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INEQUALITY IN TRIANGLE 867
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1. Let ABC be a triangle. Prove that:

$$\sum r_a(h_b+h_c)^2 \geq 12sS.$$

Proposed by Mehmet Şahin - Ankara - Turkey

Proof.

We prove the following lemma:

Lemma 1.
2) In $\Delta ABC : \sum r_a(h_b+h_c)^2 = \frac{s^2(s^2-3r^2)}{R}$.

Proof.

Using $r_a = \dfrac{S}{s-a}$ *and* $h_a = \dfrac{2S}{a}$ *we obtain:*

$$\sum r_a(h_b+h_c)^2 = \sum \frac{S}{s-a}\left(\frac{2S}{b}+\frac{2S}{c}\right)^2 = 4S^3\sum\frac{(b+c)^2}{b^2c^2(s-a)} = 4r^3s^3\cdot\frac{s^2-3r^2}{4sRr^3} = = \frac{s^2(s^2-3r^2)}{R}.$$

In the above equality we've used: $\sum \dfrac{(b+c)^2}{b^2c^2(s-a)} = \dfrac{s^2-3r^2}{4sRr^3}$, *which follows from:*

$$\sum a^2(b+c)^2(s-b)(s-c) = 4s^2Rr(s^2-3r^2), abc = 4Rrs \text{ and } \prod(s-a) = r^2s.$$

□

Let's get back to the main problem:

*Using **Lemma 1** the inequality can be written:*

$\dfrac{s^2(s^2-3r^2)}{R} \geq 12rs^2 \Leftrightarrow s^2 \geq 12Rr+3r^2$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr-5r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasises an inequality having an opposite sense.

3) In $\Delta ABC : \sum r_a(h_b+h_c)^2 \leq 6Rs^2$

Proposed by Marin Chirciu - Romania

Proof.

*Using **Lemma 1** we write the inequality:*

$\frac{s^2(s^2-3r^2)}{R} \le 6Rs^2 \Leftrightarrow s^2 \le 6R^2+3r^2$, *which follows from Gerretsen's inequality* $s^2 \le 4R^2+4Rr+3r^2$ *and Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

4) In $\Delta ABC : 12rs^2 \le \sum r_a(h_b+h_c)^2 \le 6Rs^2$.

Proof.

*See inequalities **1)** and **3)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Changing r_a with h_a we can build inequalities similar to those above.

5) In $\Delta ABC : \sum h_a(r_a+r_c)^2 \ge 12sS$

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Proof.

We prove the following lemma:

Lemma 2.

6) In $\Delta ABC : \sum h_a(r_b+r_c)^2 = 4s^2(2R-r)$.

Proof.

Using $r_a = \frac{S}{s-a}$ *and* $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum h_a(r_b+r_c)^2 = \sum \frac{2S}{a}\Big(\frac{S}{s-b}+\frac{S}{s-c}\Big)^2 = 2S^3\sum \frac{a}{(s-b)^2(s-c)^2} =$$

$$= 2r^3s^3 \cdot \frac{2(2R-r)}{sr^3} = 4s^2(2R-r)$$

In the above inequality we've used: $\sum \frac{a}{(s-b)^2(s-c)^2} = \frac{2(2R-r)}{sr^3}$

which follows from: $\sum a(s-a)^2 = 2sr(2R-r)$ *and* $\prod(s-a) = r^2s$.

□

Let's get back to the main problem:

*Using **Lemma 2** we write the inequality:*

$4s^2(2R-r) \ge 12rs^2 \Leftrightarrow R \ge 2r$ *(Euler's inequality* $R \ge 2r$*).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasises an inequality having an opposite sense.

7) In $\Delta ABC : \sum h_a(r_b + r_c)^2 \leq 2R(4R + r)^2$.

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Proof.

*Using **Lemma 2** we write the inequality:*

$4s^2(2R-r) \leq 2R(4R+r)^2 \Leftrightarrow s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$, *which is Blundon-Gerretsen's inequality.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

8) In $\Delta ABC : 12rs^2 \leq \sum h_a(r_b + r_c)^2 \leq 2R(4R + r)^2$.

