\left(1+\frac{a+b}{c}\right)^2}=$$

$$=4\cdot\sum\frac{c^2}{4p^2\cdot\sin^2\frac{C}{2}}=\frac{1}{p^2}\cdot\sum\left(\frac{c\cdot 2\cos\frac{C}{2}}{\sin\frac{C}{2}\cdot 2\cdot\cos\frac{C}{2}}\right)^2=$$

$$=\frac{1}{p^2}\cdot 16R^2\cdot\sum\cos^2\frac{C}{2}$$

$$\left(\sum\cos^2\frac{C}{2}=\frac{4R+r}{2R}\right)=\frac{16R^2}{p^2}\cdot\frac{4R+r}{2r}=\frac{8R\cdot(4R+r)}{p^2}$$

$$\sum\sec^2\frac{A}{2}\cdot\sec^2\frac{B}{2}=\frac{8R\cdot(4R+r)}{p^2}\quad (*)$$

$$LHS:\frac{16}{3}\leq\frac{8R\cdot(4R+r)}{p^2}\Leftrightarrow 2p^2\leq 3R\cdot(4R+r)$$

$$2p^2\overset{GERRETSEN}{\leq}8R^2+8Rr+6r^2\underbrace{\leq}_{ASSURE}3R\cdot(4R+r)$$

$$4R^2-5Rr-6r^2\geq 0;$$

$$(R-2r)\cdot(4R+3r)\geq 0$$

Euler (LHS)

$$RHS:\frac{8R\cdot(4R+r)}{p^2}\leq\frac{4}{3}\cdot\left(\frac{R}{2}\right)^2\quad \textit{(ASSURE)}$$

$$6r^2\cdot(4R+r)\leq R\cdot p^2$$

$$\Rightarrow R\cdot p^2\overset{Euler}{\geq}2r\cdot p^2\geq 6r^2\cdot(4R+r)$$

$$p^2\geq 3r(4R+r)$$

$$p^2\geq 12Rr+3r^2\quad \textit{(RHS)}$$

418. If in $\Delta\ ABC, x, y, z \in (0,1)$ then:

$$\frac{a^6}{x(1-x^2)} + \frac{b^6}{y(1-y^2)} + \frac{c^6}{z(1-z^2)} \geq 6776\sqrt{3}r^6$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

$$\textbf{By AM-GM: } x^3 + \frac{1}{\sqrt{27}} + \frac{1}{\sqrt{27}} \geq 3 \cdot \sqrt[3]{\frac{x^3}{27}} = x$$

$$\Rightarrow x - x^3 \leq \frac{2}{\sqrt{27}} \leftrightarrow \sum \frac{a^6}{x(1-x^2)} \geq \sum \frac{a^6}{\frac{2}{\sqrt{29}}} = \frac{\sqrt{27}}{2} \sum a^6$$

$$\textbf{We prove: } \frac{\sqrt{27}}{2} \sum a^6 \geq 6776\sqrt{3}r^6 \Leftrightarrow 3\sum a^6 \geq 13552r^6 = 13552 \cdot \left(\frac{2s}{\sum a}\right)^6$$

$$\Leftrightarrow 3\left(\sum a^6\right) \geq \frac{1669}{8} \cdot \frac{(16S^2)^3}{(\sum a)^6} = \frac{1669}{8} \cdot \frac{(\sum a)^3 \prod (b+c-a)^3}{(\sum a)^6}$$

$$\Leftrightarrow 24\left(\sum a^6\right)\left(\sum a\right)^3 \geq 1669 \prod (b+c-a)^3$$

$$\textbf{By AM-GM: } \sum a^6 \geq 3a^2b^2c^2; (\sum a)^3 \geq 27abc$$

$$\Rightarrow 24\left(\sum a^6\right)\left(\sum a\right)^3 \geq 1944(abc)^3 > 1669 \prod (b+c-a)^3$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\textbf{Let } f(x) = x(1-x^2) \textbf{ for all } x \in (0,1) \textbf{ then}$$

$$f'(x) = 1 - 3x^2 \textbf{ for all } x \in (0,1), \textbf{ for } f'(x_0) = 0 \Rightarrow x_0 = \frac{1}{\sqrt{3}}$$

$$\textbf{Now, } f''(x_0) = -2\sqrt{3} < 0, f \textbf{ attains maximum at } x = x_0, \textbf{ so}$$

$$\therefore f(x) \leq f(x_0) = \frac{2}{3\sqrt{3}}, \therefore \sum_{cyc} \frac{a^6}{x(1-x^2)} \geq \frac{3\sqrt{3}}{2} \sum_{cyc} a^6$$

$$\geq \frac{3\sqrt{3}}{2}\cdot\frac{1}{3^5}\cdot(a+b+c)^6\left[\because \frac{a^6+b^6+c^6}{3}\geq\left(\frac{a+b+c}{3}\right)^6\right]$$

$$\geq \frac{3\sqrt{3}}{2}\cdot\frac{1}{3^5}\cdot\left(6\sqrt{3}r\right)^6 = 6^5\sqrt{3}r^6 = 7776\sqrt{3}r^6$$

419. In $\Delta\ ABC$:

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq\sqrt{1+\frac{(a+b)(b+c)(c+a)}{abc}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un $\Delta\ ABC\ \frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq\sqrt{1+\frac{(a+b)(b+c)(c+a)}{abc}}$

Realizamos los siguientes cambios de variables

$$x = s-a > 0,\ \ y = s-b > 0,\ \ z = s-c > 0 \Leftrightarrow$$

$$\Leftrightarrow x+y = c, y+z = a, z+x = b$$

La desigualdad propuesta es equivalente

$$\frac{s-b}{s-a}+\frac{s-c}{s-b}+\frac{s-a}{s-c}\geq\sqrt{1+\frac{ab(a+b)+bc(b+a)+ca(c+a)+2abc}{abc}}$$

$$\frac{s-b}{s-a}+\frac{s-c}{s-b}+\frac{s-a}{s-c}\geq\sqrt{3+\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}} = \sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

$$\Leftrightarrow \frac{y}{x}+\frac{z}{y}+\frac{x}{z}\geq\sqrt{(x+y+z)\left(\frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x}\right)}$$

Lemma

Siendo $x, y, z > 0$ ***se cumple la siguiente desigualdad***

$$\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\geq\sqrt{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}$$

Es necesario probar lo siguiente

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x}$$

Aplicando la desigualdad de Cauchy

$$\frac{1}{x}+\frac{1}{y}\geq\frac{4}{x+y}\quad (A),$$

$$\frac{1}{y}+\frac{1}{z}\geq\frac{4}{y+z}\quad (B),$$

$$\frac{1}{z}+\frac{1}{x}\geq\frac{4}{z+x}\quad (C)$$

Sumando (A) + (B) + (C)

$$\Rightarrow\frac{2}{x}+\frac{2}{y}+\frac{2}{z}\geq\frac{4}{x+y}+\frac{4}{y+z}+\frac{4}{z+x}\Leftrightarrow$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x}$$

420. In $\Delta\, ABC, I$ – incentre, R_a, R_b, R_c – circumradii of $\Delta\, BIC, \Delta\, AIC, \Delta\, BIA$:

$$R_a^2+R_b^2+R_c^2\geq 2R\cdot\min(m_a,m_b,m_c)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam, Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia, Solution 3 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

Prove that: $r_a^3+r_b^3+r_c^3+24rs^2\leq\left(\frac{9R}{2}\right)^3$ ***(1)***

Let S = ***area of triangle*** ABC

(1) $\Leftrightarrow \sum\left(\frac{2S}{b+c-a}\right)^3 + 24\cdot\frac{2S}{a+b+c}\cdot\left(\frac{a+b+c}{2}\right)^2 \le \left(\frac{9}{2}\cdot\frac{abc}{4S}\right)^3$

$$\Leftrightarrow 8S^3\cdot\sum\left(\frac{1}{b+c-a}\right)^3 + 12S(a+b+c) \le \left(\frac{9abc}{8S}\right)^3$$

$$\Leftrightarrow 8^4\cdot S^6\cdot\sum\left(\frac{1}{b+c-a}\right)^3 + 8^3\cdot 12S^4(a+b+c) \le 9^3(abc)^3$$

$$\Leftrightarrow (16S^2)^3\cdot\sum\left(\frac{1}{b+c-a}\right)^3 + 24(16S^2)^2\left(\sum a\right) \le 9^3(abc)^3$$

$$\Leftrightarrow \sum\left(16S^2\cdot\frac{1}{b+c-a}\right)^3 + 24\left(\sum a\right)^3\cdot\prod(b+c-a)^2 \le 9^3(abc)^3$$

$$\Leftrightarrow \sum(a+b+c)^3\cdot(c+a-b)^3(a+b-c)^3 + 24(\sum a)^3\cdot\prod(b+c-a)^2 \le$$

$$\le 9^3(abc)^3$$

$$\Leftrightarrow (\sum a)^3\cdot[\sum(c+a-b)^3(a+b-c)^3 + 24\prod(b+c-a)^2] \le 9^3(abc)^3$$

- ***Because:*** $x^3+y^3+z^3+24xyz \le (x+y+z)^3$

$$\Rightarrow \sum(c+a-b)^3(a+b-c)^3 + 24\prod(b+c-a)^2 \le$$

$$\le \left[\sum(c+a-b)(a+b-c)\right]^3$$

$$\Rightarrow \left(\sum a\right)^3\cdot\left[\sum(c+a-b)^3(a+b-c)^3 + 24\prod(b+c-a)^2\right] \le$$

$$\le \left(\sum a\right)^3\cdot\left[\sum(c+a-b)(a+b-c)\right]^3$$

We prove: $(\sum a)^3\cdot[\sum(c+a-b)(a+b-c)]^3 \le 9^3(abc)^3$

$$\Leftrightarrow \left(\sum a\right)\cdot\left[\sum\left(a^2-(b-c)^2\right)\right] \le 9abc$$

$\Leftrightarrow \sum a^3 + 3abc \ge \sum ab(a+b) \Leftrightarrow \sum(a-b)(a-c) \ge 0$ ***(true)***

$\Rightarrow$ ***Q.E.D.***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$R_a^2 + R_b^2 + R_c^2 \ge 2R\cdot[\min(m_a, m_b, m_c)]$$

$$\sum R_a^2 = 4R^2 \cdot \sum \sin^2\frac{A}{2} \geq 2R \cdot [\min(m_a; m_b; m_c)]$$

If $\left.\begin{matrix} a \geq b \geq c \\ m_c \geq m_b \geq m_a \end{matrix}\right\}$ *(*)*

$$\sum R_a^2 = 4R^2 \cdot \sum \sin^2\frac{A}{2} \geq 2R \cdot m_a$$

$2R \cdot \sum \sin^2\frac{A}{2} \geq m_a$ ***(**)***

$$2R \cdot \sum \sin^2\frac{A}{2} = 2R \cdot \sum \frac{(p-b)(p-c)}{bc} = \frac{2R}{abc}\sum a(p-b)(p-c) =$$

$$= \frac{2R}{4p \cdot Rr} \cdot \left[\sum ap^2 - 2\sum ab \cdot p + 3abc\right] =$$

$$= \frac{1}{2p \cdot r} \cdot [2p^3 - 2(p^2 + 4Rr + r^2)p + 12p \cdot Rr] =$$

$= \frac{2R}{2pr}[p^3 - p^3 - 4Rr - r^2 + 6Rr] = \frac{1}{r} \cdot [2Rr - r^2] = 2R - r$ ***(***)***

(**); (***) $\Rightarrow 2R - r \geq m_a \Leftrightarrow \left(2 \cdot (2R - r)\right)^2 \geq 2(b^2 + c^2) - a^2 \Leftrightarrow$

$$\Leftrightarrow 4 \cdot (4R^2 - 4Rr + r^2) \geq 2 \cdot (a^2 + b^2 + c^2) - 3a^2 \Leftrightarrow$$

$$\Leftrightarrow 3a^2 \geq 2 \cdot \sum a^2 - 4(4R - 4Rr + r^2) = 4(p^2 - 4Rr - r^2) - 4(4R^2 - 4Rr + r^2) =$$

$$3a^2 \overset{(*)}{\geq} a^2 + b^2 + c^2 \geq 4p^2 - 16R^2 - 8r^2$$

$$\Rightarrow 2p^2 - 8Rr - 2r^2 \geq 4p^2 - 16R^2 - 8r^2$$

$$2p^2 \leq 16R^2 - 8Rr + 6r^2$$

$$p^2 \leq 8R^2 - 4Rr + 3r^2$$

$$p^2 \overset{GERRETSEN}{\leq} 4R^2 + 4Rr + 3r^2 \leq 8R^2 - 4Rr + 3r^2$$

$8Rr \leq 8R^2 - 4R^2 \Rightarrow 2r \leq R$ ***(Euler)***

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Let $\min(m_a, m_b, m_c) = m_a$ ***and*** $r + 4R \geq \sum_{cyc} m_a \geq 3m_a$

$$\Rightarrow \frac{r+4R}{3} \geq m_a, \sum_{cyc}(p-a)(p-b) = r(r+4R), R_a = 2R\sin\frac{A}{2},$$

$$R_b = 2R\sin\frac{B}{2}, R_c = 2R\sin\frac{C}{2}, \sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}},$$

$$\sin\frac{B}{2} = \sqrt{\frac{(p-c)(p-a)}{ca}} \text{ and } \sin\frac{C}{2} = \sqrt{\frac{(p-a)(p-b)}{ab}}$$

we need to prove, $\sum_{cyc} R_a^2 \geq 2Rm_a \Leftrightarrow \frac{4R^2}{abc}\left(\sum_{cyc} a(p-b)(p-c)\right) \geq 2Rm_a$

$$\Leftrightarrow \frac{R}{\Delta}(2p^3 - 2p(p^2 + r^2 + 4Rr) + 12Rrp) \geq 2Rm_a$$

$$\Leftrightarrow \frac{R}{\Delta}\cdot 2rp(2R-r) \geq 2Rm_a \Leftrightarrow (2R-r)^2 \geq m_a^2$$

we need to show, $9(2R-r)^2 \geq (r+4R)^2$

$\Leftrightarrow (6R-3r^2) - (r+4R)^2 \geq 0 \Leftrightarrow 4(R-2r)(5R-r) \geq 0$, ***which is true***

421. In $\Delta\, ABC$:

$$\sqrt{w_a s_a r_a} + \sqrt{w_b s_b r_b} + \sqrt{w_c s_c r_c} \leq s\sqrt{s\sqrt{3}}$$

Proposed by Daniel Sitaru – Romania

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

By AM-GM we have $s_a = \frac{bc\sqrt{2b^2+2c^2-a^2}}{b^2+c^2} \leq \frac{bc\sqrt{2b^2+2c^2-a^2}}{2bc} = \frac{\sqrt{2b^2+2c^2-a^2}}{2}$

Similarly, we have $s_b \leq \frac{\sqrt{2a^2+2c^2-b^2}}{2}$ ***and*** $s_c \leq \frac{\sqrt{2a^2+2b^2-c^2}}{2}$

$$\Rightarrow s_a + s_b + s_c \leq \frac{\sqrt{2b^2+2c^2-a^2} + \sqrt{2a^2+2c^2-b^2} + \sqrt{2a^2+2b^2-c^2}}{2}$$

On the other hand, by BCS we have

$$\frac{\sqrt{2b^2+2c^2-a^2}+\sqrt{2a^2+2c^2-b^2}+\sqrt{2a^2+2b^2-c^2}}{2} \leq \frac{\sqrt{3(2b^2+2c^2-a^2+2a^2+2c^2-b^2+2a^2+2b^2-c^2)}}{2} \Rightarrow$$

$$\Rightarrow s_a + s_b + s_c \leq \frac{3\sqrt{a^2+b^2+c^2}}{2}$$

By BCS we have $w_a r_a = \frac{\sqrt{bc(b+c-a)(a+b+c)}}{b+c}\cdot\frac{1}{2}\cdot\sqrt{\frac{(a+b+c)(a+c-b)(a+b-c)}{b+c-a}} =$

$$=\frac{\sqrt{bc}\cdot\sqrt{(a+c-a)(a+b-c)}\cdot(a+b+c)}{2(b+c)}\le\frac{a+b+c}{4}\cdot\sqrt{a^2-(b-c)^2}$$

Similarly, we have $w_br_b\le\frac{a+b+c}{4}\cdot\sqrt{b^2-(a-c)^2}$ ***and*** $w_cr_c\le\frac{a+b+c}{2}\cdot\sqrt{c^2-(a-b)^2}$

$$\Rightarrow w_ar_a+w_br_b+w_cr_c\le\frac{a+b+c}{4}\left(\sqrt{a^2-(b-c)^2}+\sqrt{b^2-(a-c)^2}+\sqrt{c^2-(a-b)^2}\right)$$

On the other hand, by BCD we have

$$\frac{a+b+c}{4}\left(\sqrt{a^2-(b-c)^2}+\sqrt{b^2-(a-c)^2}+\sqrt{c^2-(a-b)^2}\right)\le$$

$$\le\frac{a+b+c}{4}\sqrt{3[a^2-(b-c)^2+b^2-(a-c)^2+c^2-(a-b)^2]}=$$

$$=\frac{a+b+c}{4}\sqrt{3[2(ab+bc+ca)-a^2-b^2-c^2]}$$

$$\Rightarrow w_ar_a+w_br_b+w_cr_c\le\frac{a+b+c}{4}\sqrt{3[2(ab+bc+ca)-a^2-b^2-c^2]}$$

By BCS, we have:

$$\sqrt{w_as_ar_a}+\sqrt{w_as_ar_a}+\sqrt{w_as_ar_a}\le\sqrt{(s_b+s_b+s_c)(w_ar_a+w_br_b+w_cr_c)}\le$$

$$\le\sqrt{\frac{3\sqrt{a^2+b^2+c^2}}{2}\cdot\frac{a+b+c}{4}\cdot\sqrt{3[2(ab+bc+ca)-a^2-b^2-c^2]}}\quad \textbf{(1)}$$

By AM-GM we have

$$\sqrt{a^2+b^2+c^2}\cdot\sqrt{2(ab+bc+ca)-a^2-b^2-c^2}\le\frac{a^2+b^2+c^2+2(ab+bc+ca)-a^2-b^2-c^2}{2}=$$

$$=ab+bc+ca\quad \textbf{(2)}$$

$$\textbf{(1), (2)}\Rightarrow\sqrt{w_as_ar_a}+\sqrt{w_as_ar_a}+\sqrt{w_as_ar_a}\le\sqrt{\frac{3\sqrt{3}(a+b+c)(ab+bc+ca)}{8}}\le$$

$$\le\sqrt{\frac{3\sqrt{3}(a+b+c)\cdot\frac{(a+b+c)^2}{3}}{8}}=\sqrt{\frac{\sqrt{3}(a+b+c)^3}{8}}=s\sqrt{s\sqrt{3}}$$

(QED). The equality occurs when $a=b=c$.

