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ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

PROPERTIES OF THE EIGENVALUES OF SOME CLASSES OF REAL MATRICES

By Marian Ursărescu – Romania

In this article we will prove some interesting properties about the eigenvalues of some real matrices, followed by applications. For start, we will remember some of the classic results from matrices theory and their determinants.

Definition:

Let be $A \in M_n(\mathbb{R})$ and $X \in M_{n,1}(\mathbb{R})$. If it exists $\lambda \in \mathbb{C}$ such that $AX = \lambda X$, then X is called own vector, and λ eigenvalue for the matrix A.

Observation:

The matricial equation $(A - \lambda I_n)X = O_n$ is equivalent with the system:

$$(1)\begin{cases}(a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0\\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n = 0\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.\\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n = 0\end{cases}$$

The system (1) has nonzero solutions $\Leftrightarrow \det(A - \lambda I_n) = 0$.

Definition:

The polynom $p_\lambda(x) = \det(A - \lambda I_n)$ is called characteristic polynom of the matrix A, and the equation $p_\lambda(x) = \det(A - \lambda I_n) = 0$ is called characteristic equation of the matrix A.

Observation. The eigenvalues of matrix A are the solutions of the characteristic equation.

Theorem:

The characteristic polynom has the expression $p_\lambda(x) = \lambda^n - \Delta_1\lambda^{n-1} + \Delta_2\lambda^{n-2} + \cdots + (-1)^n\Delta_n$, where Δ_i represents the sum of the principal minors having the order i of the matrix $A - \lambda I_n$.

Observation:

1. $\lambda_1\lambda_2 \dots \lambda_n = \det A$
2. $\lambda_1 + \lambda_2 + \cdots + \lambda_n = Tr\, A$
3. $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n = Tr\, A^*$

Cayley – Hamilton Theorem.

If $p_\lambda(x)$ is the characteristic polynom of matrix A, then $p_\lambda(A) = O_n$.

Definitions:

Let be $A \in M_n(\mathbb{R})$

1. Matrix A is called symmetric if: $A^T = A$

2. Matrix A is called antisymmetric if $A^T = -A$.

3. Matrix A is called orthogonal if: $A \cdot A^T = I_n$ (we have denoted A^T = transposed matrix).

Theorem 1:

The eigenvalues of some real symmetric matrix are real.

Proof:

Let be $A \in M_n(\mathbb{R})$ with $A^T = A$. Let's suppose that it exists an eigenvalue λ such that:

$\lambda \in \mathbb{C} \Rightarrow AX = \lambda X$ (1)

We multiply to the left relation (1) with $\overline{X}^T \Rightarrow \overline{X}^T AX = \lambda \overline{X}^T \cdot X$ (2)

We conjugate relation (1) $\Rightarrow A\overline{X} = \overline{\lambda}\overline{X}$ (3) $\left(A \in M_n(\mathbb{R})\right)$

We multiply to the left relationship (3) with $X^T \Rightarrow X^T A\overline{X} = \overline{\lambda} X^T \cdot \overline{X} \Rightarrow$

$$\Rightarrow (X^T A\overline{X})^T = \overline{\lambda}(X^T \cdot \overline{X})^T \Rightarrow$$

$$\overline{X}^T AX = \overline{\lambda}\overline{X}^T \cdot X \quad (4)$$

From (2) and (4) $\Rightarrow \overline{\lambda}\overline{X}^T \cdot X = \lambda \overline{X}^T \cdot X \Rightarrow$

$$\left.\begin{matrix}(\overline{\lambda} - \lambda)(\overline{X}^T \cdot X) = O_n \\ But\ \overline{X}^T \cdot X = \overline{X_1} \cdot X_1 + \overline{X_2}X_2 + \cdots + \overline{X_n} \cdot X_n = (x_1)^2 + (x_2)^2 + \cdots + (x_n)^2 > 0\end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \overline{\lambda} - \lambda = 0 \Rightarrow \overline{\lambda} = \lambda \Rightarrow \lambda \in \mathbb{R}.$$

Theorem 2:

The eigenvalues of some real antisymmetric matrix are or nonzero or purely imaginary.