Proof.

*See inequalities **5)** and **7)**.*

Equality holds if and only if the triangle is equilateral.

□

9) In $\Delta ABC : 324r^3 \leq \sum r_a(r_b + r_c)^2 \leq \frac{81R^3}{2}$

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Proof.

We prove the following lemma:

Lemma 3.

10) In $\Delta ABC : \sum r_a(r_b + r_c)^2 = 4s^2(R + r)$.

Proof.

Using $r_a = \frac{S}{s-a}$ *we obtain:*

$$\sum r_a(r_b + r_c)^2 = \sum \frac{S}{s-a}\left(\frac{S}{s-b} + \frac{S}{s-c}\right)^2 = S^3 \sum \frac{1}{s-a} \cdot \frac{a^2}{(s-b)^2(s-c)^2} =$$

$$= \frac{S^3}{\prod(s-a)} \sum \frac{a^2}{(s-b)(s-c)} = \frac{r^3s^3}{r^2s} \cdot \frac{4(R+r)}{r} = 4s^2(R+r).$$

In the above inequality we've used: $\sum \frac{a^2}{(s-b)(s-c)} = \frac{4(R+r)}{r}$

which follows from: $\sum a^2(s-a) = 2sr(R+r)$ *and* $\prod(s-a) = r^2s$.

□

Let's get back to the main problem:

Using ***Lemma 3*** *the double inequality can be written:*

$$324r^3 \leq 4s^2(R+r) \leq \frac{81R^3}{2}, \text{ which follows from Gerretsen's inequality:}$$

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2 \text{ and Euler's inequality } R \geq 2r.$$

Equality holds if and only if the triangle is equilateral.

□

11) In $\Delta ABC : 48s^2 \cdot \frac{r^3}{R^2} \leq \sum h_a(h_b + h_c)^2 \leq 12s^2r.$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 4.
12) In $\Delta ABC : \sum h_a(h_b + h_c)^2 = \frac{r}{R^2} \cdot s^2(s^2 + r^2 + 10Rr).$

Proof.

$$\text{Using } h_a = \frac{2S}{a} \text{ we obtain:}$$

$$\sum h_a(h_b+h_c)^2 = \sum \frac{2S}{a}\left(\frac{2S}{b} + \frac{2S}{c}\right)^2 = 8S^3 \sum \frac{1}{a} \cdot \frac{(b+c)^2}{b^2c^2} = \frac{8S^3}{abc} \sum \frac{(b+c)^2}{bc} =$$

$$= \frac{8r^3s^3}{4Rrs} \cdot \frac{s^2 + r^2 + 10Rr}{2Rr} = \frac{r}{R^2} \cdot s^2(s^2 + r^2 + 10Rr).$$

$$\text{In the above equality we've used: } \sum \frac{(b+c)^2}{bc} = \frac{s^2 + r^2 + 10Rr}{2Rr}$$

$$\text{which follows from: } \sum a(b+c)^2 = 2s(s^2 + r^2 + 10Rr) \text{ and } abc = 4Rrs.$$

□

Let's get back to the main problem:

Using ***Lemma 4*** *the double inequality can be written:*

$$48s^2 \cdot \frac{r^3}{R^2} \leq \frac{r}{R^2} \cdot s^2(s^2+r^2+10Rr) \leq 12s^2r, \text{ which follows from Gerretsen's inequality}$$

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2 \text{ and Euler's inequality } R \geq 2r.$$

Equality holds if and only if the triangle is equilateral.

□

13) In $\Delta ABC : \sum r_a^2(h_b + h_c)^2 \geq 36S^2$

Proposed by Marin Chirciu - Romania

Proof.

