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**1. Let $ABC$ be a triangle, $h_a, h_b, h_c$ denote the lengths of altitudes, $l_a, l_b, l_c$ denote the lengths of inner bisectors, and $r_a, r_b, r_c$ be its exradii. Prove that:**

$$\frac{h_a r_a}{l_a^2} + \frac{h_b r_b}{l_b^2} + \frac{h_c r_c}{l_c^2} \geq 3$$

***Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam***

*Proof.*

*We prove the following lemma:*

**Lemma.**
**2) In $\Delta ABC$:**

$$\frac{h_a r_a}{l_a^2} + \frac{h_b r_b}{l_b^2} + \frac{h_c r_c}{l_c^2} = \frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr}$$

*Proof.*

*Using* $h_a = \frac{2S}{a}, r_a = \frac{S}{s-a}, l_a = \frac{2bc}{b+c}\cos\frac{A}{2}, \cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$, *we obtain:*

$$\sum \frac{h_a r_a}{l_a^2} = \sum \frac{\frac{2S}{a} \cdot \frac{S}{s-a}}{(\frac{2bc}{b+c}\cos\frac{A}{2})^2} = \frac{2S^2}{4abcs} \sum \frac{(b+c)^2}{(s-a)^2} =$$

$$= \frac{r}{8R} \cdot \frac{2(8R^2 + 8Rr + 3r^2 - s^2)}{r^2} = \frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr}$$

□

*Let's return to the main problem:*

*The inequality we have to prove:* $\frac{8R^2 + 8Rr + 3r^2 - s^2}{4Rr} \geq 3 \Leftrightarrow s^2 \leq 8R^2 - 4Rr + 3r^2$

*which follows from Gerretsen's inequality:* $s^2 \leq 4R^2 + 4Rr + 3r^2$

*It remains to prove that:*

$4R^2 + 4Rr + 3r^2 \leq 8R^2 - 4Rr + 3r^2 \Leftrightarrow 4R^2 \geq 8Rr \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

□

**Remark.**

*The inequality can be strengthened.*

**3) In $\Delta ABC$ :**

$$\frac{h_a r_a}{l_a^2}+\frac{h_b r_b}{l_b^2}+\frac{h_c r_c}{l_c^2}\geq\frac{R}{r}+1$$

***Proposed by Marin Chirciu - Romania***

*Proof.*

*Using **Lemma** and Gerretsen's inequality:* $s^2\leq 4R^2+4Rr+3r^2$ *we obtain:*

$$\sum\frac{h_a r_a}{l_a^2}=\frac{8R^2+8Rr+3r^2-s^2}{4Rr}\geq\frac{8R^2+8Rr+3r^2-4R^2-4Rr-3r^2}{4Rr}=$$

$$=\frac{4R^2+4Rr}{4Rr}=\frac{R}{r}+1$$

*Equality holds if and only if the triangle is equilateral.*

□

**Remark.**

*Inequality 3) is stronger than inequality 1):*

**4) In $\Delta ABC$:**

$$\frac{h_a r_a}{l_a^2}+\frac{h_b r_b}{l_b^2}+\frac{h_c r_c}{l_c^2}\geq\frac{R}{r}+1\geq 3.$$

*Proof.*

*See inequality 3) is* $\frac{R}{r}+1\geq 3\Leftrightarrow R\geq 2r$ *(Euler's inequality).*

*Equality holds if and only if the triangle is equilateral.*

□

**Remark.**

*Let's emphasises an inequality having an opposite sense:*

**5) In $\Delta ABC$ :**

$$\frac{h_a r_a}{l_a^2}+\frac{h_b r_b}{l_b^2}+\frac{h_c r_c}{l_c^2}\leq 2\Big(\frac{R}{2}-\frac{r}{R}\Big)$$

*Proof.*

*Using **Lemma** and Gerretsen's inequality:* $s^2\geq 16Rr-5r^2$ *we obtain:*

$$\sum\frac{h_a r_a}{l_a^2}=\frac{8R^2+8Rr+3r^2-s^2}{4Rr}\leq\frac{8R^2+8Rr+3r^2-16Rr+5r^2}{4Rr}=$$

$$=\frac{8R^2-8Rr+8r^2}{4Rr}=\frac{2(R^2-Rr+r^2)}{Rr}\leq\frac{2(R^2-r^2)}{Rr}=2\Big(\frac{R}{r}-\frac{r}{R}\Big).$$

*Equality holds if and only if the triangle is equilateral.*

□

**Remark.**

*We can write the double inequality:*

**6) In $\Delta ABC$:**

$$\frac{R}{r}+1\le\frac{h_ar_a}{l_a^2}+\frac{h_br_b}{l_b^2}+\frac{h_cr_c}{l_c^2}\le 2\left(\frac{R}{r}-\frac{r}{R}\right)$$

***Proposed by Marin Chirciu - Romania***

*Proof.*

*See inequalities 3) and 5).*

*Equality holds if and only if the triangle is equilateral.*

□

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