422. In $\Delta\ ABC$**:**

$$\sqrt{1+\frac{8m_am_bm_c}{h_ah_bh_c}}\ge\sqrt{\frac{r_a}{r_b}}+\sqrt{\frac{r_b}{r_c}}+\sqrt{\frac{r_c}{r_a}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$\sqrt{1+\frac{8m_am_bm_c}{h_ah_bh_c}} \geq \sqrt{\frac{r_a}{r_b}}+\sqrt{\frac{r_b}{r_c}}+\sqrt{\frac{r_c}{r_a}}$$

Recordar las siguientes identidades y desigualdades conocidas en un $\Delta\ ABC$

$$h_a=\frac{2S}{a},\quad h_b=\frac{2S}{b},\quad h_c=\frac{2S}{c} \Leftrightarrow h_ah_bh_c=\frac{8S^3}{abc}=\frac{8S^3}{4RS}=\frac{2S^2}{R}$$

$$m_a \geq \sqrt{s(s-a)},\quad m_b \geq \sqrt{s(s-b)},\quad m_c \geq \sqrt{s(s-c)} \Leftrightarrow$$

$$\Leftrightarrow m_am_bm_c \geq Sp = S\cdot\frac{S}{r}=\frac{S^2}{r}$$

$$\Leftrightarrow \frac{m_am_bm_c}{h_ah_bh_c} \geq \frac{\frac{S^2}{r}}{\frac{2S^2}{R}}=\frac{R}{2r},\ r_a+r_b+r_c=4R+r,\ \frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$$

Aplicando la desigualdad de Cauchy

$$\sqrt{\frac{r_a}{r_b}}+\sqrt{\frac{r_b}{r_c}}+\sqrt{\frac{r_c}{r_a}} \leq \sqrt{(r_a+r_b+r_c)\left(\frac{1}{r_b}+\frac{1}{r_c}+\frac{1}{r_a}\right)}=$$

$$=\sqrt{\frac{4R+r}{r}}=\sqrt{1+\frac{4R}{r}} \leq \sqrt{1+\frac{8m_am_bm_c}{h_ah_bh_c}}$$

423. In $\Delta\ ABC$:

$$\left(\frac{1}{a}\right)^{\frac{a}{s}}\cdot\left(\frac{1}{b}\right)^{\frac{b}{s}}\cdot\left(\frac{1}{c}\right)^{\frac{c}{s}} \leq \frac{1}{12r^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru, Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam, Solution 3 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo s semiperímetro. Probar en un triángulo ABC

$$\left(\frac{1}{a}\right)^{\frac{a}{s}}\cdot\left(\frac{1}{b}\right)^{\frac{b}{s}}\cdot\left(\frac{1}{c}\right)^{\frac{c}{s}}\le\frac{1}{12r^2}$$

Tener en cuenta la siguiente desigualdad en un $\Delta\ ABC$

$$s\ge 3\sqrt{3}\ \textbf{(Mitrinovic)}$$

Aplicando la desigualdad ponderada $MA\ge MG$

$$\frac{\frac{1}{a}\cdot\frac{a}{s}+\frac{1}{b}\cdot\frac{b}{s}+\frac{1}{c}\cdot\frac{c}{s}}{\frac{a}{s}+\frac{b}{s}+\frac{c}{s}}\ge\sqrt[\frac{a}{s}+\frac{b}{s}+\frac{c}{s}]{\left(\frac{1}{a}\right)^{\frac{a}{s}}\left(\frac{1}{b}\right)^{\frac{b}{s}}\left(\frac{1}{c}\right)^{\frac{c}{s}}}$$

$$\Leftrightarrow\frac{3}{2s}\ge\sqrt[2]{\left(\frac{1}{a}\right)^{\frac{a}{s}}\left(\frac{1}{b}\right)^{\frac{b}{s}}\left(\frac{1}{c}\right)^{\frac{c}{s}}}\Leftrightarrow\left(\frac{1}{a}\right)^{\frac{a}{s}}\left(\frac{1}{b}\right)^{\frac{b}{s}}\left(\frac{1}{c}\right)^{\frac{c}{s}}\le\frac{9}{4s^2}\le\frac{9}{4\cdot 27r^2}=\frac{1}{12r^2}$$

Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam

Prove that: $\left(\frac{1}{a}\right)^{\frac{a}{s}}\cdot\left(\frac{1}{b}\right)^{\frac{b}{s}}\cdot\left(\frac{1}{c}\right)^{\frac{c}{s}}\le\frac{1}{12r^2}$

$$\Leftrightarrow\left(\frac{1}{a}\right)^{\frac{2a}{a+b+c}}\cdot\left(\frac{1}{b}\right)^{\frac{2b}{a+b+c}}\cdot\left(\frac{1}{c}\right)^{\frac{2c}{a+b+c}}\le\frac{1}{12r^2}$$

$$\Leftrightarrow\ln\left(\left(\frac{1}{a}\right)^{\frac{2a}{a+b+c}}\cdot\left(\frac{1}{b}\right)^{\frac{2b}{a+b+c}}\cdot\left(\frac{1}{c}\right)^{\frac{2c}{a+b+c}}\right)\le\ln\left(\frac{1}{12r^2}\right)$$

$$\Leftrightarrow\frac{2a\cdot\ln a}{a+b+c}+\frac{2b\ln b}{a+b+c}+\frac{2c\ln c}{a+b+c}\ge\ln(12r^2)$$

$$\Leftrightarrow 2\sum\frac{a\ln a}{a+b+c}\ge\ln\left(\frac{3\prod(b+c-a)}{a+b+c}\right)\quad\textbf{(1)}$$

Popose: $a+b+c=3$, ***(1)*** $\Leftrightarrow\frac{2}{3}\sum a\ln a\ge\sum\ln(3-2a)$ ***(2)***

We prove: $\frac{2}{3}a\ln a-\ln(3-2a)\ge\frac{8}{3}(a-1)$

$$f'(a) = \frac{2}{3}(\ln a + 1) + \frac{2}{3-2a} - \frac{8}{2}$$

$$f'(a) = 0 \Leftrightarrow a = 1$$

0	1	3
−	0	+
↘	0	↗

$$f(a)_{Min} \leftrightarrow a = 1$$

$$\Rightarrow \frac{2}{3}a\ln a - \ln(3-2a) \geq \frac{8}{3}(a-1)$$

$$\Rightarrow \sum \frac{2}{3}a\ln a \geq \sum \ln(3-a) + \frac{8}{3}\underbrace{\left(\sum a - 3\right)}_{0}$$

$$\Rightarrow \frac{2}{3}\sum a\ln a \geq \sum \ln(3-2a) \Rightarrow \textit{(2) true} \Rightarrow \textit{QED}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Applying Weighted A.M ≥ G.M;

$$\frac{a\cdot\left(\frac{1}{a}\right) + b\cdot\left(\frac{1}{b}\right) + c\cdot\left(\frac{1}{c}\right)}{a+b+c} \geq \left\{\left(\frac{1}{a}\right)^a\cdot\left(\frac{1}{b}\right)^b\cdot\left(\frac{1}{c}\right)^c\right\}^{\frac{1}{a+b+c}}$$

$$\Rightarrow \left(\frac{3}{a+b+c}\right)^2 \geq \prod_{cyc}\left(\frac{1}{a}\right)^{\frac{a}{p}}$$

$$\Rightarrow \frac{9}{4p^2} \geq \prod_{cyc}\left(\frac{1}{a}\right)^{\frac{a}{p}} \Rightarrow \frac{9}{4}\cdot\frac{1}{27r^2} \geq \prod_{cyc}\left(\frac{1}{a}\right)^{\frac{a}{p}}$$

$$\Rightarrow \frac{1}{12r^2} \geq \left(\frac{1}{a}\right)^{\frac{a}{p}}\cdot\left(\frac{1}{b}\right)^{\frac{b}{p}}\cdot\left(\frac{1}{c}\right)^{\frac{c}{p}}$$

(proved)

424. In $\Delta\ ABC$:

$$\frac{R^2}{ms^2 + nr^2 + tRr} \geq \frac{4}{27m + n + 2t}, m, n, t > 0$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian, Solution 2 by SK Rejuan-West Bengal-India

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$27mR^2 + nR^2 + 2tR^2 \geq 4mS^2 + 4nr^2 + 4tRr$$

$$27mR^2 \geq 4ms^2, \qquad nR^2 \geq 4nr^2, \qquad 2tR^2 \geq 4tRr$$

$27R^2 \geq 4S^2$ **(from Lemma),** $r \geq 2r$ **(Euler),** $R^2 \geq 2Rr \Rightarrow (R - 2r) \geq 0$

(Euler)

Solution 2 by SK Rejuan-West Bengal-India

$$\Delta\ ABC: \frac{R^2}{ms^2+nr^2+tRr} \geq \frac{4}{27m+n+2t}$$

$$\Leftrightarrow 27mR^2 + nR^2 + 2R^2t \geq 4mS^2 + 4nr^2 + 4trR$$

$$\Leftrightarrow (27mR^2 + 4mr^2) + (nR^2 - 4nr^2) + (2R^2t - 4tnr) \geq 0 \quad \textit{(A)}$$

$$R \geq 2r \Rightarrow 2R^2t \geq 4rRt \quad [t > 0] \Rightarrow (2R^2t - 4Rrt) \geq 0 \quad \textit{(1)}$$

$$R \geq 2r \Rightarrow nR^2 \geq 4nr^2 \Rightarrow (nR^2 - 4nr^2) \geq 0 \quad \textit{(2)}$$

$$\textit{Again, } 3S^2 \leq (r + 4R)^2 \leq \left(\frac{R}{2} + 4R\right)^2$$

$$\Rightarrow S^2 \leq \frac{1}{3}\left(\frac{9R}{2}\right)^2 = \frac{27R^2}{4} \Rightarrow 4s^2 \leq 27R^2$$

$$\Rightarrow 27mR^2 \geq 4ms^2 \Rightarrow (27mR^2 - 4ms^2) \geq 0 \quad \textit{(3)}$$

$\Rightarrow$ ***Adding (1), (2), (3) we get the result (A)***

425. In any scalene $\Delta\, ABC$:

$$\sum \frac{(1+\sin^4 A)^3}{\sin^6 A} + \sum \frac{(1+\cos^4 A)^3}{\cos^6 A} > 93$$

Proposed by Daniel Sitaru – Romania

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

$$\textit{By AM-GM: } 1+\sin^4 A = \sin^4 A + \frac{1}{2} + \frac{1}{2} \geq 3\sqrt[3]{\frac{\sin^4 A}{4}}$$

$$\Rightarrow \sum \frac{(1+\sin^4 A)^3}{\sin^6 A} \geq \sum \frac{\left(3\sqrt[3]{\frac{\sin^4 A}{4}}\right)^3}{\sin^6 A} = \frac{27}{4}\sum \frac{1}{\sin^4 A}$$

$$\Rightarrow \sum \frac{(1+\sin^4 A)^3}{\sin^6 A} \geq \frac{27}{4}\cdot 3\cdot \sqrt[3]{\frac{1}{\prod \sin^2 A}} \quad (1)$$

$$\textit{- Other: } \prod \sin A \leq \frac{(\sum \sin A)^3}{27} \leq \frac{\left(\frac{3\sqrt{3}}{2}\right)^3}{27} = \frac{3\sqrt{3}}{8} \quad (2)$$

$$(1),(2) \Rightarrow \sum \frac{(1+\sin^4 A)^3}{\sin^6 A} \geq \frac{81}{4}\cdot \sqrt[3]{\frac{1}{\frac{27}{64}}} = 27 \quad (3)$$

$$\textit{* Similar: } \sum \frac{(1+\cos^4 A)^3}{\cos^6 A} \geq \frac{81}{4}\cdot \sqrt[3]{\frac{1}{\prod \cos^2 A}} \quad (4)$$

$$\textit{- Other: } \prod \cos A \leq \frac{1}{8}, (4) \Rightarrow \sum \frac{(1+\cos^4 A)^3}{\cos^6 A} \geq 81 \quad (5)$$

$$(3),(5) \Rightarrow \sum \frac{(1+\sin^4 A)^3}{\sin^6 A} + \sum \frac{(1+\cos^4 A)^3}{\cos^6 A} \geq 27 + 81 > 93 \Rightarrow \textit{Q.E.D.}$$

426. In $\Delta\, ABC$:

$$\frac{2m_a m_b m_c}{h_a h_b h_c} + 1 \geq \frac{(r_a+r_b+r_c)^2}{r_a r_b + r_b r_c + r_c r_a}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

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ROMANIAN MATHEMATICAL MAGAZINE

Probar en triángulo ABC: $\frac{2m_am_bm_c}{h_ah_bh_c}+1\ge\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}$

Se demostro anteriormente que

$$\frac{m_am_bm_c}{h_ah_bh_c}+1\ge\frac{R}{2r}\Leftrightarrow\frac{2m_am_bm_c}{h_ah_bh_c}+1\ge\frac{R}{r}+1$$

Es sufieciente probar que $\frac{R}{r}+1\ge\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}$

Realizamos los siguientes cambios de variables

$$x=r_a>0, y=r_b>0, z=r_c>0$$

Se verifica lo siguiente

$$(x+y)(y+z)(z+x)=4p^2R, \qquad xyz=Sp=p^2r$$

La desigualdad propuesta es equivalente $\frac{(x+y)(y+z)(z+x)}{4xyz}+1\ge\frac{(x+y+z)^2}{xy+yz+zx}$

Ahora bien

$$\frac{(x+y)(y+z)(z+x)}{4xyz}+1=\frac{x+y}{4z}+\frac{y+z}{4x}+\frac{z+x}{4y}+\frac{1}{2}+1=$$

$$=\frac{1}{4}\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{1}{4}\left(\frac{y}{z}+\frac{z}{y}\right)+\frac{1}{4}\left(\frac{z}{x}+\frac{x}{z}\right)+\frac{3}{2}$$

Aplicando la desigualdad de Cauchy

$$\frac{(x+y)(y+z)(z+x)}{4xyz}+1=\frac{x^2+y^2}{4xy}+\frac{y^2+z^2}{4yz}+\frac{z^2+x^2}{4zx}+\frac{3}{2}\ge$$

$$\ge\frac{\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2}{4(xy+yz+zx)}+\frac{1}{2}=$$

$$=\frac{2(x^2+y^2+z^2)+2\sum\left(\sqrt{x^2+y^2}\right)\left(\sqrt{x^2+z^2}\right)}{4(xy+yz+zx)}+\frac{3}{2}\ge$$

$$\ge\frac{2\sum x^2+2\sum(x^2+yz)}{4(xy+yz+zx)}+\frac{3}{2}=\frac{4\sum x^2+2\sum xy}{4\sum xy}+\frac{3}{2}=$$

$$=\frac{\sum x^2}{\sum xy}+2=\frac{(x+y+z)^2}{xy+yz+zx}$$

427. In acute $\Delta\ ABC$:

$$\sum \frac{\sin^2 A}{\tan A + (\tan B + \tan C)\cos^2 A} \le \frac{1}{2}(\cot A + \cot B + \cot C)$$

Proposed by Daniel Sitaru – Romania

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

$$\sum \frac{\frac{1}{\cot^2 A + 1}}{\frac{1}{\cot A} + \left(\frac{1}{\cot B} + \frac{1}{\cot C}\right) \cdot \frac{\cot^2 A}{\cot^2 A + 1}} \le \frac{1}{2}\sum \cot A$$

$$\Leftrightarrow \sum \frac{\frac{1}{x^2 + 1}}{\frac{1}{x} + \left(\frac{1}{y} + \frac{1}{z}\right) \cdot \frac{x^2}{x^2 + 1}} \le \frac{1}{2}\sum x$$

$$\left(x = \cot A\,; y = \cot B\,; z = \cot C \Rightarrow \sum xy = 1\right)$$

$$\Leftrightarrow \sum \frac{\frac{1}{x^2 + 1}}{\frac{yz(x^2 + 1) + x^3(y + z)}{xyz(x^2 + 1)}} \le \frac{1}{2}\sum x$$

$$\Leftrightarrow \sum \frac{xyz}{yz(x^2 + 1) + x^3(y + z)} \le \frac{1}{2}\sum x$$

$$\Leftrightarrow \sum \frac{xyz}{yz + x^2(xy + xz + yz)} \le \frac{1}{2}\sum x + \sum \frac{xyz}{x^2 + y^2} \le \frac{1}{2}\sum x \quad \textit{(1)}$$

Because; by AM-GM: $\sum \frac{xyz}{x^2 + yz} \le \sum \frac{xyz}{2x\sqrt{yz}} = \frac{1}{2}\sum \sqrt{yz} \le \frac{1}{2}\sum x$

$\Rightarrow$ *(1) true* $\Rightarrow$ *Q.E.D.*

428. Prove that in any triangle ABC,

$$\frac{1}{a(s-a)} + \frac{1}{b(s-b)} + \frac{1}{c(s-c)} \ge \frac{1}{Rr}$$

where s denotes the semi-perimeter.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$\frac{1}{a(s-a)}+\frac{1}{b(s-b)}+\frac{1}{c(s-c)}\geq\frac{1}{Rr}$$

Siendo a,b,c los lados de un triángulo se cumple lo siguiente

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{a+b+c}{abc}=\frac{2p}{4pRr}=\frac{1}{2Rr}$$

La desigualdad propuesta es equivalente

$$\frac{1}{a(s-a)}+\frac{1}{b(s-b)}+\frac{1}{c(s-c)}\geq\frac{2}{ab}+\frac{2}{bc}+\frac{2}{ca}$$

$$\Leftrightarrow\frac{bc}{s-a}+\frac{ca}{s-b}+\frac{ab}{s-c}\geq 2(a+b+c)$$

Realizamos las siguientes sustituciones algebraicas

$$x=s-a>0, y=s-b>0, z=s-c>0\Leftrightarrow x+y=c, y+z=a, z+x=b$$

$$\Leftrightarrow\frac{(x+z)(x+y)}{x}+\frac{(y+x)(y+z)}{y}+\frac{(z+y)(z+x)}{z}\geq 4(x+y+z)$$

$$\Leftrightarrow\left(x+\frac{xy+yz+zx}{x}\right)+\left(y+\frac{xy+yz+zx}{y}\right)+\left(z+\frac{xy+yz+zx}{z}\right)=$$

$$=x+y+z+(xy+yz+zx)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=$$

$$=x+y+z+(xy+yz+zx)\left(\frac{xy+yz+xz}{xyz}\right)=x+y+z+\frac{(xy+yz+zx)^2}{xyz}\geq$$

$$\geq x+y+z+3(x+y+z)=4(x+y+z)\ \textbf{(LQQD)}$$

429. In $\Delta\ ABC$:

$$\frac{a^2\cos(B-C)}{\sin A}+\frac{b^2\cos(C-A)}{\sin B}+\frac{c^2\cos(A-B)}{\sin C}\leq 4sR$$