Proof:

Let $A \in M_n(\mathbb{R})$ with$A^T = -A$. For start let's prove that the only eigenvalues are nonzero.

Let $\lambda \in \mathbb{R}$ be an eigenvalue $\Rightarrow AX = \lambda X \Rightarrow$ by multiplying to the left with $X^T \Rightarrow$

$$X^T AX = \lambda X^T X \ (1)$$

$$\Rightarrow (X^T AX)^T = \lambda (X^T X)^T \Rightarrow -X^T AX = \lambda X^T X \quad (2)$$

From (1)+(2)$\Rightarrow \lambda X^T X = -\lambda X^T X \Rightarrow 2\lambda \underbrace{X^T \cdot X}_{>0} = 0 \Rightarrow \lambda = 0$

Let $\lambda \in \mathbb{C}$ be an eigenvalue $\Rightarrow AX = \lambda X \Rightarrow \overline{X}^T AX = \lambda \overline{X}^T X$ (3) $\Rightarrow \overline{\overline{X}^T AX} = \overline{\lambda \overline{X}^T X} \Rightarrow$

$$X^T A\overline{X} = \overline{\lambda} X^T \cdot \overline{X} \Rightarrow (X^T A\overline{X})^T = \overline{\lambda}(X^T \cdot \overline{X})^T \Rightarrow$$

$$-\overline{X}^T AX = \overline{\lambda}\overline{X}^T \cdot X \quad (4)$$

$$\text{From (3)+(4)} \Rightarrow \lambda \overline{X}^T \cdot X = -\overline{\lambda}\overline{X}^T \cdot X \Rightarrow$$

$(\lambda + \overline{\lambda}) \underbrace{\overline{X}^T \cdot X}_{>0} = 0 \Rightarrow \lambda + \overline{\lambda} = 0 \Rightarrow \overline{\lambda} = -\lambda \Rightarrow \lambda$ is purely imaginary, namely $\lambda = bi$

Theorem 3:

The eigenvalues of an orthogonal real matrix have an absolute value equal with 1.

Proof:

Let be $A \in M_n(\mathbb{R})$ with $AA^T = I_n$. Let λ be an eigenvalue $\Rightarrow AX = \lambda X \Rightarrow$ by multiplying to the left with $\overline{X}^T \Rightarrow \overline{X}^T AX = \lambda \overline{X}^T X$ (1)

$$AX = \lambda X \Rightarrow \overline{AX} = \overline{\lambda X} \Rightarrow A\overline{X} = \overline{\lambda}\overline{X} \Rightarrow$$

by multiplying to the left with $X^T \Rightarrow X^T A\overline{X} = \overline{\lambda} X^T \overline{X} \Rightarrow (X^T A\overline{X})^T = \overline{\lambda}(X^T \cdot \overline{X})^T \Rightarrow$

$$\Rightarrow \overline{X}^T A^T X = \overline{\lambda}\overline{X}^T \cdot x \quad (2)$$

$$\left.\begin{array}{c} From\ (1) + (2) \Rightarrow \overline{X}^T AX \cdot \overline{X}^T \cdot A^T \cdot X = |\lambda|^2 \cdot (\overline{X}^T \cdot X)^2 \\ But\ X\overline{X}^T = |X_1|^2 + |X_2|^2 + \cdots + |X_n|^2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow X \cdot \overline{X}^T \overline{X}^T A \cdot A^T = X = |\lambda|^2 (\overline{X}^T \cdot X)^2 \Rightarrow$$

$$(X \cdot \overline{X}^T)^2 = |\lambda|^2 (\overline{X}^T \cdot X)^2 \Rightarrow |\lambda|^2 = 1 \Rightarrow |\lambda| = 1$$

Applications:

1. Let $A \in M_n(\mathbb{R})$ an antisymmetric matrix

a) If n is odd, then $\det A = 0$

b) If n is even, then $\det A$ is a perfect square.

