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ABOUT PROBLEM JP.192.

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1) If $x, y, z > 1$ then:

$$\log_x\left(\frac{y^5+z^5}{y^3+z^3}\right)+\log_y\left(\frac{z^5+x^5}{z^3+x^3}\right)+\log_z\left(\frac{x^5+y^5}{x^3+y^3}\right)\ge 6$$

Proposed by Marian Ursărescu – Romania

Solution.

We prove the following lemma:

Lemma 1.

2) If $x, y > 0$ then:

$$\frac{x^5+y^5}{x^3+y^3}\ge xy$$

Solution.

Inequality is equivalent with $(x-y)(x^4-y^4)\ge 0$, true because the factors $(x-y)$ and (x^4-y^4) have the same sign.

Let's get back to the main problem.

Using Lemma 1 and the monotony of the logarithmic function in supraunitary base, we obtain:

$$\sum\log_x\left(\frac{y^5+z^5}{y^3+z^3}\right)\ge\sum\log_x(yz)=\sum(\log_x y+\log_x y)=$$

$$=\sum\left(\log_x y+\frac{1}{\log_x y}\right)\ge\sum 2=6$$

Equality holds if and only if the $x=y=z$.

Remark.

The inequality can be developed.

3) If $x, y, z > 1$ and $n\in N$ then:

$$\log_x\left(\frac{y^{n+2}+z^{n+2}}{y^n+z^n}\right)+\log_y\left(\frac{z^{n+2}+x^{n+2}}{z^n+x^n}\right)+\log_z\left(\frac{x^{n+2}+y^{n+2}}{x^n+y^n}\right)\ge 6$$

Solution.

We prove the following lemma:

Lemma 2.

4) If $x,y>0$ then:

$$\frac{x^{n+2}+y^{n+2}}{x^n+y^n}\ge xy$$

Proof.

The inequality is equivalent with $(x-y)(x^{n+1}-y^{n+1})\ge 0$, true because the factors $(x-y)$ and $(x^{n+1}-y^{n+1})$ have the same sign.

Let's get back to the main problem:

Using Lemma 2 and the monotony of the logarithmic function in supraunitary base, we obtain:

$$\sum\log_x\left(\frac{y^{n+2}+z^{n+2}}{y^n+z^n}\right)\ge\sum\log_x(yz)=\sum(\log_x y+\log_x y)=$$
$$=\sum\left(\log_x y+\frac{1}{\log_x y}\right)\ge\sum 2=6$$

Equality holds if and only if $x=y=z$.

Remark.

The inequality can be generalized:

5) If $x_1,x_2,\dots,x_k>1,k\ge 3$ and $n\in\mathbb{N}$ then:

$$\log_{x_1}\left(\frac{x_2^{n+2}+x_3^{n+2}}{x_2^n+x_3^n}\right)+\log_{x_2}\left(\frac{x_3^{n+2}+x_4^{n+2}}{x_3^n+x_4^n}\right)+\dots+\log_{x_k}\left(\frac{x_1^{n+2}+x_2^{n+2}}{x_1^n+x_2^n}\right)\ge 6k$$

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Solution.

Using Lemma 2 and the monotony of the logarithmic function in supraunitary base, we obtain:

$$\sum\log_{x_1}\left(\frac{x_2^{n+2}+x_3^{n+2}}{x_2^n+x_3^n}\right)\ge\sum\log_{x_1}(x_2x_3)=\sum(\log_{x_1}x_2+\log_{x_1}x_3)\ge 2k$$

which follows from the means inequality applied to the $2k$ positive numbers.

Equality holds if and only if $x_1 = x_2 = \cdots = x_k$.

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