Proposed by Daniel Sitaru, Marin Chirciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru, Solution 2 by Nirapada Pal-Jhargram-India, Solution 3 by Ravi Prakash-New Delhi-India, Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

$$\frac{a^2\cos(B-C)}{\sin A}+\frac{b^2\cos(C-A)}{\sin B}+\frac{c^2\cos(A-B)}{\sin C}\le 4sR$$

Tener en cuenta lo siguiente

$$a=2R\sin A,\qquad b=2R\sin B,\qquad c=2R\sin C$$

$$\Leftrightarrow \frac{a}{\sin A}=2R,\frac{b}{\sin B}=2R,\frac{c}{\sin C}=2R$$

En la desigualdad propuesta

$$\Leftrightarrow \frac{a^2\cos(B-C)}{\sin A}+\frac{b^2\cos(C-A)}{\sin B}+\frac{c^2\cos(A-B)}{\sin C}\le \frac{a^2}{\sin A}+\frac{b^2}{\sin B}+\frac{c^2}{\sin C}=$$

$$=2R(a+b+c)=4sR$$

Solution 2 by Nirapada Pal-Jhargram-India

$$a\cos(B-C)=2R\sin\{\pi-(B+C)\}\cos(B-C)=2R\sin(B+C)\cos(B-C)$$

$$=R(\sin 2B+\sin 2C)=R(2\sin B\cos B+2\sin C\cos C)=b\cos B+c\cos C$$

$$\text{So }\sum\frac{a^2\cos(B-C)}{\sin A}=2R\sum a\cos(B-C)=2R\sum(b\cos B+c\cos C)$$

$$\le 2R\sum(b+c)\quad \text{as}\quad \cos B\le 1,\cos C\le 1$$

$$=2R\times 2s=4sR$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\frac{a^2\cos(B-C)}{\sin A}+\frac{b^2\cos(C-A)}{\sin B}+\frac{c^2\cos(A-B)}{\sin C}\le 4sR$$

$$LHS=4R^2[\sin A\cos(B-C)+\sin B\cos(C-A)+\sin C\cos(A-B)]$$

$$=4R^2[\sin(B+C)\cos(B-C)+\sin(C+A)\cos(C-A)+\sin(A+B)\cos(A-B)]$$

$$=2R^2[(\sin 2B+\sin 2C)+(\sin 2C+\sin 2A)+(\sin 2A+\sin 2B)]$$

$$=4R^2(\sin 2A+\sin 2B+\sin 2C)=16R^2\sin A\sin B\sin C$$

$$= 16R^2\left(\frac{a}{2R}\right)\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right) = \frac{2}{R}(abc) = \frac{2}{R}(4sRr) = 4s(2r) \le 4sR \quad [\because 2r \le R]$$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{a^2\cos(B-C)}{\sin A} + \frac{b^2\cos(C-A)}{\sin B} + \frac{c^2\cos(A-B)}{\sin C} \le 4pR$$

1) $\cos(B-C) = 2\cos^2\frac{B-C}{2} - 1 = 2\cdot\left(\frac{b+c}{a}\right)^2\cdot\sin^2\frac{A}{2} - 1$

2) $\sin A = 2\cdot\sin\frac{A}{2}\cdot\cos\frac{A}{2}$

3) $\tan\frac{A}{2} = \frac{r}{p-a}$

$$\sum\frac{a^2\cdot\cos(B-C)}{\sin A} = \sum\frac{a^2\left[2\cdot\frac{(b+c)^2}{a^2}\cdot\sin^2\frac{A}{2} - 1\right]}{\sin A} =$$

$$= \sum\frac{2\cdot(b+c)^2\cdot\sin^2\frac{A}{2} - a^2}{\sin A} = \sum\frac{2\cdot(b+c)^2\cdot\sin^2\frac{A}{2}}{2\cdot\sin\frac{A}{2}\cdot\cos\frac{A}{2}} - \sum\frac{a^2}{\sin A}$$

$$= \sum(b+c)^2\cdot\tan\frac{A}{2} - \sum\frac{a^2}{\frac{a}{2R}} = \sum(2p-a)^2\cdot\frac{r}{p-a} - \sum 2a$$

$$= r\cdot\sum\frac{(p+p-a)^2}{p-a} - 4pR = r\cdot\sum\frac{p^2 + 2p(p-a) + (p-a)}{p-a}$$

$$= 4pR = r\cdot\sum\left(\frac{p^2}{p-a} + 2p + p - a\right) - 4pR =$$

$$= r\cdot\left[p^2\cdot\left(\frac{4Rr + r^2}{(p-a)(p-b)(p-c)}\right) + 6p + 3p - 2p\right] - 4pR =$$

$$= r\cdot\left[\frac{p^3}{S^2}\cdot(4Rr + r^2) + 7p\right] - 4pR = r\cdot\left[\frac{p}{r}\cdot(4R + r) + 7p\right] - 4rp =$$

$$= p(4R + r) + 7pr - 4Rp = p(4R + r + 7r - 4R) = 8pr \underset{Euler}{\le} 4pR$$

430. In $\Delta\, ABC$:

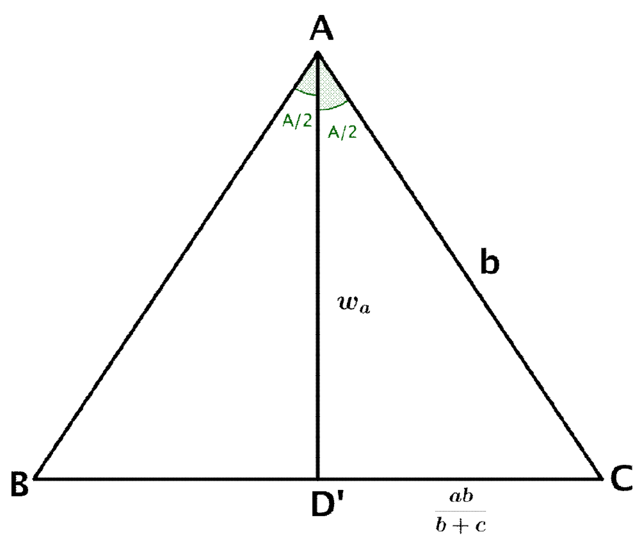
$$m_a w_a \cos(\sphericalangle(m_a, w_a)) + m_b w_b \cos(\sphericalangle(m_b, w_b)) + m_c w_c \cos(\sphericalangle(m_c, w_c)) \le \frac{27R^2}{4}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India,

Solution 2 by Mehmet Sahin-Ankara-Turkey

Solution 1 by Ravi Prakash-New Delhi-India



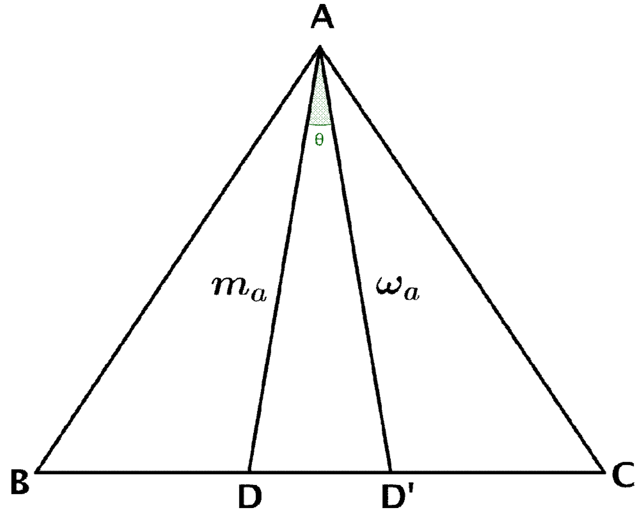
$$\cos\left(\frac{A}{2}\right) = \frac{b^2 + w_a^2 - \left(\frac{ab}{b+c}\right)^2}{2bw_a} \Rightarrow 2bw_a \cos\left(\frac{A}{2}\right) = b^2 + w_a^2 - \frac{(ab)^2}{(b+c)^2}$$

Also $2cw_a \cos\left(\frac{A}{2}\right) = c^2 + w_a^2 - \frac{a^2c^2}{(b+c)^2}$

Substracting, we get $2(b-c)w_a \cos\left(\frac{A}{2}\right) = b^2 - c^2 - \frac{a^2}{(b+c)^2}(b^2 - c^2)$

$$\Rightarrow 2w_a \cos\left(\frac{A}{2}\right) = b + c - \frac{a^2}{(b+c)} = \frac{b^2 + c^2 - a^2 + 2bc}{(b+c)} = \frac{2bc(1 + \cos A)}{b+c}$$

$$\Rightarrow w_a \cos\left(\frac{A}{2}\right) = \frac{bc \cdot 2\cos^2\left(\frac{A}{2}\right)}{b+c} \Rightarrow w_a = \frac{2bc}{b+c}\cos\left(\frac{A}{2}\right)$$



$$2m_a w_a \cos\theta = m_a^2 + w_a^2 - (DD')^2$$

$$= \frac{1}{2}(b^2 + c^2 - 2a^2) + \left(\frac{2bc}{b+c}\right)^2 \cos^2\frac{A}{2} - \left(\frac{a}{2} - \frac{ab}{b+c}\right)^2$$

$$= \frac{1}{2}(b^2 + c^2 - 2a^2) + \frac{4b^2c^2}{(b+c)^2}\cos^2\frac{A}{2} - \left\{\frac{a^2}{u} + \frac{a^2b^2}{(b+c)^2} - \frac{a^2b}{b+c}\right\}$$

$$= \frac{1}{2}(b^2 + c^2 - a^2) + \frac{4b^2c^2}{(b+c)^2}\cos^2\frac{A}{2} - \frac{a^2b}{(b+c)^2}\{b - (b+c)\}$$

$$= \frac{1}{2}(b^2 + c^2 - a^2) + \frac{4b^2c^2}{(b+c)^2}\cos^2\frac{A}{2} + \frac{a^2bc}{(b+c)^2}$$

$$= \frac{4bc}{(b+c)^2}s(s-a) + \frac{a^2bc}{(b+c)^2} + \frac{1}{2}(b^2 + c^2 - a^2)$$

$$= \frac{bc}{(b+c)^2}\{(a+b+c)(b+c-a) + a^2\} + \frac{1}{2}(b^2 + c^2 - a^2)$$

$$= \frac{bc}{(b+c)^2}\{(b+c)^2 - a^2 + a^2\} + bc\cos A$$

$$= bc(1 + \cos A) = 2bc\cos^2\left(\frac{A}{2}\right) = 2s(s-a) \Rightarrow m_a w_a \cos\theta = s(s-a)$$

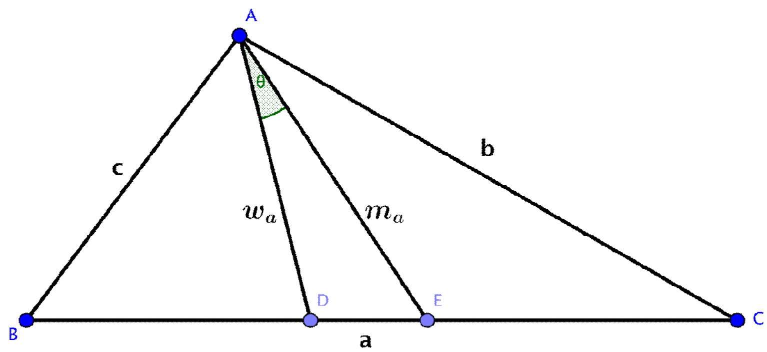
Now,

$$E = m_a w_a \cos(\angle m_a, w_a) + m_b w_b \cos(\angle m_b, w_b) + m_c w_c \cos(\angle m_c, w_c)$$

$$= s(s-a)+s(s-b)+s(s-c)=s(3s-2s)=s^2 \le \left(\frac{3\sqrt{3}}{2}R\right)^2$$

$$\Rightarrow E \le \frac{27}{4}R^2$$

Solution 2 by Mehmet Sahin-Ankara-Turkey

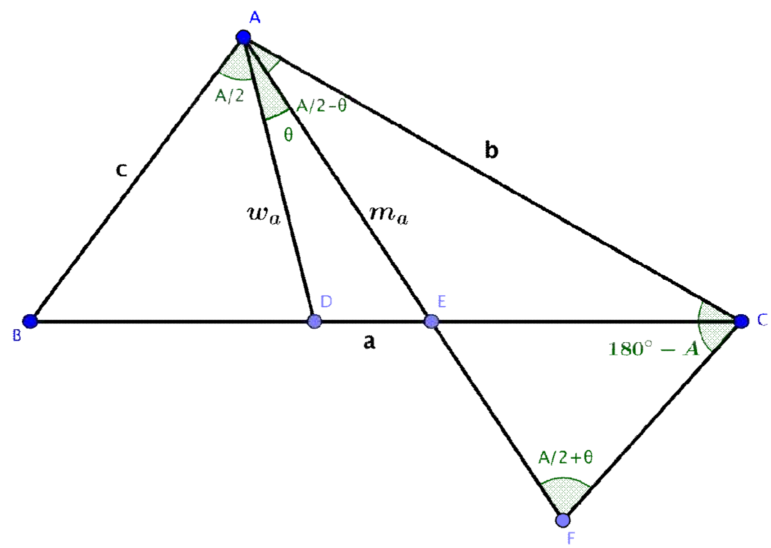


Let ABC be a triangle and $[AD]$ is bisector $[AE]$ is median.

If $m(\widehat{DAE})=\theta$ then $w_a m_a \cdot \cos\theta = s(s-a)$

where s is semiperimeter of ABC

Proof:



Let's use the sine theorem in the triangle AFC.

$$\frac{c}{\sin\left(\frac{A}{2}-\theta\right)}=\frac{b}{\sin\left(\frac{A}{2}+\theta\right)}=\frac{2\cdot m_a}{\sin(180^\circ-A)}$$

$$\Rightarrow\frac{c+b}{\sin\left(\frac{A}{2}-\theta\right)+\sin\left(\frac{A}{2}+\theta\right)}=\frac{2\cdot m_a}{\sin A}\Rightarrow\frac{c+b}{2\sin\frac{A}{2}\cdot\cos\theta}=\frac{2\cdot m_a}{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}$$

$$\Rightarrow m_a\cdot\cos\theta=\frac{b+c}{2}\cdot\cos\frac{A}{2}\quad (1)$$

In triangle ABC: $|AD|^2=|AB|\cdot|AC|-|BD|\cdot|DC|$

$$w_a^2=c\cdot b-\frac{a\cdot c}{b+c}\cdot\frac{ab}{b+c}$$

$$w_a^2=bc\left[1-\frac{a^2}{(b+c)^2}\right]=bc\cdot\frac{(b+c-a)(b+c+a)}{(b+c)^2}$$

$$w_a^2=\frac{4bc}{(b+c)^2}\cdot s(s-a)$$

$$w_a=\frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)}\quad (2)$$

From (1) and (2)

$$m_aw_a\cdot\cos\theta=\frac{b+c}{2}\cdot\sqrt{\frac{s(s-a)}{bc}}\cdot\frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)}=s(s-a)\ \therefore$$

$$m_a\cdot w_a\cdot\cos(\sphericalangle(m_a,w_a))+m_bw_b\cdot w_a(\sphericalangle(m_b,w_b))+m_cw_c\cdot\cos(\sphericalangle(m_c,w_b))$$

$$s(s-a)+s(s-b)+s(s-c)$$

$$3s^2-s(a+b+c)=3s^2-2s^2=s^2$$

$$2s\le 3\sqrt{3}R\Rightarrow T=s^2\le\frac{27R^2}{4}\ \therefore$$

431. In $\Delta\ ABC$:

$$3r\le\sqrt[3]{m_am_bm_c}\le\frac{3R}{2}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en en triángulo ABC

$$3r \leq \sqrt[3]{m_a m_b m_c} \leq \frac{3R}{2}$$

Utilizando las siguientes identidades y desigualdades conocidas en un $\Delta\ ABC$

$$S = sr = \sqrt{s(s-a)(s-b)(s-c)}, \qquad m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

$$s \geq 3\sqrt{3}r, a^2 + b^2 + c^2 \leq 9R^2, m_a \geq \sqrt{s(s-a)}, m_b \leq \sqrt{s(s-b)},$$

$$m_c \leq \sqrt{s(s-c)}$$

Ahora bien

$$\sqrt[3]{m_a m_b m_c} \geq \sqrt[3]{s\sqrt{(s-a)(s-b)(s-c)}} = \sqrt[3]{s^2 r} \geq \sqrt[3]{27r^3} = 3r$$

(LQQD)

Por ultimo: Aplicando $MA \geq MG$ y Cauchy

$$\sqrt[3]{m_a m_b m_c} \leq \frac{m_a + m_b + m_c}{3} \leq \frac{1}{3} \cdot \sqrt{3(m_a^2 + m_b^2 + m_c^2)} =$$

$$= \frac{1}{3} \cdot \sqrt{\frac{9}{4}(a^2 + b^2 + c^2)} \leq \frac{1}{3} \cdot \frac{9R}{2} = \frac{3R}{2}$$

(LQQD)

432. $\left(\frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c}\right)\left(\frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c}\right) \geq \frac{4R+r}{r}$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$\left(\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}\right)\left(\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)\geq\frac{4R+r}{r}=\frac{4R}{r}+1$$

Tener en cuenta las siguientes identidades en un $\Delta\ ABC$

$$r_a=\frac{S}{s-a},r_b=\frac{S}{s-b},r_c=\frac{S}{s-c},h_a=\frac{2S}{a},h_b=\frac{2S}{b},h_c=\frac{2S}{c}$$