Proof:

a) $n = 2k + 1 \Rightarrow p_A(x)$ has an odd number of pairs $\Rightarrow$ at least one is real $\Rightarrow$ from theorem 2 $\Rightarrow$ the real root is $\Rightarrow \det A = \lambda_1 \lambda_2 \dots \lambda_n = 0$.

b) $n = 2k \Rightarrow p_A(x)$ **has an even number of pairs,** $p_A(X) \in \mathbb{R}[X] \Rightarrow$ **the roots are complex conjugate, from theorem 2** $\Rightarrow \lambda_1 = b_{1i}$ **and** $\overline{\lambda_1} = -b_{1i}, \lambda_2 = b_{2i}$ **and** $\overline{\lambda_2} = -b_{2i} \dots \lambda_k = b_k i$ **and** $\overline{\lambda_k} = -b_k i \Rightarrow \det A = \lambda_1 \lambda_2 \dots \lambda_n = (b_1 \cdot b_2 \dots b_k)^2 > 0$

2. Let $A \in M_3(\mathbb{R})$ **such that** $AA^T = I_3$ **and** $TrA = 0$**. Prove that:** $|Tr\ A| = 3$

(Mathematical Gazette)

Proof:

Let $\lambda_1, \lambda_2, \lambda_3$ **be the eigenvalues of matrix** A **from theorem 3** $\Rightarrow |\lambda_1| = |\lambda_2| = |\lambda_3| = 1$

$Tr\ A = 0 \Leftrightarrow \lambda_1 + \lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 3\lambda_1\lambda_2\lambda_3$ **(known identity)**$\Rightarrow$

$$|\lambda_1^3 + \lambda_2^3 + \lambda_3^3| = 3|\lambda_1||\lambda_2||\lambda_3| \Rightarrow (Tr\ A^3) = 3$$

3) Let be $A \in M_3(\mathbb{R})$**. If** $A \cdot A^T = I_3$ **and** $Tr\ A^2 = 0 \Rightarrow Tr\ A \in \{-2, 0, 2\}$

(Mathematical Gazette)

Proof:

Let be $\lambda_1, \lambda_2, \lambda_3$ **the eigenvalues of matrix** A**. From theorem 3**$\Rightarrow |\lambda_1| = |\lambda_2| = |\lambda_3| = 1$

$$Tr\ A = \lambda_1 + \lambda_2 + \lambda_3, Tr\ A^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$$

$$\det(A \cdot A^T) = \det I_3 \Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1$$

$$\left.\begin{array}{c} \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0 \Rightarrow (\lambda_1 + \lambda_2 + \lambda_3)^2 = 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) \\ but\ |\lambda_i|^2 = 1 \Rightarrow \lambda_1 \cdot \overline{\lambda_i} = 1, i = \overline{1,3} \end{array}\right\} \Rightarrow$$

$$(\lambda_1 + \lambda_2 + \lambda_3)^2 = 2\lambda_1\lambda_2\lambda_3\left(\frac{1}{\lambda_3} + \frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) \Rightarrow$$

$$(\lambda_1 + \lambda_2 + \lambda_3)^2 = 2\lambda_1\lambda_2\lambda_3(\overline{\lambda_1 + \lambda_2 + \lambda_3}) \Rightarrow$$

$$(Tr\ A)^2 = 2\det A\,\overline{Tr\ A}; Tr\ A \in \mathbb{R} \Rightarrow \overline{Tr\ A} = Tr\ A$$

$$\Rightarrow Tr\ A(Tr\ A - 2\det A) = 0 \Rightarrow Tr\ A \text{ or } Tr\ A = \pm 2$$