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ABOUT SOME SPECIAL CLASS OF TRIANGLES

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Abstract: *In this paper are presented some applications for a special class of triangles with diameter of circumcircle 1*

$$2R = 1 \leftrightarrow \begin{cases} a = \sin A \\ b = \sin B \\ c = \sin C \end{cases}$$

$\sin(A+B) = \sin(\pi - C) = \sin C$ and $\cos A = \sqrt{1-a^2}, \cos B = \sqrt{1-b^2}$ (using the fact that ΔABC is acute).

Hence: $c = a\sqrt{1-b^2} + b\sqrt{1-a^2}$ and analogous.

Let be $f:(0,1) \to \mathbb{R}$, $f(x) = \sqrt{1-x^2}$, $f''(x) < 0 \to f - concave$

$$4m_a^2 = 2\left(af(c) + cf(a)\right)^2 + 2\left(af(b) + bf(a)\right)^2 - \left(c\sqrt{1-b^2} + b\sqrt{1-c^2}\right)^2 \quad (*)$$

$$af(c) + cf(a) \le (a+c)f\left(\frac{2ac}{a+c}\right) \leftrightarrow af(c) + cf(a) \le \sqrt{(a+c)^2 - 4a^2c^2} \quad (1)$$

By Jensen's inequality:

$$c\sqrt{(1-b)(1+b)} + b\sqrt{(1-c)(1+c)} \overset{AM-GM}{\geq} \frac{2b(1-c^2)}{2} + \frac{2c(1-b^2)}{2} = (b+c)(1-bc) \ (2),$$

By (1), (2), (*):

$$4\sum m_a^2 \le 4\sum(a+b)^2 - 8\sum a^2(b^2+c^2) - \sum(a+b)^2(1-ab)^2 \quad (3).$$

$$b^2 + c^2 \ge 2bc \to -8\sum a^2(b^2+c^2) \le -16\sum a^2bc = -16abc\sum a \quad (4).$$

From $\begin{cases} 4\sum m_a^2 = 3\sum a^2 \\ 4\sum(a+b)^2 = 8\sum a^2 + 8\sum ab \end{cases}$ we have:

$$4\sum(a+b)^2-4\sum m_a^2=4\left(\sum a\right)^2+\sum a^5 \quad \textbf{(5)}$$

Using (4), (5) in (3) $\rightarrow \sum(a+b)^2(1-ab)^2 \le \sum a^2+4\left(\sum a\right)\left(\sum a-4abc\right)$.

Another proof for last inequality:

By (2) $\rightarrow a^2 \ge (b+c)^2(1-bc)^2 \rightarrow \sum(a+b)^2(1-ab)^2 \le \sum a^2$.

Remains to prove: $4\left(\sum a\right)\left(\sum a-4abc\right) \ge 0 \leftrightarrow 4abc \le \sum a$.

$\sum a \ge 3\sqrt[3]{abc}$ **by AM-GM.**

$3\sqrt[3]{abc} \ge 4abc \leftrightarrow abc \le \frac{3\sqrt{3}}{8} \leftrightarrow \sin A\sin B\sin C \le \frac{3\sqrt{3}}{8}$.

Let be $g:\left(0,\frac{\pi}{2}\right) \rightarrow \mathbb{R}$, $g(x)=\ln\sin x$, $g''(x)=\frac{2}{\cos 2x-1}<0-g\ concave$

By Jensen's inequality:

$\sum \ln\sin A \le 3\ln\sin\frac{\pi}{3} \leftrightarrow \sin A\sin B\sin C \le \frac{3\sqrt{3}}{8}$.

By (1) $\rightarrow c^2+4a^2b^2 \le a^2+b^2+2ab \rightarrow 4\sum a^2b^2 \le \sum a^2+2\sum ab$,

Hence: $\sqrt{a^2b^2+b^2c^2+c^2a^2} \le s$

By (2) $\rightarrow 2\sum m_a^2 \ge \sum a^4+\sum a^2b^2$.

Proposed problems:

If $R=\frac{1}{2}\ then:\begin{cases}\sum(a+b)^2(1-ab)^2 \le \sum a^2+4\left(\sum a\right)\left(\sum a-4abc\right)\\ \sqrt{a^2b^2+b^2c^2+c^2a^2} \le s\\ 2\sum m_a^2 \ge \sum a^4+\sum a^2b^2\end{cases}$.