La desigualdad propuesta s equivalente

$$\left(\frac{a}{2(s-a)}+\frac{b}{2(s-b)}+\frac{c}{2(s-c)}\right)\left(\frac{2(s-a)}{a}+\frac{2(s-b)}{b}+\frac{2(s-c)}{c}\right)=$$

$$=\frac{abc}{(s-a)(s-b)(s-c)}+1$$

Siendo a,b,c ***los lados de triángulo*** ABC***, realizamos los siguientes cambios de variables***

$$x=s-a>0,y=s-b>0,z=s-c>0\Leftrightarrow$$

$$\Leftrightarrow y+z=a,\qquad z+x=b,\qquad x+y=c$$

$$\Leftrightarrow\left(\frac{y+z}{2x}+\frac{z+x}{2y}+\frac{x+y}{2z}\right)\left(\frac{2x}{y+z}+\frac{2y}{z+x}+\frac{2z}{x+y}\right)\geq\frac{(x+y)(y+z)(z+x)}{xyz}+1=$$

$$=\frac{x+y}{z}+\frac{y+z}{x}+\frac{z+x}{y}+3$$

Como $\rightarrow\frac{2x}{y+z}+\frac{2y}{z+x}+\frac{2z}{x+y}\geq 3$ ***(Inequality Nesbit). Es suficiente probar que***

$$\Rightarrow 3\left(\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}\right)\geq 2\left(\frac{x+y}{z}+\frac{y+z}{x}+\frac{z+x}{y}+3\right)\Leftrightarrow$$

$$\Leftrightarrow\frac{x+y}{z}+\frac{y+z}{x}+\frac{z+x}{y}\geq 6$$ ***(Válido por*** $MA\geq MG$***)***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{r_a}{h_a}=\frac{2R-r}{r}\quad (1)$$

$$\sum\frac{h_a}{r_a}=\frac{p^2-8Rr+r^2}{2Rr}\quad (2)$$

$$(1),(2)\Rightarrow\frac{2R-r}{r}\cdot\left(\frac{p^2-8Rr+r^2}{2Rr}\right)\ge\frac{4R+r}{r}\ (ASSURE)\Leftrightarrow$$

$$\Leftrightarrow p^2\ge r\left(12R+2r+\frac{3r^2}{2R-r}\right)\overset{GERRETSEN}{\Rightarrow}$$

$$p^2\ge 16Rr-5r^2\ge r\cdot\left(12R+2r+\frac{3r^2}{2R-r}\right)$$

$$\Rightarrow 4R-7r\ge\frac{3r^2}{2r-r}\Leftrightarrow\underbrace{(4R-7r)\cdot(2r-r)}_{Euler}\ge 3r^2$$

$$(4R-7r)(2R-r)\ge(8r-7r)(4R-r)=3r^2$$

433. In $\Delta\, ABC$:

$$\frac{6r^2}{R}\le\frac{m_a^2+m_b^2+m_c^2}{m_a+m_b+m_c}\le\frac{3R}{2},\ m_a,m_b,m_c\ \text{medians}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{6r^2}{R}\le\frac{m_a^2+m_b^2+m_c^2}{m_a+m_b+m_c}\le\frac{3R}{2}$

Recordar las siguientes identidades y desigualdades conocidas en un triángulo ABC

$$S=sr=\sqrt{s(s-a)(s-b)(s-c)},\ m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$$

$$m_a\ge\sqrt{s(s-a)},\ m_b\ge\sqrt{s(s-b)},\ m_c\ge\sqrt{s(s-c)},\ s\ge 3\sqrt{3}r,\ R\ge 2r$$

Aplicando la desigualdad de Cauchy y $MA\ge MG$

$$\frac{m_a^2+m_b^2+m_c^2}{m_a+m_b+m_c}\ge\frac{m_a+m_b+m_c}{3}\ge\sqrt[3]{m_am_bm_c}\ge$$

$$\geq \sqrt[3]{s\sqrt{s(s-a)(s-b)(s-c)}} = \sqrt[3]{s^2 r} \geq \sqrt[3]{27r^3} = 3r \geq \frac{6r^2}{R}$$

Además $\Rightarrow m_a \geq \frac{b^2+c^2}{4R}, m_b \geq \frac{c^2+a^2}{4R}, m_c \geq \frac{a^2+b^2}{4R}$

$$\Rightarrow m_a + m_b + m_c \geq \frac{b^2+c^2}{4R} + \frac{c^2+a^2}{4R} + \frac{c^2+a^2}{4R} = \frac{a^2+b^2+c^2}{2R}$$

$$\Leftrightarrow m_a + m_b + m_c \geq \frac{2(m_a^2+m_b^2+m_c^2)}{3R} \Leftrightarrow \frac{m_a^2+m_b^2+m_c^2}{m_a+m_b+m_c} \leq \frac{3R}{2}$$

434. In $\Delta\, ABC, AA', BB', CC'$ - bisectors, AA'', BB'', CC'' - symedians:

$$8\prod\left(\frac{a^2}{b^2+c^2}\right) \leq \frac{area\,(A''B''C'')}{area\,(A'B'C')} \leq \frac{1}{8}\prod\left(\frac{a+b}{c}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

En un triángulo ABC, siendo AA', BB', CC' - biscetrices y AA'', BB'', C'' - simedians

$$8\left(\frac{a^2}{b^2+c^2}\right)\left(\frac{b^2}{c^2+a^2}\right)\left(\frac{c^2}{a^2+b^2}\right) \leq \frac{[A''B''C'']}{[A'B'C']} \leq \frac{1}{8}\left(\frac{a+b}{c}\right)\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right)$$

Ahora bien

$\frac{[A'B'C']}{[ABC]} = \frac{1+xyz}{(1+x)(1+y)(1+z)}, \frac{[A''B''C'']}{[ABC]} = \frac{1+mnp}{(1+m)(1+n)(1+p)}$, ***donde***

$$x = \frac{A'B}{A'C}, y = \frac{B'C}{B'A}, z = \frac{C'A}{C'A}, A' \in BC, B' \in CA, C' \in AB$$

$$m = \frac{A''B}{A''C}, n = \frac{B''C}{B''A}, p = \frac{C''A}{C''B}, A'' \in BC, B'' \in CA, C'' \in AB$$

Para bisecrices y simedianas

$$x = \frac{A'B}{A'C} = \frac{AB}{AC} = \frac{c}{b}, y = \frac{B'C}{B'A} = \frac{a}{c}, z = \frac{C'A}{C'B} = \frac{b}{a} \Leftrightarrow xyz = 1$$

$$m=\frac{A''B}{A''C}=\frac{AB^2}{AC^2}=\frac{c^2}{b^2},n=\frac{B''C}{B''A}=\frac{a^2}{c^2},p=\frac{C''A}{C''B}=\frac{b^2}{a^2}\Leftrightarrow mnp=1$$

$$\Rightarrow\frac{[A'B'C']}{[ABC]}=\frac{2abc}{(a+b)(b+c)(c+a)}\wedge\frac{[A''B''C'']}{[ABC]}=\frac{2a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

Por lo tanto

$$\frac{\lfloor A''B''C''\rfloor}{[A'B'C']}=\frac{a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\cdot\frac{(a+b)(b+c)(c+a)}{abc}\geq$$

$$\geq\frac{8a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\quad\text{(Válido por } MA\geq MG\text{)}$$

$$\frac{\lfloor A''B''C''\rfloor}{[A'B'C']}=\frac{a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\cdot\frac{(a+b)(b+c)(c+a)}{abc}\leq$$

$$\leq\frac{(a+b)(b+c)(c+a)}{8abc}\quad\text{(Válido por } MA\geq MG\text{)}$$

435. In $\Delta\ ABC$:

$$\frac{m_am_bm_c}{h_ah_bh_c}\geq\frac{abc+a^3+b^3+c^3}{4abc}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{m_am_bm_c}{h_ah_bh_c}\geq\frac{abc+a^3+b^3+c^3}{4abc}$

Recordar las siguientes identidades en un triángulo ABC

$$a+b+c=2s,\qquad abc=4sRr,\qquad ab+bc+ca=s^2+r^2+4Rr,$$

$$a^2+b^2+c^2=2(s^2-r^2-4Rr)$$

$$\Rightarrow a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc=$$

$$=2s(s^2-3r^2-12Rr)+12sRr\Rightarrow a^3+b^3+c^3=2s(s^2-3r^2-6Rr)$$

Se demostro anteriormente que $\frac{m_am_bm_c}{h_ah_bh_c}\geq\frac{R}{2r}$

Es suficiente demostrar

$$\frac{R}{2r} \geq \frac{a^3+b^3+c^3+abc}{4abc} \Leftrightarrow \frac{R}{2r} \geq \frac{2s(s^2-3r^2-4Rr)}{16sRr} \Leftrightarrow$$

$$\Leftrightarrow s^2 \leq 4R^2+3r^2+4Rr \text{ (Gerretsen's inequality)}$$

436. In $\Delta\, ABC$:

$$r_a^3+r_b^3+r_c^3+24rs^2 \leq \left(\frac{9R}{2}\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru, Solution 2 by Soumitra Mandal-Chandar Nagore-India, Solution 3 by Francisco Javier Garcia Capitan-Spain

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo $ABC: r_a^3+r_b^3+r_c^3+24rs^2 \leq \left(\frac{9R}{2}\right)^3$

Recordar la siguiente identidad y desigualdad en un $\Delta\, ABC$

$$r_a+r_b+r_c = 4R+r \leq 4R+\frac{R}{2} = \frac{9R}{2}, \qquad r_ar_br_c = Ss = s^2r$$

Es suficiente probar $x^3+y^3+z^3+24xyz \leq (x+y+z)^3$***, donde***

$$x = r_a > 0;\ y = r_b > 0;\ z = r_c > 0$$

$$\Leftrightarrow x^3+y^3+z^3+24xyz \leq x^3+y^3+z^3+3(x+y)(y+z)(z+x)$$

$$\Leftrightarrow 24xyz \leq 3(x+y)(y+z)(z+x) \text{ (Válido por } MA \geq MG)$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$r_a = \frac{\Delta}{p-a}, r_b = \frac{\Delta}{p-b}, r_c = \frac{\Delta}{p-c} \text{ and } 2r \leq R$$

we know, $(x+y+z)^3 \geq x^3+y^3+z^3+24xyz$***. Now,***

$$\prod_{cyc} r_a = \frac{\Delta^3}{(p-a)(p-b)(p-c)} = \frac{p\Delta^3}{p(p-a)(p-b)(p-c)}$$

$$= \frac{p\Delta^3}{\Delta^2} = rp^2.$$ ***Putting*** $x = r_1, y = r_b, z = r_c$

$$\therefore \sum_{cyc} r_a^3 + 24r_ar_br_c \le \left(\sum_{cyc} r_a\right)^3 = \left(\sum_{cyc} \frac{\Delta}{p-a}\right)^3$$

$$\Rightarrow \sum_{cyc} r_a^3 + 24rp^2 \le \frac{\Delta^3}{(p-a)^3(p-b)^3(p-c)^3}\left(\sum_{cyc}(p-a)(p-b)\right)^3$$

$$= \frac{p^3\Delta^3}{\Delta^6} r^3(r+4R)^3 = \frac{p^3}{\Delta^3} r^3(r+4R)^3 = (r+4R)^3 \le \left(\frac{9R}{2}\right)^3$$

Solution 3 by Francisco Javier Garcia Capitan-Spain

$$r_a + r_b + r_c = 4R + r$$

$$r_br_c + r_cr_a + r_ar_b = s^2$$

$$r_ar_br_c = s^2r$$ ***and the identity***

$$x^3 + y^3 + z^3 = (x+y+z)^3 - 3(x+y+z)(yz+zx+xy) + 3xyz$$

we get $r_a^3 + r_b^3 + r_c^3 + 24s^2r \le \left(\frac{9R}{2}\right)^3$

$$\Leftrightarrow (r_a + r_b + r_c)^3 - 3(r_a + r_b + r_c)(r_br_c + r_cr_a + r_ar_b) + 3r_ar_br_c + 24s^2r \le \left(\frac{9R}{2}\right)^3$$

$$\Leftrightarrow (4R+r)^3 - 3(4R+r)s^2 + 27s^2r \le \left(\frac{9R}{2}\right)^3$$

$$\Leftrightarrow (4R+r)^3 - \left(\frac{9R}{2}\right)^3 \le 3(4R+r)s^2 - 27s^2r$$

$$\Leftrightarrow -\frac{1}{8}(R-2r)(4r^2 + 50Rr + 217R^2) \le 12(R-2s)s^2,$$

which is true.

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437. If in $\Delta\ ABC, m(\sphericalangle A) = 90^\circ$ then: $h_a\sqrt{2bc} \le 2sR(\sqrt{2}-1)$

Proposed by Daniel Sitaru – Romania

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$R \ge (\sqrt{2}+1)r \to \textbf{Lemma}$$

$$h_a\sqrt{2bc} = 2sr,\ h_a\sqrt{2bc} \le 2S,\ \sqrt{2bc} \le a,\ 2bc \le a^2$$

$$a^2 = b^2 + c^2 \ge 2bc \quad \textbf{(Proved)}$$

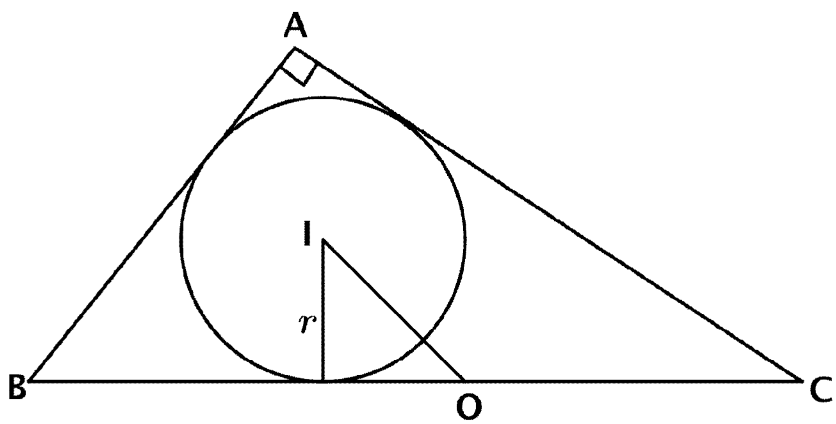
$$R = \frac{a}{2},\ r = \frac{b+c-a}{2},\ \frac{a}{2} \ge (\sqrt{2}+1)\left(\frac{b+c-a}{2}\right)$$

$$a \ge (\sqrt{2}+1)(a\cdot\sin b + a\cdot\cos b - a)$$

$$\sqrt{2}-1 \ge \sin b + \cos b - 1,\ \sqrt{2} \ge \sin b + \cos b$$

$$\sin 2b \le 1 \ \textbf{(Proved)}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia



$$a^2 = b^2 + c^2$$

$$a \ge \sqrt{2bc}\ (*)$$

$$\left.\begin{matrix} OI \ge r \\ OI^2 = R^2 - 2Rr \end{matrix}\right\} \Rightarrow R^2 - 2Rr \ge r^2$$

$$\left(\frac{r}{R}\right)^2 \le 1 - \frac{2r}{R}$$

$$\left(\frac{r}{R}+1\right)^2 \le 2, \frac{r}{R} \le \sqrt{2}-1 \quad (**)$$

Prove that: $h_a \cdot \sqrt{bc} \le 2p \cdot R(\sqrt{2}-1)$

$$h_a \cdot \sqrt{2bc} \overset{(*)}{\le} h_a \cdot a = \frac{2S}{a} \cdot a = 2S = 2pr = 2p \cdot \frac{r}{R} \cdot R \overset{(**)}{\le} 2pR(\sqrt{2}-1)$$

438. **In $\Delta\, ABC$:**

$$\frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c} + \frac{2r}{R} \ge 4$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru, Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c} + \frac{2r}{R} \ge 4$

Recordar la siguientes identidades en un triángulo ABC

$$h_a = \frac{2S}{a}, h_b = \frac{2S}{b}, h_c = \frac{2S}{c}, r_a = \frac{S}{s-a}, r_b = \frac{S}{s-b}, r_c = \frac{S}{s-c}$$

La desigualdad propuesta es equivalente

$$\frac{2(s-a)}{a} + \frac{2(s-b)}{b} + \frac{2(s-c)}{c} + \frac{8(s-a)(s-b)(s-c)}{abc} \ge 4$$

$$\Leftrightarrow \frac{s-a}{a} + \frac{s-b}{b} + \frac{s-c}{c} + \frac{4(s-a)(s-b)(s-c)}{abc} \ge 2$$

Como a, b, c son lados de un triángulo ABC, realizamos las siguientes sustituciones

$$x = s-a > 0, y = s-b > 0, z = s-c > 0, x+y = c, y+z = a, z+x = c,$$

$$x+y+z = s$$

$$\Rightarrow \frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}+\frac{4xyz}{(x+y)(y+z)(z+x)}\geq 2$$

$$\Leftrightarrow x(x+y)(x+z)+y(y+z)(y+x)+z(z+x)(z+y)+4xyz\geq$$

$$\geq 2(x+y)(y+z)(z+x)$$

$$\Leftrightarrow x^3+x^2(y+z)+y^3+y^2(z+x)+z^3+z^2(x+y)+7xyz\geq$$

$$\geq 2x^2(y+z)+2y^2(z+x)+2z^2(x+y)+4xyz$$

$$\Leftrightarrow x^3+y^3+z^3-x^2(y+z)-y^2(z+x)-z^2(z+x)+3xyz=$$

$$= x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y)\geq 0 \text{ (Schur)}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{h_a}{r_a}+\frac{2r}{R}\geq 4 \Leftrightarrow \sum_{cyc}\frac{\frac{2\Delta}{a}}{\frac{\Delta}{p-a}}+\frac{2r}{R}\geq 4 \Leftrightarrow \sum_{cyc}\frac{p-a}{a}+\frac{r}{R}\geq 2$$

$$\Leftrightarrow p\left(\sum_{cyc}\frac{1}{a}\right)+\frac{r}{R}\geq 5 \Leftrightarrow \frac{p(ab+bc+ca)}{abc}+\frac{r}{R}\geq 5 \Leftrightarrow \frac{p^2+r^2+4Rr}{4Rr}+\frac{r}{R}\geq 5$$

$$\Leftrightarrow p^2\geq 16Rr-5r^2, \text{ which is true}$$

$$\therefore \sum_{cyc}\frac{h_a}{r_a}+\frac{2r}{R}\geq 4$$

439. Let ABC be a triangle. Prove that:

$$2(ab+bc+ca)-(a^2+b^2+c^2)\geq 2\left(ab\sin\frac{C}{2}+bc\sin\frac{A}{2}+ca\sin\frac{B}{2}\right)\geq$$

$$\geq 6S\sqrt[3]{\frac{4R}{p}}\geq 4S\sqrt{3}$$

Proposed by Vasile Jiglau – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru,

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo ABC un triángulo. Probar que

$$2(ab+bc+ca)-(a^2+b^2+c^2)\geq 2\left(ab\sin\frac{C}{2}+bc\sin\frac{A}{2}+ca\sin\frac{B}{2}\right)\geq$$

$$\geq 6S\sqrt[3]{\frac{4R}{p}}\geq 4S\sqrt{3}$$

Tener en cuenta las siguientes identidades y desigualdades en triángulo ABC

$$ab=2S\csc C,\ bc=2S\csc A,\ ca=2S\csc B,$$

$$a^2+b^2+c^2=4S(\cot A+\cot B+\cot C)$$

$$\frac{4R}{p}=\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2},\ \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1,$$

$$2p\leq 3\sqrt{3}R;\ \csc A-\cot A=\frac{1-\cos A}{\sin A}=\frac{2\sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\tan\frac{A}{2},$$

$$\csc B-\cot B=\tan\frac{B}{2},\ \csc C-\cot C=\tan\frac{C}{2}$$

Ahora bien probaremos

$$2(ab+bc+ca)-(a^2+b^2+c^2)\geq 2\left(ab\sin\frac{C}{2}+bc\sin\frac{A}{2}+ca\sin\frac{B}{2}\right)$$

$$\Leftrightarrow 4S(\csc A-\cot A)+4S(\csc B-\cot B)+4S(\csc C-\cot C)\geq$$

$$\geq 4S\csc A\sin\frac{A}{2}+4S\csc B\sin\frac{B}{2}+4S\csc C\sin\frac{C}{2}$$

$$\Leftrightarrow 4S\tan\frac{A}{2}+4S\tan\frac{B}{2}+4S\tan\frac{C}{2}\geq 2S\sec\frac{A}{2}+2S\sec\frac{B}{2}+2S\sec\frac{C}{2}$$

$$\Leftrightarrow 2\tan\frac{A}{2}+2\tan\frac{B}{2}+2\tan\frac{C}{2}\geq\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}$$