With means inequality we have:

$$(1)\quad \sum r_a^2(h_b+h_c)^2 \geq \sum r_a^2 \cdot 4h_bh_c = 4\sum r_a^2h_bh_c = \frac{8r}{R}\cdot s^2(8R^2+2Rr-s^2)$$

which follows from: $\sum r_a^2h_bh_c = \frac{2r}{R}\cdot s^2(8R^2+2Rr-s^2)$, *because:*

$$\sum r_a^2h_bh_c = \sum\left(\frac{S}{s-a}\right)^2\cdot\frac{2S}{b}\cdot\frac{2S}{c} = 4S^4\sum\frac{1}{bc(s-a)^2},$$

$$\sum\frac{1}{bc(s-a)^2} = \frac{\sum a(s-b)^2(s-c)^2}{abc\prod(s-a)},$$

$$\sum a(s-b)^2(s-c)^2 = 2sr^2(8R^2+2Rr-s^2), abc = 4Rrs, \prod(s-a) = sr^2.$$

In order to prove $\sum r_a^2(h_b+h_c)^2 \geq 36S^2$ *using* (1) *it suffices to prove that:*

$$\frac{8r}{R}\cdot s^2(8R^2+2Rr-s^2) \geq 36S^2 \Leftrightarrow 2s^2 \leq 16R^2-5Rr,$$ *true from*

Gerretsen's inequality $s^2 \leq 4R^2+4Rr+3r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

14) In $\Delta ABC : \sum h_a^2(r_b+r_c)^2 \geq 36S^2$

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Proof.

With means inequality we have:

(1)

$$\sum h_a^2(r_b+r_c)^2 \geq \sum h_a^2\cdot 4r_br_c = 4\sum h_a^2r_br_c = \frac{s^2}{R^2}\cdot[s^4+s^2(2r^2-12Rr)+r^3(4Rr+r)]$$

which follows from: $\sum h_a^2r_br_c = \frac{s^2}{4R^2}\cdot[s^4+s^2(2r^2-12Rr)+r^3(4R+r)]$, *because*

$$\sum h_a^2r_br_c = \sum\left(\frac{2S}{a}\right)^2\cdot\frac{S}{s-b}\cdot\frac{S}{s-c} = 4S^4\sum\frac{1}{a^2(s-b)(s-c)},$$

$$\sum\frac{1}{a^2(s-b)(s-c)} = \frac{\sum b^2c^2(s-a)}{(abc)^2\prod(s-a)},$$

$$\sum b^2c^2(s-a) = s[s^4+s^2(2r^2-12Rr)+r^3(4Rr+r)], abc = 4Rrs, \prod(s-a) = sr^2.$$

In order to prove $\sum h_a^2(r_b+r_c)^2 \geq 36S^2$ *using* (1) *it suffices to prove that:*

$$\frac{s^2}{R^2}\cdot[s^4+s^2(2r^2-12Rr)+r^3(4R+r)] \geq 36S^2 \Leftrightarrow$$

$s^4+s^2(2R^2-12Rr)+r^3(4R+r) \geq 36R^2r^2$, *true from Gerretsen's inequality*

$s^2 \geq 16Rr-5r^2$ *and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

15) In $\Delta ABC : \sum r_a^2(r_b+r_c)^2 \geq 36Sr^2$.

Proof.

With means inequality we have:

$$\sum r_a^2(r_b+r_c)^2 \geq \sum r_a^2 \cdot 4r_br_c = 4r_ar_br_c \sum r_a = 4 \cdot s^2r(4R+r) \geq 4 \cdot s^2r \cdot 9r = 36Sr^2.$$

Equality holds if and only if the triangle is equilateral.

□

16) In $\Delta ABC : \sum h_a^2(h_b+h_c)^2 \geq \left(\frac{12Sr}{R}\right)^2$

Proof.

With means inequality we have:

$$\sum h_a^2(h_b+h_c)^2 \geq \sum h_a^2 \cdot 4h_bh_c = 4h_ah_bh_c \sum h_a = 4 \cdot \frac{s^2r^2}{R} \cdot \frac{s^2+r^2+4Rr}{2r} \geq$$

$$\geq 4 \cdot \frac{s^2r^2}{R} \cdot \frac{36r^2}{2R} = \frac{144S^2r^2}{R^2} = \left(\frac{12Sr}{R}\right)^2.$$

Equality holds if and only if the triangle is equilateral.

□

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