Siendo $x=\tan\frac{A}{2},\ y=\tan\frac{B}{2},\ z=\tan\frac{C}{2}>0\Leftrightarrow xy+yz+zx=1$

$$x^2+1=x^2+xy+yz+zx=(x+y)(x+z),\ y^2+1=(y+z)(y+x),$$

$$z^2+1=(z+y)(z+x)$$

La desigualdad propuesta es equivalente

$$2(x+y+z)\geq\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}=$$

$$=\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)}$$

Como $x,y,z>0$***; Aplicando*** $MA\geq MG$

$$(x+y)+(x+z)\geq 2\sqrt{(x+y)(x+z)},$$

$$(y+z)+(y+x)\geq 2\sqrt{(y+z)(y+x)},$$

$$(z+x)+(z+y)\geq 2\sqrt{(z+x)(z+y)}$$

Sumando y simplificando las desigualdades se obtiene

$$\Rightarrow 2(x+y+z)\geq\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)}$$

(LQQD)

Nuevamente por $MA\geq MG$

$$2S\,bc\sin\frac{A}{2}+2S\,ca\sin\frac{B}{2}+2S\,ab\sin\frac{C}{2}=2S\sec\frac{A}{2}+2S\sec\frac{B}{2}+2S\sec\frac{C}{2}\geq$$

$$6S\sqrt[3]{\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}}=6S\sqrt[3]{\frac{4R}{p}}\ \textit{(LQQD)}$$

$$\Rightarrow 6S\sqrt[3]{\frac{4R}{p}}=6S\sqrt[3]{\frac{8R}{2p}}\geq 6S\sqrt[3]{\frac{8R}{3\sqrt{3}R}}=4S\sqrt{3}\ \textit{(LQQD)}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$2\sum_{cyc}ab\sin\frac{C}{2}=2\sum_{cyc}ab\sqrt{\frac{(p-a)(p-b)}{ab}}=2\sum_{cyc}\sqrt{(ap-a^2)(bp-b^2)}$$

$$\leq\frac{2}{3}\left(\sum_{cyc}\sqrt{ap-a^2}\right)^2\left[\because\left(\sum_{cyc}x\right)^2\geq 3\sum_{cyc}xy\right]$$

$$\leq 2\sum_{cyc}(ap-a^2)\left[\because 3\sum_{cyc}x \geq \left(\sum_{cyc}\sqrt{x}\right)^2\right]$$

$$= 2p(a+b+c) - 2\sum_{cyc}a^2 = 2\sum_{cyc}ab - \sum_{cyc}a^2$$

$$\therefore \sum_{cyc}ab - \sum_{cyc}a^2 \geq 2\sum_{cyc}ab\sin\frac{C}{2}$$

$$2\sum_{cyc}ab\sin\frac{C}{2} \overset{AM\geq GM}{\geq} 2\cdot\sqrt[3]{(abc)^2\prod_{cyc}\sin\frac{A}{2}} = 6\sqrt[3]{abc\prod_{cyc}(p-a)} = 6\sqrt[3]{4R\Delta\cdot pr^2}$$

$$= 6\Delta\sqrt[3]{\frac{4Rpr^2}{\Delta^2}} = 6\Delta\sqrt[3]{\frac{4R}{p}}.$$ **So,** $2\sum_{cyc}ab\sin\frac{C}{2} \geq 6\Delta\sqrt[3]{\frac{4R}{p}}$

we need to prove,

$$6\Delta\sqrt[3]{\frac{4R}{p}} \geq 4\Delta\sqrt{3} \Leftrightarrow 3\sqrt[3]{\frac{4R}{p}} \geq 2\sqrt{3} \Leftrightarrow \frac{3\sqrt{3}}{2}R \geq p,$$ ***which is true***

$$\therefore 2\sum_{cyc}ab - \sum_{cyc}a^2 \geq 2\sum_{cyc}ab\sin\frac{C}{2} \geq 6\Delta\sqrt[3]{\frac{4R}{p}} \geq 4\sqrt{3}\Delta$$

440. The incircle of a right triangle touch the hypotenuse in N and one of the sides of triangle in M. If c is the hypotenuse then:

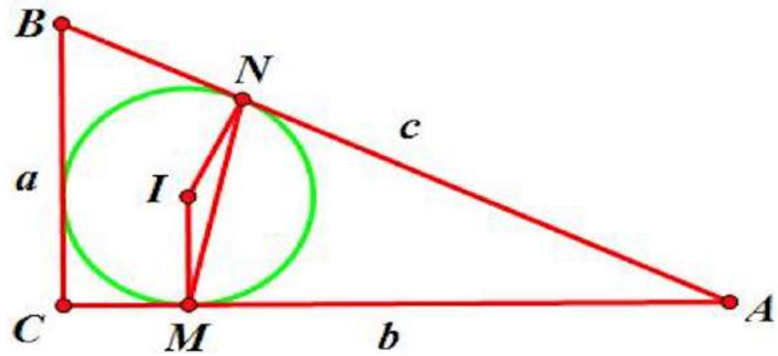
$$MN \leq \frac{2\sqrt{3}}{9}c$$

Proposed by Boris Colakovic-Belgrade-Serbia

Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam

Solution 2 by Ravi Prakash-New Delhi-India

Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam



The incircle of a right triangle touch the hypotenuse in N and one of the sides of triangle in M. If c is the hypotenuse then $MN \le \frac{2\sqrt{3}}{9}c$

We put the length of the edges as shown on the left $\Rightarrow MA = NA = \frac{b+c-a}{2}$

We have $MN^2 = MA^2 + NA^2 - 2MA \cdot NA \cdot \cos A$

$$\Rightarrow MN^2 = \left(\frac{b+c-a}{2}\right)^2 + \left(\frac{b+c-a}{2}\right)^2 - 2\left(\frac{b+c-a}{2}\right)\left(\frac{b+c-a}{2}\right)\cdot\frac{b}{c}$$

$$\Rightarrow MN^2 = 2\left(\frac{b+c-a}{2}\right)^2\left(1-\frac{b}{c}\right) \Rightarrow MN^2 = \frac{(b+c-a)^2(c-b)}{2c} \Rightarrow$$

$$\Rightarrow MN^2 = \frac{\left(b+c-\sqrt{c^2-b^2}\right)^2(c-b)}{2c}$$

$$f(b) = \frac{\left(b+c-\sqrt{c^2-b^2}\right)^2(c-b)}{2c}$$

$$\frac{d}{db}f(b) = \frac{d}{db}\left(\frac{\left(b+c-\sqrt{c^2-b^2}\right)^2(c-b)}{2c}\right) = \frac{1}{2c}\cdot\frac{d}{db}\left[\left(b+c-\sqrt{c^2-b^2}\right)^2(c-b)\right]$$

$$\frac{d}{db}f(b) = \frac{1}{dc}\cdot\left[\left(\frac{d}{db}\left[\left(b+c-\sqrt{c^2-b^2}\right)^2\right]\right)\cdot(c-b) + \left(b+c-\sqrt{c^2-b^2}\right)^2\cdot\left(\frac{d}{db}(c-b)\right)\right] =$$

$$= \frac{1}{2c}\left[2\left(1+\frac{2b}{2\sqrt{c^2-b^2}}\right)\left(b+c-\sqrt{c^2-b^2}\right)(c-b) - \left(b+c-\sqrt{c^2-b^2}\right)^2\right]$$

$$\Rightarrow \frac{d}{db}f(b) = \frac{b+c-\sqrt{c^2-b^2}}{2c}\left[\frac{2(c-b)\left(b+\sqrt{c^2-b^2}\right)}{\sqrt{c^2-b^2}} - \left(b+c-\sqrt{c^2-b^2}\right)\right] =$$

$$= \frac{b+c-\sqrt{c^2-b^2}}{2c}\left[\frac{2(c-b)\left(b+\sqrt{c^2-b^2}\right)-(b+c)\sqrt{c^2-b^2}+c^2-b^2}{\sqrt{c^2-b^2}}\right]$$

$$\Rightarrow \frac{d}{db}f(b) = \frac{b+c-\sqrt{c^2-b^2}}{2c}\left[\frac{(-3b+c)\sqrt{c^2-b^2}-3b^2+2bc+c^2}{\sqrt{c^2-b^2}}\right] =$$

$$= \frac{(b+c)(-3b+c)\sqrt{c^2-b^2}-(-3b+c)(c^2-b^2)+(b+c)(-3b^2+2bc+c^2)-(-3b^2+2bc+c^2)\sqrt{c^2-b^2}}{2c\sqrt{c^2-b^2}}$$

$$\Rightarrow \frac{d}{db}f(b) = \frac{-4bc\sqrt{c^2-b^2}+6b(c^2-b^2)}{2c\sqrt{c^2-b^2}} = \frac{b\left(6\sqrt{c^2-b^2}-4c\right)}{2c}$$

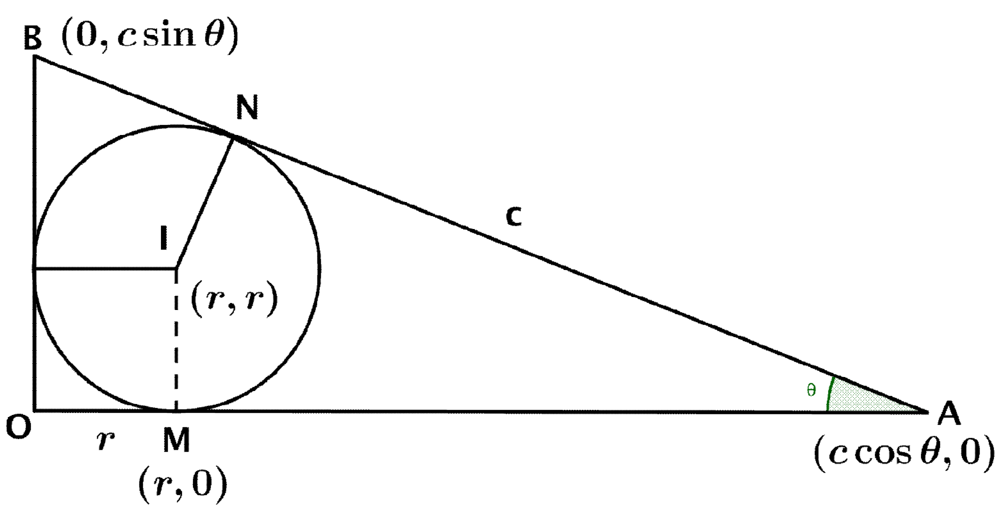
We have $\frac{d}{db}f(b) = 0 \Rightarrow \frac{b\left(6\sqrt{c^2-b^2}-4c\right)}{2c} = 0 \Rightarrow 6\sqrt{c^2-b^2} = 4c \Rightarrow$

$$\Rightarrow 36c^2-36b^2 = 16c^2 \Rightarrow 36b^2 = 20c^2 \Rightarrow b = \frac{\sqrt{5}}{3}c$$

We have $f(b) \le \frac{4}{27}c^2 \Rightarrow MN^2 \le \frac{4}{27}c^2 \Rightarrow MN \le \frac{2\sqrt{3}}{9}c$ ***(QED).***

The equality occurs when $a = \frac{2}{3}c$ ***and*** $b = \frac{\sqrt{5}}{3}c$

Solution 2 by Ravi Prakash-New Delhi-India



Let $0 < \theta < \frac{\pi}{2}; r = \frac{c^2\cos\theta\sin\theta}{c\cos\theta+c\sin\theta+c} = \frac{c\cos\theta\sin\theta}{\cos\theta+\sin\theta+1}$

Equaltion of AB ***is*** $x\sec\theta + y\csc\theta = c$

$$\Rightarrow x\sec\theta + y\csc\theta = \frac{r(\cos\theta+\sin\theta+1)}{\cos\theta\sin\theta}$$

$$\Rightarrow (x-r)\sin\theta + (y-r)\cos\theta = r \quad \textbf{(1)}$$

slope of IN $\cot\theta$***, its eqn is*** $y - r = \frac{\cos\theta}{\sin\theta}(x - r)$

$$\Rightarrow (x - r)\cos\theta - (y - r)\sin\theta = 0 \quad \textbf{(2)}$$

For coordinates of N***, we solve (1), (2) to obtain***

$$x - r = r\sin\theta, y - r = r\cos\theta \Rightarrow N \equiv \left(r(1 + \sin\theta), r(1 + \cos\theta)\right)$$

coordinates of $M \equiv (r, 0)$

$$\therefore MN^2 = (r + r\sin\theta - r)^2 + (r + r\cos\theta)^2 = r^2\{\sin^2\theta + (1 + \cos\theta)^2\}$$

$$= 2r^2(1 + \cos\theta) = \frac{2c^2(1 + \cos\theta)\cos^2\theta\sin^2\theta}{(1 + \cos\theta + \sin\theta)^2}$$

$$= \frac{2c^2(1 + \cos\theta)\cos^2\theta\sin^2\theta}{2(1 + \cos\theta)(1 + \sin\theta)} = \frac{c^2(1 - \sin^2\theta)\sin^2\theta}{1 + \sin\theta}$$

$$= c^2(1 - \sin\theta)\sin^2\theta = c^2(\sin^2\theta - \sin^3\theta) = f(\theta) \quad \textit{(say)}$$

$$f'(\theta) = c^2(2\sin\theta - 3\sin^2\theta)\cos\theta = c^2(2 - 3\sin\theta)\sin\theta\cos\theta$$

$$f'(\theta) = 0 \Rightarrow \sin\theta = \frac{2}{3}$$

$$f'(\theta) > 0 \textit{ if } 0 < \theta < \sin^{-1}\left(\frac{2}{3}\right) < 0 \textit{ if } \sin^{-1}\left(\frac{2}{3}\right) < \theta < \frac{\pi}{2}$$

$$\therefore \max_{0<\theta<\frac{\pi}{2}}(MN) = \sqrt{f\left(\sin^{-1}\left(\frac{2}{3}\right)\right)} = \frac{2c\sqrt{3}}{9} \therefore MN \le \frac{2c\sqrt{3}}{9}$$

441. In $\Delta\, ABC - N -$ **ninepoint center**

$$12r^2 \le AN^2 + BN^2 + CN^2 \le 3R^2$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Solution 2 by Ravi Prakash-New Delhi-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo N ***- nine point center. Probar en un*** $\Delta\, ABC$

$$12r^2 \le NA^2 + NB^2 + NC^2 \le 3R^2$$

Teorema Leibniz

Para cualquier punto P en el plano de triángulo ABC teniendo centroide G, se cumple

$$9PG^2 + a^2 + b^2 + c^2 = 3(PA^2 + PB^2 + PC^2)$$

Sea $P = N$, donde $NG = \frac{1}{6}OH = \frac{1}{6}\sqrt{9R^2 - (a^2 + b^2 + c^2)} \ge 0 \Leftrightarrow$

$$\Leftrightarrow 9R^2 \ge a^2 + b^2 + c^2$$

$$\Rightarrow 9NG^2 + a^2 + b^2 + c^2 = 3(NA^2 + NB^2 + NC^2)$$

$$\Rightarrow 3(NA^2 + NB^2 + NC^2) = 9NG^2 + a^2 + b^2 + c^2 \ge a^2 + b^2 + c^2 \ge$$

$$\ge ab + bc + ca \ge 18Rr \ge 36r^2$$

$\Rightarrow NA^2 + NB^2 + NC^2 \ge 12r^2$. ***Por último***

$$3(NA^2 + NB^2 + NC^2) = 9 \cdot \frac{1}{36}\left(9R^2 - (a^2 + b^2 + c^2)\right) + a^2 + b^2 + c^2$$

$$\Rightarrow 3(NA^2 + NB^2 + NC^2) = \frac{9R^2 + 3(a^2 + b^2 + c^2)}{4} \le \frac{9R^2 + 27R^2}{4} = 9R^2$$

$\Rightarrow NA^2 + NB^2 + NC^2 \le 3R^2$ ***(LQQD)***

Solution 2 by Ravi Prakash-New Delhi-India

Let's take O, the circumcentre as origin.

Let points A, B, C be z_1, z_2, z_3 respectively, then orthocentre of $\Delta\, ABC$ is

$$H(z_1 + z_2 + z_3)$$

Also, N is the mid-point of OH, i.e. affix of N is $\frac{1}{2}(z_1 + z_2 + z_3)$

Note that $|z_1| = |z_2| = |z_3| = R$

We have $AN^2 = \left|\frac{1}{2}(z_1 + z_2 + z_3) - z_1\right|^2 = \frac{1}{4}|z_2 + z_3 - z_1|^2$

$BN^2 = \frac{1}{4}|z_1 + z_3 - z_2|^2$ ***and*** $CN^2 = \frac{1}{4}|z_1 + z_2 - z_3|^2$

Now, $AN^2 + BN^2 + CN^2 =$

$$= \frac{1}{4}\left\{\begin{array}{c} |z_1|^2 + |z_2|^2 + |z_3|^2 + z_2\overline{z_3} + \overline{z_2}z_3 - z_1\overline{z_2} - \overline{z_1}z_2 - z_1\overline{z_3} - \overline{z_1}z_3 + \\ +|z_1|^2 + |z_2|^2 + |z_3|^2 + z_2\overline{z_3} + \overline{z_2}z_3 - \overline{z_1}z_3 - z_1\overline{z_3} - z_2\overline{z_3} - \overline{z_2}z_3 + \\ +|z_1|^2 + |z_2|^2 + |z_3|^2 + z_1\overline{z_2} + \overline{z_1}z_2 - \overline{z_1}z_3 - z_1\overline{z_3} - z_2\overline{z_3} - \overline{z_2}z_3 \end{array}\right\}$$

$$= \frac{3}{4}(|z_1|^2 + |z_2|^2 + |z_3|^2) - \frac{1}{4}(\overline{z_1}z_2 + z_1\overline{z_2} + \overline{z_1}z_3 + z_1\overline{z_3} + \overline{z_2}z_3 + z_2\overline{z_3}) \quad \textit{(1)}$$

$$= \frac{3}{4}(3R^2) + \frac{1}{4}(|z_1|^2 + |z_2|^2 + |z_3|^2 - |z_1 + z_2 + z_3|^2)$$

$$= 3R^2 - \frac{1}{4}|z_1 + z_2 + z_3|^2 \le 3R^2. \textit{ Also, } AN^2 + BN^2 + CN^2 =$$

$$= \frac{1}{4}(|z_1|^2 + |z_2|^2 + |z_3|^2) + \frac{1}{4}(|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2)$$

$$= \frac{3}{4}R^2 + \frac{1}{4}(a^2 + b^2 + c^2)$$

where $a = BC, b = CA, c = AB$

$$= \frac{3}{4}R^2 + \frac{1}{8S}(a + b + c)(a^2 + b^2 + c^2)$$

$$\ge \frac{3}{4}R^2 + \frac{1}{8S}3(abc)^{\frac{1}{3}}(abc)^{\frac{2}{3}} \quad \textit{(3)}$$

$$\ge \frac{3}{4}(2r)^2 + \frac{1}{8S}(9)(abc) = 3r^2 + \left(\frac{9}{2}\right)\frac{abc}{4\Delta}\cdot\frac{\Delta}{s}$$

$$= 3r^2 + \frac{9}{2}(R)r \ge 3r^2 + \frac{9}{2}(2r)r = 12r^2$$

Thus, $12r^2 \le AN^2 + BN^2 + CN^2 \le 3R^2$

442. **In $\Delta\, ABC, \Omega -$ first Brocard point, $\omega -$ Brocard's angle:**

$$\left(\sum \Omega A^2\right)\left(\sum \frac{1}{\sin^2 A}\right) \ge \frac{1}{\sin^2 \omega}\cdot\frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo Ω primer punto de Brocard y ω el ángulo de Brocard. Probar en un triángulo ABC

$$(\Omega A^2 + \Omega B^2 + \Omega C^2)(\csc^2 A + \csc^2 B + \csc^2 C) \geq \csc^2 \omega \cdot \frac{a^2b^2+b^2c^2+c^2a^2}{a^2+b^2+c^2}$$

Como ω es el ángulo de Brocard, se verifica lo siguiente

$$\cot\omega = \cot A + \cot B + \cot C \Leftrightarrow \cot^2\omega = (\cot A + \cot B + \cot C)^2$$

$$\Leftrightarrow \cot^2\omega = \cot^2 A + \cot^2 B + \cot^2 C + 2(\cot A\cot B + \cot B\cot C + \cot C\cot A)$$

$$\Leftrightarrow \cot^2\omega = \cot^2 A + \cot^2 B + \cot^2 C + 2 \Leftrightarrow$$

$$\Leftrightarrow 1 + \cot^2\omega = 1 + \cot^2 A + 1 + \cot^2 B + 1 + \cot^2 C$$

$\Leftrightarrow \csc^2\omega = \csc^2 A + \csc^2 B + \csc^2 C$ ***Es necesario probar lo siguiente***

$$\Omega A^2 + \Omega B^2 + \Omega C^2 \geq \frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2}$$

Theorem Leibniz: Siendo P un punto interior en el plano de ΔABC se cumple

$PA^2 + PB^2 + PC^2 \geq \frac{a^2+b^2+c^2}{3}$, ***sea*** $P = \Omega \Leftrightarrow \Omega A^2 + \Omega B^2 + \Omega C^2 \geq \frac{a^2+b^2+c^2}{3}$

Es necesario probar lo siguiente

$$\frac{a^2 + b^2 + c^2}{3} \geq \frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2} \Leftrightarrow (a^2 + b^2 + c^2)^2 \geq 3(a^2b^2 + b^2c^2 + c^2a^2)$$

(Lo cual es cierto)

443. In $\Delta\ ABC$:

$$\frac{m_am_bm_c}{h_ah_bh_c} \geq \left(\frac{a^2 + b^2 + c^2}{ab + bc + ca}\right)^2$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Lemma: En un triángulo ABC se cumple la siguiente desigualdad

$$\frac{R}{2r} \geq \left(\frac{a^2 + b^2 + c^2}{ab + bc + ca}\right)^2$$

Como a, b, c son los lados de un triángulo ABC, realizamos los siguientes cambios de variables

$$x = s - a > 0, y = s - b > 0, z = s - c > 0, x + y = c, y + z = a, z + x = b$$

$$\Rightarrow \frac{R}{2r} = \frac{abc}{8(s-a)(s-b)(s-c)} = \frac{(x+y)(y+z)(z+x)}{8xyz} =$$

$$= \frac{(x+y)}{8z} + \frac{(y+z)}{8x} + \frac{(z+x)}{8y} + \frac{1}{4}$$

$$\Rightarrow \frac{a^2+b^2+c^2}{ab+bc+ca} = \frac{(x+y)^2+(y+z)^2+(z+x)^2}{(x+y)(y+z)+(y+z)(z+x)+(z+x)(x+y)} =$$

$$= \frac{2(x^2+y^2+z^2+xy+yz+zx)}{x^2+y^2+z^2+3xy+3yz+3zx}$$

La desigualdad propuesta es equivalente

$$\frac{(x+y)}{8z} + \frac{(y+z)}{8x} + \frac{(z+x)}{8y} + \frac{1}{4} \geq \left(\frac{2(x^2+y^2+z^2+xy+yz+zx)}{x^2+y^2+z^2+3xy+3yz+3zx}\right)^2$$

Ahora bien por la desigualdad de Cauchy

$$\frac{(x+y)}{8z} + \frac{(y+z)}{8x} + \frac{(z+x)}{8y} = \left(\frac{x}{8y}+\frac{y}{8x}\right) + \left(\frac{y}{8z}+\frac{z}{8y}\right) + \left(\frac{z}{8x}+\frac{x}{8z}\right) =$$

$$= \frac{x^2+y^2}{8xy} + \frac{y^2+z^2}{8yz} + \frac{z^2+x^2}{8zx}$$

$$\frac{x^2+y^2}{8xy} + \frac{y^2+z^2}{8yz} + \frac{z^2+x^2}{8zx} = \frac{\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2}{8(xy+yz+zx)} =$$

$$= \frac{2(x^2+y^2+z^2) + 2\sum\sqrt{(x^2+y^2)(x^2+z^2)}}{8(xy+yz+zx)} \geq$$

$$\geq \frac{2(x^2+y^2+z^2)+2\sum(x^2+yz)}{8(xy+yz+zx)} = \frac{4(x^2+y^2+z^2)+2(xy+yz+zx)}{8(xy+yz+zx)} = \frac{x^2+y^2+z^2}{2(xy+yz+zx)} + \frac{1}{4}$$

Por transitividad $\frac{(x+y)}{8z} + \frac{(y+z)}{8x} + \frac{(z+x)}{8y} + \frac{1}{4} \geq \frac{x^2+y^2+z^2}{2(xy+yz+zx)} + \frac{1}{2}$

Por último demostraremos

$$\frac{x^2+y^2+z^2}{2(xy+yz+zx)} + \frac{1}{2} \geq \left(\frac{2(x^2+y^2+z^2+xy+yz+zx)}{x^2+y^2+z^2+3xy+3yz+3zx}\right)^2$$

$$\Leftrightarrow \frac{x^2+y^2+z^2}{2(xy+yz+zx)} + \frac{1}{2} \ge \left(\frac{\frac{2(x^2+y^2+z^2)}{xy+yz+zx}}{\frac{x^2+y^2+z^2}{xy+yz+zx}+3}\right)^2$$ ***, donde*** $\rightarrow m = \frac{x^2+y^2+z^2}{xy+yz+zx} \ge 1 > 0$

$$\Leftrightarrow \frac{m+1}{2} \ge \left(\frac{2(m+1)}{m+3}\right)^2 \Leftrightarrow (m+3)^2 \ge 8(m+1) \Leftrightarrow$$

$$\Leftrightarrow (m+3)^2 - 8(m+1) = (m-1)^2 \ge 0$$ ***(Lo cual es cierto)***

Probar en un triángulo ABC: $\frac{m_a m_b m_c}{h_a h_b h_c} \ge \left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2$

Recordar las siguientes identidades y desigualdades conocidas en un

$\Delta\ ABC.$

$$h_a = \frac{2S}{a},\quad h_b = \frac{2S}{b},\quad h_c = \frac{2S}{c} \Leftrightarrow h_a h_b h_c = \frac{8S^3}{abc} = \frac{8S^3}{4RS} = \frac{2S^2}{R}$$

$$m_a \ge \sqrt{s(s-a)}, m_b \ge \sqrt{s(s-b)}, m_c \ge \sqrt{s(s-c)} \Leftrightarrow m_a m_b m_c = Sp = S\cdot\frac{S}{r} = \frac{S^2}{r}$$

Luego $\frac{m_a m_b m_c}{h_a h_b h_c} \ge \frac{\frac{S^2}{r}}{\frac{2S^2}{R}} = \frac{R}{2r} \ge \left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2$ ***(LQQD)***

444. In $\Delta\ ABC$**:**

$$27r^2 \le m_a w_a + m_b w_b + m_c w_c \le 3r^2 + 6R^2$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$27r^2 \le \sum m_a \cdot w_a \le 3r^2 + 6R^2$$

I. $m_a \ge w_a$ ***(*)***

II. $w_a \ge h_a$ ***(**)***

III. $\sum h_a \ge 9r$ ***(***)***

1. $\sum m_a \cdot w_a \overset{(*)}{\le} \sum m_a^2 = \frac{3}{4}\sum a^2 =$

$$= \frac{3}{4} \cdot 2 \cdot (p^2 - 4Rr - r^2) \overset{p^2 \le 4R^2 + 4Rr + 3r^2}{\le}$$

$$\le \frac{3}{2} \cdot (4R^2 + 4Rr + 3r^2 - 4Rr - r^2) = \frac{3}{2} \cdot (4R^2 + 2r^2) = 6R^2 + 3r^2 \quad RHS$$

2. $\sum m_a \cdot w_a \overset{(**)}{\ge} \sum w_a^2 \ge \frac{1}{3} \cdot \left(\sum w_a\right)^2 \overset{(**)}{\ge}$

$$\ge \frac{1}{3}\left(\sum h_a\right)^2 \overset{(***)}{\ge} \frac{1}{3} \cdot 81r^2 = 27r^2 \quad LHS$$

445. In acute $\Delta\, ABC$:

$$1 + \left(\sum \tan A\right)\left(\sum \tan A \tan^2 C\right) > 2 \sum \tan A \tan C$$

Proposed by Gheorghe Alexe, Șerban George – Florin – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo acutángulo ABC

$$1 + \left(\sum \tan A\right)\left(\sum \tan A \tan^2 C\right) > 2 \sum \tan A \tan C$$

Como es un triángulo acutángulo $\Leftrightarrow \tan A, \tan B, \tan C > 0$

Tener en cuenta lo siguiente

$$\tan A = \frac{\tan B + \tan C}{\tan B \tan C - 1} > 0, \tan B = \frac{\tan C + \tan A}{\tan C \tan A - 1}, \tan C = \frac{\tan A + \tan B}{\tan A \tan B - 1} > 0$$

Lo cual implica $\rightarrow$

$$\rightarrow \tan B \tan C - 1 > 0, \tan C \tan A - 1 > 0, \tan A \tan B - 1 > 0$$

Sumando las desigualdades se obtiene

$$\tan A \tan C + \tan B \tan C + \tan A \tan B > 3$$

Aplicando la desigualdad de Cauchy

$$1 + \left(\sum \tan A\right)\left(\sum \tan A \tan^2 C\right) \ge 1 + \left(\sum \tan A \tan C\right)^2$$

Es suficiente probar $1 + \left(\sum \tan A \tan C\right)^2 > 2 \sum \tan A \tan C \Leftrightarrow$

$$\Leftrightarrow (\tan A \tan C + \tan B \tan C + \tan A \tan B - 1)^2 > 4 > 0$$

(Lo cual es cierto)

446. In $\Delta\ ABC$:

$$\frac{1}{r^3}\sum a^3 \cos B \cos C \geq 16\left(\sum \sin A\right)\left(\sum \cos^2 A\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo ABC. Probar que:

$$\frac{1}{r^3}\sum a^3 \cos B \cos C \geq 16(\sum \sin A)(\sum \cos^2 A) \rightarrow r \textit{ (Inradio)}$$

$$\frac{R^3}{r^3}(8\sin^3 A \cos B \cos C + 8\sin^3 B \cos A \cos C + 8\sin^3 C \cos A \cos B) \geq$$

$$\geq 16(\sin A + \sin B + \sin C)(\cos^2 A + \cos^2 B + \cos^2 C)$$

Tener presente en un triángulo ABC:

1) $\sin 4A + \sin 4B + \sin 4C = -4\sin 2A \sin 2B \sin 2C$

2) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

3) $\frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

4) $\sin(B + C) = \sin A \wedge$ ***5)*** $\sin(2B + 2C) = -\sin 2A$

$$T_1 = 8\sin^3 A \cos B \cos C \rightarrow T_1 = (2\sin^2 A)(2\sin A)(\cos(B + C) + \cos(B - C))$$

$$T_1 = (1 - \cos 2A)(2\sin(B + C))(\cos(B + C)\cos(B - C))$$

$$T_1 = (1 - \cos 2A)(\sin(2B + 2C) + \sin 2B + \sin 2C)$$

$$T_1 = (-\sin 2A + \sin 2B + \sin 2C) - \sin 2B \cos 2A - \sin 2 \cos 2A + (0,5)2\sin 2A \cos 2A$$

$$T_2 = 8\sin^3 B \cos A \cos C \rightarrow$$

$$\rightarrow T_2 = (-2\sin 2B + \sin 2A + \sin 2C) - \sin 2A \cos 2B - \sin 2C \cos 2B + (0.5)2\sin 2B \cos 2B$$

$$T_3 = 8\sin^3 C \cos A \cos B \rightarrow$$

$$\to T_3 = (-\sin 2C + \sin 2A + \sin 2B) - \sin 2A\cos 2C - \sin 2B\cos 2C + (0,5)2\sin 2C\cos 2C$$

$$T_1 + T_2 + T_3 = 2(\sin 2A + \sin 2B + \sin 2C) - 2\sin 2A\sin 2B\sin 2C$$

$$T_1 + T_2 + T_3 = 2(4\sin A\sin B\sin C)(1 - 2\cos A\cos B\cos C) \geq$$

$$\geq 16(\sin A + \sin B + \sin C)(\cos^2 A + \cos^2 B + \cos^2 C)\left(4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^3$$

$$8\sin A\sin B\sin C \geq 16\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\left(2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)8\ \times 4\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2$$

$\to$

$$\to \frac{1}{64} \geq \left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2 \to \frac{1}{8} \geq \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$a^3\cos B\cos C = R^3(8\sin^3 A\cos B\cos C)$$

$$8\sin^3 A\cos B\cos C = (2\sin^2 A)(4\sin A\cos B\cos C)$$

$$= (1 - \cos 2A)(2\cos C)(2\sin A\cos B)$$

$$= (1 - \cos 2A)(2\cos C)\{\sin(A + B) + \sin(A - B)\}$$

$$= (1 - \cos 2A)\{2\cos C\sin C - 2\cos(A + B)\sin(A - B)\}$$

$$= (1 - \cos 2A)\{\sin 2C - (\sin 2A - \sin 2B)\}$$

$$= (1 - \cos 2A)(\sin 2B + \sin 2C - \sin 2A)$$

$$= \sin 2B + \sin 2C - \sin 2A - \cos 2A\sin 2B - \cos 2A\sin 2C + \cos 2A\sin 2A \quad (1)$$

Similarly, $8\sin^3 B\cos C\cos A = (1 - \cos 2B)(\sin 2C + \sin 2A - \sin 2B)$

$$= \sin 2C + \sin 2A - \sin 2B - \cos 2B\sin 2C - \cos 2B\sin 2A + \sin 2B\cos 2B \quad (2)$$

and $8\sin^3 C\cos A\cos B = (1 - \cos 2C)(\sin 2A + \sin 2B - \sin 2C)$

$$= \sin 2A + \sin 2B - \sin 2C - \cos 2C\sin 2A - \cos 2C\sin 2B + \sin 2C\cos 2C \quad (3)$$

(1)+(2)+(3)$\Rightarrow \frac{1}{r^3}\sum a^3\cos B\cos C$

$$= \frac{R^3}{r^3}\big((\sin 2A + \sin 2B + \sin 2C) - \sin 2C\,(\cos 2A + \cos 2B)$$

$$- \sin 2B\,(\cos 2C + \cos 2A) - \sin 2A\,(\cos 2B + \cos 2C) + \cos 2A\sin 2A$$

$$+ \cos 2B\sin 2B + \cos 2C\sin 2C - \sin 2C\,(\cos 2A + \cos 2B)\big) =$$

$$= -\sin 2C\,\{2\cos(A + B)\cos(A - B)\} = -2\sin C\cos C\,(-2\cos C\cos(A - B))$$

$$= 4\cos^2 C\sin(A + B)\cos(A - B) = 2\cos^2 C\,(\sin 2A + \sin 2B)$$

$$= (1+\cos 2C)(\sin 2A+\sin 2B)$$

$$= (\sin 2A+\sin 2B)+\cos 2C\,(\sin 2A+\sin 2B) \quad \textbf{(4)}$$

Similarly, $-\sin 2B\,(\cos 2C+\cos 2A)$

$$= (\sin 2C+\sin 2A)+\cos 2B\,(\sin 2C+\sin 2A) \quad \textbf{(5)}$$

and $-\sin 2A\,(\cos 2B+\cos 2C)$

$$= (\sin 2B+\sin 2C)+\cos 2A\,(\sin 2B+\sin 2C) \quad \textbf{(6)}$$

$$\therefore \frac{1}{r^3}\sum a^3\cos B\cos C$$

$$= \frac{R^3}{r^3}\left\{3\sum \sin 2A+(\cos 2A+\cos 2B+\cos 2C)\left(\sum \sin 2A\right)\right\}$$

$$= \frac{R^3}{r^3}\left(\sum \sin 2A\right)\{(1+\cos 2A)+(1+\cos 2B)+(1+\cos 2C)\}$$

$$= \frac{R^3}{r^3}\left(\sum \sin 2A\right)(2)\left(\sum \cos^2 A\right) = \frac{2R^3}{r^3}\left(\sum \sin 2A\right)\left(\sum \cos^2 A\right) \quad \textbf{(A)}$$

$$\sum \sin 2A = \sin 2A+\sin 2B+\sin 2C = 2\sin(A+B)\cos(A-B)+2\sin C\cos C$$

$$= 2\sin C\{\cos(A-B)-\cos(A+B)\} = 2\sin C\cdot 2\sin A\sin B = 4\sin A\sin B\sin C$$

$$= 4\cdot 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$= \left(8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right) \quad \textbf{(7)}$$

Now, $\sin A+\sin B+\sin C = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$

$$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right) = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\therefore \textbf{(7)} \Rightarrow \sum \sin 2A = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\left(\sum \sin A\right)$$

$$= 8\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}\left(\sum \sin A\right)$$

$$= \frac{8s(s-a)(s-b)(s-c)}{s\,abc}\left(\sum \sin A\right) = \left(\frac{8\Delta^2}{sabc}\right)\left(\sum \sin A\right)$$

$$\Delta = \frac{abc}{4R} \textbf{ and } \Delta = rs,\ \therefore \Delta^2 = \frac{(sabc)r}{4R}$$

$$\therefore \sum \sin 2A = \frac{8(sabc)r}{4R(sabc)}\left(\sum \sin A\right) = \frac{2r}{R}\left(\sum \sin A\right)$$

$\therefore$ ***(A)*** $\Rightarrow \frac{1}{r^3}(\sum a^3 \cos B \cos C) = \left(\frac{2R^3}{r^3}\right)\left(\frac{2r}{R}(\sum \sin A)\right)(\sum \cos^2 A)$

$$= 4\left(\frac{R^2}{r^2}\right)\left(\sum \sin A\right)\left(\sum \cos^2 A\right) \geq 4(2^2)\left(\sum \sin A\right)\left(\sum \cos^2 A\right) (\because R \geq 2r)$$

$= 16(\sum \sin A)(\sum \cos^2 A)$ ***(Hence proved)***

447. In $\Delta\ ABC$:

$$\frac{s}{R} \leq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{s}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru, Solution 2 by Rozeta Atanasova-Skopje, Solution 3 by Soumitra Mandal-Chandar Nagore-India, Solution 4 by Soumava Chakraborty-Kolkata-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{s}{R} \leq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{s}{2r}$

Recordar las siguientes identidades y desigualdades en un triángulo ABC

$$\frac{s}{R} = \frac{a+b+c}{2R} = \sin A + \sin B + \sin C, \frac{s}{r} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \geq 3\sqrt{3}$$

Ahora bien $\sin A + \sin B = 2\cos\frac{C}{2}\cos\left(\frac{B-C}{2}\right) \leq 2\cos\frac{C}{2}$,

$$\sin B + \sin C \leq 2\cos\frac{A}{2}, \sin C + \sin A \leq 2\cos\frac{B}{2}$$

Sumando dichas desigualdades se obtiene

$$\Rightarrow \sin A + \sin B + \sin C \leq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}$$

Por ultimo $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2} \leq \frac{s}{2r}$

Solution 2 by Rozeta Atanasova-Skopje-Macedonia

$$LHS=\frac{s}{R}=\frac{a+b+c}{2R}=\sin A+\sin B+\sin C$$

$$=2\left(\sin\frac{A}{2}\cos\frac{A}{2}+\sin\frac{B}{2}\cos\frac{B}{2}+\sin\frac{C}{2}\cos\frac{C}{2}\right)$$

$$\overset{Chebyshev}{\leq}\frac{2}{3}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)$$

$$\overset{Jensen}{\leq}\frac{2}{3}\cdot 3\sin\frac{A+B+C}{6}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)$$

$$=\frac{2}{3}\cdot 3\cdot\frac{1}{2}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)$$

$$\overset{Jensen}{\leq}3\cos\frac{A+B+C}{6}=\frac{3\sqrt{3}}{2}\overset{Jensen}{\leq}\frac{1}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)$$

$$=\frac{1}{2}\cdot\frac{s}{r}=\frac{2}{2r}=RHS$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\cos\frac{A}{2}=\sqrt{p}\sum_{cyc}\sqrt{\frac{p-a}{bc}}\overset{Cauchy-Schwarz}{\leq}\sqrt{p}\sqrt{\left(\sum_{cyc}(p-a)\right)\left(\sum_{cyc}\frac{1}{ab}\right)}$$

$$=\sqrt{p}\sqrt{p\cdot\left(\frac{a+b+c}{abc}\right)}=\sqrt{p}\sqrt{p\cdot\frac{2p}{4Rrp}}=\frac{p}{\sqrt{2Rp}}\leq\frac{p}{2r}\ \textbf{[where } R\geq 2r\textbf{]}$$

$$\sum_{cyc}\cos\frac{A}{2}=\sqrt{p}\sum_{cyc}\sqrt{\frac{p-a}{bc}}\overset{AM\geq GM}{\geq}3\sqrt{p}\sqrt[3]{\frac{\sqrt{(p-a)(p-b)(p-c)}}{abc}}$$

$$=3\sqrt{p}\sqrt[3]{\frac{\sqrt{pr^2}}{4Rrp}}=3\sqrt[3]{\frac{p^{\frac{3}{2}}}{4R\sqrt{p}}}=3\sqrt[3]{\frac{p}{4R}}$$

Now we need to prove, $3\sqrt[3]{\frac{p}{4R}}\geq\frac{p}{R}\Leftrightarrow\frac{27}{4}R^2\geq p^2\Leftrightarrow\frac{3\sqrt{3}}{2}R\geq p$,

which is true $\frac{p}{R}\leq\sum_{cyc}\cos\frac{A}{2}\leq\frac{p}{2r}$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\frac{s}{R}\overset{(1)}{\le}\sum\cos\frac{A}{2}\overset{(2)}{\le}\frac{s}{2r}$$

$$\sum\cos\frac{A}{2}\overset{(C-B-S)}{\le}\sqrt{3}\sqrt{\sum\cos^2\frac{A}{2}}=\sqrt{\frac{3}{2}\sum\left(2\cos^2\frac{A}{2}\right)}$$

$$=\sqrt{\frac{3}{2}\sum(1+\cos A)}=\sqrt{\frac{3}{2}\left(3+1+\frac{r}{R}\right)}=\sqrt{\frac{3}{2}\left(\frac{4R+r}{R}\right)}$$

$$\overset{(?)}{\le}\frac{s}{2R}\Leftrightarrow\frac{3}{2}\left(\frac{4R+r}{R}\right)\overset{(?)}{\le}\frac{s^2}{4r^2}\Leftrightarrow Rs^2\overset{(?)}{\ge}6r^2(4R+r)\quad \textit{(3)}$$

Now, LHS of (3) $\underset{(4)}{\overset{Gerretsen}{\ge}} R(16Rr-5r^2)$

∴ in order to prove (2) and hence (3), it suffices to prove:

$R(16Rr-5r^2)\ge 6r^2(4R+r)$ ***(using (3), (4))***

$\Leftrightarrow 16R^2-29Rr-6r^2\ge 0\Leftrightarrow(R-2r)(16R+3r)\ge 0\rightarrow$ ***true,***

$\because R\ge 2r$ ***(Euler) ∴ (2) is true ****

Now, (1) $\Leftrightarrow\sum\left(R\cos\frac{A}{2}\right)\ge s\Leftrightarrow\sum\frac{abc\sqrt{s(s-a)}}{4\sqrt{s(s-a)(s-b)(s-c)bc}}\ge s$

$\Leftrightarrow\sum\frac{4Rrs}{4\sqrt{bc(s-b)(s-c)}}\ge s\Leftrightarrow Rr\sum\frac{1}{\sqrt{bc(s-b)(s-c)}}$ ***(5)***

LHS of (5) $\underset{(6)}{\overset{Bergstrom}{\ge}}\frac{9Rr}{\sum\sqrt{bc(s-b)(s-c)}}$

Now, $\sum\sqrt{b(s-b)\cdot c(s-c)}\overset{C-B-S}{\le}\sqrt{\sum\{b(s-b)\}}\sqrt{\sum\{c(s-c)\}}$

$$=\sum\{a(s-a)\}=2s^2-\sum a^2=2s^2-2(s^2-4Rr-r^2)=2(4Rr+r^2)$$

$$\Rightarrow\frac{1}{\sum\sqrt{bc(s-b)(s-c)}}\overset{(7)}{\ge}\frac{1}{2(4Rr+r^2)}$$

$$\text{(6),(7)} \Rightarrow \text{LHS of (5)} \geq \frac{1}{2(4Rr+r^2)} \overset{(?)}{\geq} 1$$

$$\Leftrightarrow Rr \overset{(?)}{\geq} 2r^2 \Leftrightarrow R \overset{(?)}{\geq} 2r \rightarrow \text{true (Euler)} \Rightarrow \text{(5) is true} \Rightarrow \text{(1) is true *}$$

448. In $\Delta\, ABC$:

$$\prod (h_a + h_b)^6 \leq 2^{15} \prod \left(w_a^6 + w_b^6\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

Solution 1 by Nguyen Ngoc Tu-Ha Giang-Vietnam

$$\text{We have } \left(h_a^6 + h_b^6\right)(1+1)^5 \geq (h_a + h_b)^6$$

$$\Rightarrow \left(h_a^6 + h_b^6\right)\left(h_b^6 + h_c^6\right)\left(h_c^6 + h_a^6\right)\left(2^5\right)^3 \geq (h_a + h_b)^6 (h_b + h_c)^6 (h_c + h_a)^6$$

$$\Rightarrow 2^{15}\left(w_a^6 + h_b^6\right)\left(w_b^6 + h_c^6\right)\left(w_c^6 + w_a^6\right) \geq (h_a + h_b)^6 (h_b + h_c)^6 (h_c + h_a)^6$$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\text{Lemma 1: } w_a \geq h_a$$

$$\text{Lemma 2: } 2^{n-1}\left(h_a^n + h_b^n\right) \geq (h_a + h_b)^n$$

$$2^5\left(h_a^6 + h_b^6\right) \geq (h_a + h_b)^6 \text{ then:}$$

$$\prod (h_a + h_b)^6 \leq 2^{15} \prod \left(w_a^6 + w_b^6\right)$$

449. Prove that in any triangle ABC the following inequality holds:

$$\frac{\sqrt{r_b r_c}}{bc} + \frac{\sqrt{r_c r_a}}{ca} + \frac{\sqrt{r_a r_b}}{ab} \leq \sqrt{\frac{1}{Rr} + \frac{1}{4R^2}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{\sqrt{r_b r_c}}{bc}+\frac{\sqrt{r_c r_a}}{ca}+\frac{\sqrt{r_a r_b}}{ab}\le\sqrt{\frac{1}{Rr}+\frac{1}{4R^2}}$

Tener en cuenta las siguientes identidades en un $\Delta\ ABC$

$$\cos A+\cos B+\cos C=1+\frac{r}{R},\ \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}$$

$$r_a=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},\ r_b=4R\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2},\ r_c$$

$$=4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$$

$$r_a r_b=(2R\sin A)(2R\sin B)\left(\cos^2\frac{C}{2}\right)\Leftrightarrow$$

$$\Leftrightarrow r_a r_b=\frac{ab}{2}(1+\cos C),\ r_b r_c=\frac{bc}{2}(1+\cos A),\ r_c r_a=\frac{ca}{2}(1+\cos B)$$

$$\Rightarrow\frac{r_a r_b}{ab}+\frac{r_b r_c}{bc}+\frac{r_c r_a}{ca}=\frac{1+\cos C}{2}+\frac{1+\cos A}{2}+\frac{1+\cos B}{2}=\frac{4+\frac{r}{R}}{2}=$$

$$=\frac{4R+r}{2R}=2+\frac{r}{2R}$$

Aplicando la desigualdad de Cauchy

$$\frac{\sqrt{r_b r_c}}{bc}+\frac{\sqrt{r_c r_a}}{ca}+\frac{\sqrt{r_a r_b}}{ab}\le\sqrt{\left(\frac{r_b r_c}{bc}+\frac{r_c r_a}{ca}+\frac{r_a r_b}{ab}\right)\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)}=$$

$$=\sqrt{\left(2+\frac{r}{2R}\right)\left(\frac{1}{2Rr}\right)}=\sqrt{\frac{1}{Rr}+\frac{1}{4R^2}}$$

450. If $m\ge 0, x, y>0$ **then in** $\Delta\ ABC$:

$$\frac{a^{m+2}}{(xb+yc)^m}+\frac{b^{m+2}}{(xc+ya)^m}+\frac{c^{m+2}}{(xa+yb)^m}\ge\frac{4\sqrt{3}S}{(x+y)^m}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\begin{cases} a \geq b \geq c \Leftrightarrow \\ a^{m+1} \geq b^{m+1} \geq c^{m+1} \\ \dfrac{1}{(xb+yc)^m} \geq \dfrac{1}{(xc+ya)^m} \geq \dfrac{1}{(xa+yb)^m} \end{cases}$$

$$\sum \frac{a^{m+1}}{(xb+yc)^m} \cdot a \overset{Chebyshev}{\underset{RADON}{\geq}} \frac{1}{3} \cdot (a+b+c) \cdot \frac{(a+b+c)^{m+1}}{(a+b+c)^m \cdot (x+y)^m} =$$

$$= \frac{(a+b+c)^2}{3(x+y)^m} = \frac{4p^2}{3(x+y)^m} \geq \frac{4p \cdot 3\sqrt{3}r}{3(x+y)^m} = \frac{4\sqrt{3}S}{(x+y)^m}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc} \frac{a^{m+2}}{(bx+cy)^m} = \sum_{cyc} \frac{a^{2m+2}}{(abx+acy)^m}$$

$$\overset{RADON'S INEQUALITY}{\geq} \frac{(a^2+b^2+c^2)^{m+1}}{(x+y)^m(ab+bc+ca)^m} \geq \frac{a^2+b^2+c^2}{(x+y)^m} \geq \frac{4\sqrt{3}\Delta}{(x+y)^m}$$

451. In $\Delta\ ABC$:

$$\frac{m_a m_b m_c}{abc} < \left(\frac{m_a}{c} + \frac{m_c}{a}\right)\left(\frac{m_b}{a} + \frac{m_a}{b}\right)\left(\frac{m_c}{b} + \frac{m_b}{c}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\dfrac{m_a m_b m_c}{abc} < \left(\dfrac{m_a}{c} + \dfrac{m_c}{a}\right)\left(\dfrac{m_b}{a} + \dfrac{m_a}{b}\right)\left(\dfrac{m_c}{b} + \dfrac{m_b}{c}\right)$

Como $m_a, m_b, m_c, a, b, c > 0$. ***Aplicando*** $MA \geq MG$

$$\frac{m_a}{c} + \frac{m_c}{a} \geq 2\sqrt{\frac{m_a m_c}{ca}} \ \text{(A)};\ \frac{m_b}{a} + \frac{m_a}{b} \geq 2\sqrt{\frac{m_b m_a}{ab}} \ \text{(B)};\ \frac{m_c}{b} + \frac{m_b}{c} \geq 2\sqrt{\frac{m_c m_b}{bc}} \ \text{(C)}$$

Multiplicando (A), (B), (C)

$$\Rightarrow \left(\frac{m_a}{c}+\frac{m_c}{a}\right)\left(\frac{m_b}{a}+\frac{m_a}{b}\right)\left(\frac{m_c}{b}+\frac{m_b}{c}\right) \geq \frac{8m_am_bm_c}{abc} > \frac{m_am_bm_c}{abc}$$

452. In $\Delta\ ABC$:

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c} \leq \frac{1}{3}+\frac{8m_am_bm_c}{3h_ah_bh_c}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c} \leq \frac{1}{3}+\frac{8m_am_bm_c}{3h_ah_bh_c}$

Se demostro anteriormente que

$\frac{m_am_bm_c}{h_ah_bh_c} \geq \frac{R}{2r}$, $m_a+m_b+m_c \leq 4R+r$ *(Bottema inequality)*

Es suficiente demostrar lo siguiente $\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c} \leq \frac{1}{3}+\frac{4R}{3r}=\frac{4R+r}{3r}$

Supongamos sin *pérdida de generalidad*

$$a \leq b \leq c \Leftrightarrow m_a \geq m_b \geq m_c, \frac{1}{h_a} \leq \frac{1}{h_b} \leq \frac{1}{h_c}$$

Aplicando la desigualdad de Chebyshev

$$\Rightarrow \frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c} \leq \frac{1}{3}(m_a+m_b+m_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)=\frac{m_a+m_b+m_c}{3r} \leq \frac{4R+r}{3r}$$

453. If in ΔABC, $a+b+c=1$ then:

$$\sin(aA+bB+cC) \geq 3\sqrt[3]{\frac{r^2}{2R}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Solution 2 by Geanina Tudose-Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$(\sin x)'' = -\sin x < 0 \text{ for all } x \in (0,\pi), \text{ so } \sin x \text{ is concave}$$

$$\text{and } a+b+c=1 \text{ hence } \sin(aA+bB+cC) \ge a\sin A + b\sin B + c\sin C$$

$$= \frac{a^2+b^2+c^2}{2R} \overset{AM\ge GM}{\ge} \frac{3}{2R}\sqrt[3]{(abc)^2} = 3\sqrt[3]{\frac{16R^2r^2p^2}{8R^3}} = 3\sqrt[3]{\frac{2r^2p^2}{R}} = 3\sqrt[3]{\frac{r^2}{2R}},$$

$$\because p = \frac{1}{2}$$

Solution 2 by Geanina Tudose-Romania

$$\text{For } A,B,C \in (0,\pi) \sin \text{ is a concave function } a+b+c=1$$

$$\Rightarrow \sin(aA+bB+cC) \ge a\sin A + b\sin B + c\sin C$$

$$= \frac{a^2}{2R} + \frac{b^2}{2R} + \frac{c^2}{2R} \overset{AM\ge GM}{\ge} \frac{3}{2R}\sqrt[3]{a^2b^2c^2}$$

$$\text{We have } \left.\begin{matrix} S = \frac{abc}{4R} \Rightarrow abc = 4RS \\ S = s\cdot r = \frac{r}{2} \end{matrix}\right\} \Rightarrow abc = 4R\cdot\frac{r}{2} = 2Rr$$

$$\text{Therefore } \sin(aA+bB+cC) \ge 3\sqrt[3]{\frac{4R^2r^2}{8R^3}} = 3\sqrt[3]{\frac{r^2}{2R}}$$

454. In $\Delta\, ABC$:

$$(m_a+m_b+m_c)^2 + \frac{45S^2}{(m_a+m_b+m_c)^2} \ge \frac{32m_am_bm_c}{m_a+m_b+m_c}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC:

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$$(m_a + m_b + m_c)^2 + \frac{45S^2}{(m_b + m_b + m_c)^2} \geq \frac{32m_am_bm_c}{m_a + m_b + m_c}$$

Siendo m_a, m_b, m_c los lados de un triángulo ABC se cumple lo siguiente

$$9S^2 = (m_a + m_b + m_c)(m_b + m_c - m_a)(m_c + m_a - m_b)(m_a + m_b - m_c)$$

La desigualdad propuesta es equivalente

$$(m_a + m_b + m_c)^2 + \frac{5(m_b+m_c-m_a)(m_c+m_a-m_b)(m_a+m_b-m_c)}{(m_a+m_b+m_c)} \geq \frac{32m_am_bm_c}{m_a+m_b+m_c}$$

$$\Leftrightarrow (m_a + m_b + m_c)^3 + 5(m_b + m_c - m_a)(m_c + m_a - m_b)(m_a + m_b - m_c) \geq$$

$$\geq 32m_am_bm_c$$

Realizamos los siguientes cambios de variables

$$x = m_b + m_c - m_a > 0, y = m_c + m_a - m_b > 0, z = m_a + m_b - m_c > 0$$

$$\Leftrightarrow x + y + z = m_a + m_b + m_c, x + y = 2m_c, y + z = 2m_a, z + x = 2m_b$$

$$\Rightarrow (x + y + z)^3 + 5xyz \geq 4(x + y)(y + z)(z + x)$$

$$\Leftrightarrow x^3 + y^3 + z^3 + 3(x + y)(y + z)(z + x) + 5xyz \geq$$

$$\geq 4(x + y)(y + z)(z + x)$$

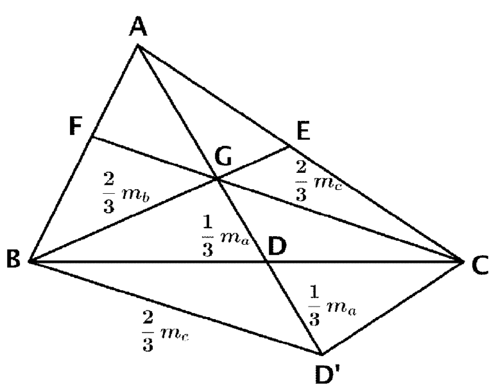
$$\Leftrightarrow x^3 + y^3 + z^3 + 5xyz \geq (x + y)(y + z)(z + x) \Leftrightarrow$$

$$\Leftrightarrow x^3 + y^3 + z^3 + 5xyz \geq xy(x + y) + yz(y + z) + zx(z + x) + 2xyz$$

$$\Leftrightarrow x^3 + y^3 + z^3 + 3xyz - xy(x + y) - yz(y + z) - zx(z + x) =$$

$$= x(x - y)(x - z) + y(y - x)(y - z) + z(z - x)(z - y) \geq 0$$ ***(Schur)***

Solution 2 by Soumava Chakraborty-Kolkata-India



In $\Delta\, BGD'$, the sides are $\frac{2}{3}m_a, \frac{2}{3}m_b, \frac{2}{3}m_c$

Semi-perimeter $s' \overset{(1)}{=} \frac{\sum m_a}{3}$ in-radius $r' \overset{(2)}{=} \frac{S}{\sum m_a}$, and circumradius

$$R' \overset{(3)}{=} \frac{2m_a m_b m_c}{9S}$$

Applying Gerretsen's inequality on $\Delta\, BGD'$, $s'^2 \ge 16R'r' - 5r'^2$

$$\Rightarrow \frac{(\sum m_a)^2}{9} \ge 16 \cdot \frac{2m_a m_b m_c}{9S} \cdot \frac{S}{\sum m_a} - 5\frac{S^2}{(\sum m_a)^2}$$

(using (1), (2), (3))$\Rightarrow (\sum m_a)^2 \ge \frac{32m_a m_b m_c}{\sum m_a} - \frac{45S^2}{(\sum m_a)^2}$

$$\Rightarrow \left(\sum m_a\right)^2 + \frac{45S^2}{(\sum m_a)^2} \ge \frac{32m_a m_b m_c}{\sum m_a}$$

455. In $\Delta\, ABC$:

$$\frac{a(m_a + w_a)}{h_a w_a} + \frac{b(m_b + w_b)}{h_b w_b} + \frac{c(m_c + w_c)}{h_c w_c} \ge 4\sqrt{3}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru,

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{a(m_a+w_a)}{h_a w_a} + \frac{b(m_b+w_b)}{h_b w_b} + \frac{c(m_c+w_c)}{h_c w_c} \ge 4\sqrt{3}$

Recordar las siguientes identidades y desigualdades en un $\Delta\, ABC$

$h_a = \frac{2S}{a}, h_b = \frac{2S}{b}, h_c = \frac{2S}{c}; m_a \ge w_a, m_b \ge w_b, m_c \ge w_c, a^2 + b^2 + c^2 \ge 4\sqrt{3}S$

(Inequality Weizenbock) Por lo tanto

$$\frac{(am_a + w_a)}{h_a w_a} + \frac{b(m_b + w_b)}{h_b w_b} + \frac{c(m_c + w_c)}{h_c w_c} \ge \frac{2a}{h_a} + \frac{2b}{h_b} + \frac{2c}{h_c} =$$

$$= \frac{a^2 + b^2 + c^2}{S} \ge 4\sqrt{3}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$m_a \geq w_a \geq h_a, m_b \geq w_b \geq h_b \text{ and } m_c \geq w_c \geq h_c$$

$$\sum_{cyc} \frac{a(m_a + w_a)}{h_a w_a} \geq \sum_{cyc} \frac{a(w_a + w_a)}{h_a w_a} = 2\sum_{cyc} \frac{a}{h_a} = \frac{a^2 + b^2 + c^2}{\Delta} \geq 4\sqrt{3}$$

456. In $\Delta\ ABC$:

$$\frac{r_a}{\sin\frac{A}{2}} + \frac{r_b}{\sin\frac{B}{2}} + \frac{r_c}{\sin\frac{C}{2}} \geq 2s\sqrt{3}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Adil Abdullayev-Baku-Azerbaidian

Lemma 1. $r_a = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}, r_b = 4R\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2},$

$$r_c = 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$$

Lemma 2. $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R}$. $\quad 27R^2 \geq 4s^2$

$$LHS = 4R\left(\cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{A}{2}\cos\frac{C}{2} + \cos\frac{A}{2}\cos\frac{B}{2}\right) \geq$$

$$\geq 4R \cdot 3\sqrt[3]{\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)^2} = 12R \cdot \sqrt[3]{\frac{s^2}{16R^2}} \geq 2s\sqrt{3} \leftrightarrow 27R^2 \geq 4s^2$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc} \frac{r_a}{\sin\frac{A}{2}} = \sum_{cyc} \frac{p\tan\frac{A}{2}}{\sin\frac{A}{2}} = \sum_{cyc} \frac{p}{\cos\frac{A}{2}} = \sqrt{p}\sum_{cyc}\sqrt{\frac{bc}{p-a}}$$

$$\overset{AM\geq GM}{\geq} 3\sqrt[3]{p}\sqrt[3]{\frac{abc}{\sqrt{(p-a)(p-b)(p-c)}}} = 3\sqrt[3]{p}\sqrt[3]{\frac{4Rrp}{\sqrt{pr^2}}} = 3\sqrt[3]{4Rp^2}$$

We need to prove, $3\sqrt[3]{4Rp^2} \geq 2\sqrt{3}p \Leftrightarrow 108Rp^2 \geq 24\sqrt{3}p^3$

$\Leftrightarrow \frac{3\sqrt{3}}{2}R \geq p$, ***which is true***

$$\therefore \sum_{cyc} \frac{r_a}{\sin\frac{A}{2}} \geq 2\sqrt{3}p$$

457. **In** $\Delta\, ABC$**:**

$$108Rr^2 \leq \sqrt{(s^2+r_a^2)(s^2+r_b^2)(s^2+r_c^2)} \leq 27R^3$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$108Rr^2 \leq \sqrt{(s^2+r_a^2)(s^2+r_b^2)(s^2+r_c^2)} \leq 27R^3 \quad \textbf{(1)}$$

I. $\quad r_a = \frac{S}{p-a} \ldots$ ***(*)***

II. $\quad \frac{r}{p-a} = \tan\frac{A}{2} \ldots$ ***(**)***

III. $\quad \cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2} = \frac{p}{4R}$ ***(***)***

$$\textbf{(1)} \Rightarrow \sqrt{\prod(p^2+r_a^2)} \overset{(*)}{=} \sqrt{\prod\left(p^2+\frac{s^2}{(p-a)^2}\right)} = p^3\cdot\sqrt{\prod\left(1+\left(\frac{r}{p-a}\right)\right)^2} \overset{(**)}{=}$$

$$= p^3\cdot\sqrt{\prod\left(1+\tan^2\frac{A}{2}\right)} = p^3\cdot\sqrt{\prod\frac{1}{\cos^2\frac{A}{2}}} = \frac{p^3}{\prod\cos\frac{A}{2}} \overset{(***)}{=}$$

$$=\frac{p}{\frac{p}{4R}}=p^2\cdot 4R \quad \textbf{(2)}$$

$$\textbf{(1), (2)}\Rightarrow 108R\cdot r^2\le p^2\cdot 4R\le 27R^3$$

$$\textbf{LHS}\ p^2\cdot 4R\overset{p\ge 3\sqrt{3}r}{\ge}\left(3\sqrt{3}\cdot r\right)^2\cdot 4R=108r^2R\ \textbf{(LHS)};\ RHS: p\le\frac{3\sqrt{3}}{2}\cdot R$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$r_a=p\tan\frac{A}{2}, r_b=p\tan\frac{B}{2}\ \textbf{and}\ r_c=p\tan\frac{C}{2},$$

$$108Rr^2\le\sqrt{\prod_{cyc}(p^2+r^2)}\le 27R^3\Leftrightarrow 108Rr^2\le p^3\prod_{cyc}\sec\frac{A}{2}\le 427$$

$$\Leftrightarrow 108Rr^2\le\frac{p^3}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\le 27R^3\Leftrightarrow 108Rr^2\le\frac{p^3abc}{p\Delta}\le 27R^3$$

$$\Leftrightarrow 108Rr^2\le 4Rp^2\le 27R^3\Leftrightarrow 27r^2\le p^2\le\frac{27}{4}R^2\Leftrightarrow$$

$$\Leftrightarrow 3\sqrt{3}r\le p\le\frac{3\sqrt{3}}{2}R\quad \textbf{(proved)}$$

458. In $\Delta\ ABC, AD=h_a, AM=m_a, BR=w_b, D,M\in(BC), R\in(AC)$, $\{P\}=BR\cap AD, \{Q\}=AM\cap BR$. If $AP=AQ$ then:

$$\frac{S[APQ]}{S[ABC]}\le\frac{\left(\sqrt{2}-1\right)^2}{2}$$

When equality holds?

Proposed by Nica Nicolae – Romania

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

Put $AL\perp PQ\ (L\in PQ)$. **We have**

$$AP=AQ\Rightarrow\angle PAQ=2\angle PAL=2\angle PBD=\angle ABC\Rightarrow$$

$$\Rightarrow \cos\angle PAQ = \cos\angle ABC \Rightarrow \frac{AD}{AM} = \frac{BD}{AB} \Rightarrow \frac{h_a}{m_a} = \frac{BD}{c} \Rightarrow BD = \frac{ch_a}{m_a}$$

We have BR is the bisector of triangle APD $\Rightarrow \frac{AP}{PD} = \frac{AB}{BD} \Rightarrow \frac{AP}{AD} = \frac{AB}{AB+BD}$ *(1)*

Similarly, we have $\frac{AQ}{AM} = \frac{AB}{AB+BM}$ *(2)*

$$(1),(2) \Rightarrow \frac{AM}{AD} = \frac{AB+BM}{AB+BD} \Rightarrow \frac{h_a}{m_a} = \frac{c+\frac{a}{2}}{c+\frac{ch_a}{m_a}} \Rightarrow \frac{m_a}{h_a} = \frac{2c+a}{2}\cdot\frac{m_a}{c(m_a+h_a)} \Rightarrow$$

$$\Rightarrow \frac{m_a+h_a}{h_a} = \frac{2c+a}{2c} \Rightarrow \frac{h_a}{m_a} = \frac{2c}{a} \Rightarrow h_a = \frac{2c}{a}\cdot m_a \Rightarrow$$

$$\Rightarrow BD = \frac{2c^2}{a} \text{ and } \frac{h_a}{c} = \frac{2m_a}{c}$$

We have $S_{ABC} = \frac{1}{2}\cdot a\cdot h_a = \frac{1}{2}\cdot a\cdot\frac{2c}{a}\cdot m_a = c\cdot m_a$ *(3)*

By Pitago theorem of triangle ADM, we have $AD^2 + DM^2 = AM^2 \Rightarrow$

$$\Rightarrow \frac{4c^2}{a^2}\cdot m_a^2 + \left(\frac{a}{2}-\frac{2c^2}{a}\right)^2 = m_a^2 \Rightarrow \left(\frac{a^2-4c^2}{2a}\right)^2 = \frac{a^2-4c^2}{a^2}\cdot m_a^2 \Rightarrow$$

$$\Rightarrow m_a^2 = \frac{a^2-4c^2}{4}$$

On the other hand, we have (2) $\Rightarrow \frac{AQ}{m_a} = \frac{c}{c+\frac{a}{2}} \Rightarrow AQ = \frac{2c\cdot m_a}{2c+a} \Rightarrow$

$$\Rightarrow S_{APQ} = \frac{1}{2}\cdot AP\cdot AQ\cdot\sin\angle PAQ = \frac{1}{2}\cdot AQ^2\cdot\sin\angle ABC =$$

$$= \frac{1}{2}\cdot\left(\frac{2c\cdot m_a}{2c+a}\right)^2\cdot\frac{h_a}{c} = \frac{1}{2}\cdot\left(\frac{2c\cdot m_a}{2c+a}\right)^2\cdot\frac{2m_a}{a} = \frac{4c^2\cdot m_a^3}{a(2c+a)^2} \quad (4)$$

$$(3) \text{ and } (4) \Rightarrow \frac{S_{APQ}}{S_{ABC}} = \frac{\frac{4c^2\cdot m_a^3}{a(2c+a)^2}}{c\cdot m_a} = \frac{4c\cdot m_a^2}{a(2c+a)^2} = \frac{4c\cdot\frac{a^2-4c^2}{4}}{a(2c+a)^2} = \frac{c(a-2c)}{a(a+2c)} \quad (5)$$

We need to prove that $\frac{c(a-2c)}{a(a+2c)} \le \frac{(\sqrt{2}-1)^2}{2}$ *(6)*

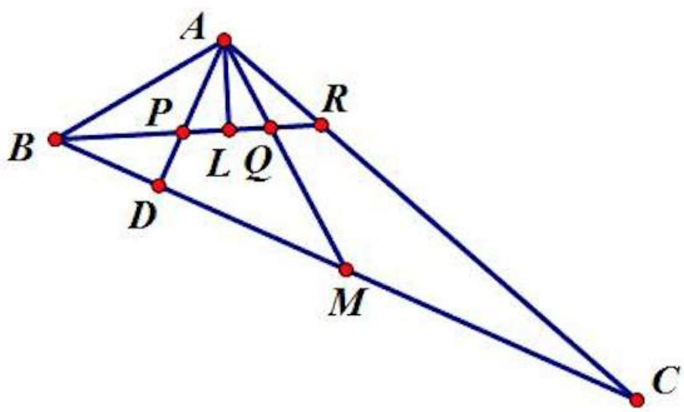
$$\Rightarrow 2ac - 4c^2 \le (3-2\sqrt{2})\cdot a^2 + 6(6-4\sqrt{2})ac \Rightarrow$$

$$\Rightarrow (3-2\sqrt{2})\cdot a^2 + (4-4\sqrt{2})ac + 4c^2 \ge 0 \Rightarrow$$

$$\Rightarrow \left[\left(\sqrt{2}-1\right)a-2c\right]^2 \geq 0 \text{ (True)} \Rightarrow \text{(6) true}$$

$$\text{(5) and (6)} \Rightarrow \frac{S_{APQ}}{S_{ABC}} \leq \frac{\left(\sqrt{2}-1\right)^2}{2}.$$

The equality occurs when $\left(\sqrt{2}-1\right)\cdot a - 2c = 0 \Rightarrow a = \left(2+2\sqrt{2}\right)c$



459. In $\Delta\ ABC$:

$$\frac{r_a^2}{r_a^2+s^2}+\frac{r_b^2}{r_b^2+s^2}+\frac{r_c^2}{r_c^2+s^2} \geq \frac{3}{4}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

Solution 1 by Kevin Soto Palacios – Huarmey – Peru, Solution 2 by Soumava Chakraborty-Kolkata-India, Solution 3 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{r_a^2}{r_a^2+s^2}+\frac{r_b^2}{r_b^2+s^2}+\frac{r_c^2}{r_c^2+s^2} \geq \frac{3}{4}$

Recordar la siguiente identidad algebraica

$$(x+y)(y+z)(z+x) = x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz$$

Siendo $x = r_a > 0, y = r_b > 0, z = r_c > 0 \Leftrightarrow xy+yz+zx = s^2$

La desigualdad propuesta es equivalente

$$\Leftrightarrow \frac{x^2}{x^2+xy+yz+zx}+\frac{y^2}{y^2+xy+yz+zx}+\frac{z^2}{z^2+xy+yz+zx}\ge\frac{3}{4}$$

$$\Leftrightarrow \frac{x^2}{(x+y)(x+z)}+\frac{y^2}{(y+x)(y+z)}+\frac{z^2}{(z+y)(z+x)}=$$

$$=\frac{x^2(y+z)+y^2(z+x)+z^2(x+y)}{(x+y)(y+z)(z+x)}=1-\frac{2xyz}{(x+y)(y+z)(z+x)}\ge$$

$$\ge 1-\frac{1}{4}=\frac{3}{4}.$$ ***(LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$r_a = s\tan\frac{A}{2}, \textit{etc}, \therefore LHS=\sum\frac{s^2\tan^2\frac{A}{2}}{s^2\tan^2\frac{A}{2}+s^2}$$

$$=\sum\frac{\tan^2\frac{A}{2}}{\sec^2\frac{A}{2}}=\sum\sin^2\frac{A}{2}\ge\frac{3}{4}$$ ***(well – known)(*) (Proved)***

(*) Proof of $\sum\sin^2\frac{A}{2}\ge\frac{3}{4}\Leftrightarrow\sum(1-\cos A)\ge\frac{3}{2}$

$\Leftrightarrow\sum\cos A\le\frac{3}{2}\Leftrightarrow 1+\frac{r}{R}\le\frac{3}{2}\Leftrightarrow\frac{R+r}{R}\le\frac{3}{2}\Leftrightarrow R\ge 2r\to$ ***true (Euler)***

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{r_a^2}{r_a^2+p^2}=\sum_{cyc}\frac{\left(p\tan\frac{A}{2}\right)^2}{\left(p\tan\frac{A}{2}\right)^2+p^2}=\sum_{cyc}\frac{\tan^2\frac{A}{2}}{\sec^2\frac{A}{2}}=\sum_{cyc}\sin^2\frac{A}{2}$$

$$=\sum_{cyc}\frac{(p-a)(p-b)}{ab}=\frac{1}{abc}\left(\sum_{cyc}c(p-a)(p-b)\right)$$

$$=\frac{1}{abc}\left(p^2\sum_{cyc}a-2p\sum_{cyc}ab+3abc\right)=\frac{12Rrp-2p(r^2+4Rr)}{4Rrp}=$$

$$=\frac{4Rrp-2pr^2}{4Rrp}=\frac{2R-r}{2R}\ge\frac{2R-\frac{R}{2}}{2R}=\frac{3}{4}$$ ***(Proved)***