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ABOUT AN INEQUALITY IN TRIANGLE FROM RMM-2019

By Marin Chirciu – Romania

1) In ΔABC, G – centroid and R_a, R_b, R_c – circumradii $\Delta BCG, \Delta CAG, \Delta ABG$ respectively. Prove that:

$$\frac{R_a}{a}+\frac{R_b}{b}+\frac{R_c}{c}\le \cot A+\cot B+\cot C$$

Proposed by Marian Ursărescu – Romania

Solution

We prove the following lemma:

Lemma.

2) In ΔABC, G – centroid and R_a, R_b, R_c – circumradii $\Delta BCG, \Delta CAG, \Delta ABG$ respectively. Prove that:

$$\frac{R_a}{a}+\frac{R_b}{b}+\frac{R_c}{c}=\frac{m_am_b+m_bm_c+m_cm_a}{3S}$$

Expressing the area of ΔBCG in two ways, we obtain:

$[BCG]=\frac{S}{3}$ *and* $[BCG]=\frac{BC\cdot BG\cdot CG}{4R_a}=\frac{a\cdot\frac{2}{3}m_b\cdot\frac{2}{3}m_c}{4R_a}=\frac{am_bm_c}{9R_a}$, *wherefrom* $\frac{am_bm_c}{9R_a}=\frac{S}{3}$

It follows $R_a=\frac{am_bm_c}{3rs}$ *and from here* $\frac{R_a}{a}=\frac{m_bm_c}{3rs}$

We have: $\frac{R_a}{a}+\frac{R_b}{b}+\frac{R_c}{c}=\frac{m_bm_c}{3rs}+\frac{m_cm_a}{3rs}+\frac{m_am_b}{3rs}$

Let's get back to the main problem:

Using the Lemma and inequality $4m_bm_c\le 2a^2+bc$ we obtain:

$$\sum\frac{R_a}{a}=\sum\frac{m_bm_c}{3rs}\le\sum\frac{2a^2+bc}{3rs}=\frac{2\sum a^2+\sum bc}{12rs}=$$

$$=\frac{2\cdot 2(s^2-r^2-4Rr)+s^2+r^2+4Rr}{12rs}=\frac{5s^2-3r^2-12Rr}{12rs}, \text{ so, } \sum\frac{R_a}{a}\le\frac{5s^2-3r^2-12Rr}{12rs}$$

Using the known identity in triangle $\cot A+\cot B+\cot C=\frac{s^2-r^2-4Rr}{2rs}$.

In order to prove the inequality from the enunciation, it suffices to prove that:

$\frac{5s^2-3r^2-12Rr}{12Rrs} \le \frac{s^2-r^2-4Rr}{2rs} \Leftrightarrow s^2 \ge 12Rr + 3r^2$, *which follows from Gerretsen's inequality:*

$s^2 \ge 16Rr - 5r^2$. *It suffices to prove that:*

$16Rr - 5r^2 \ge 12Rr + 3r^2 \Leftrightarrow R \ge 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark.

Let's find an inequality having an opposite sense:

3) In ΔABC, G – centroid and R_a, R_b, R_c – circumradii $\Delta BCG, \Delta CAG, \Delta ABG$ respectively. Prove that:

$$\frac{R_a}{a} + \frac{R_b}{b} + \frac{R_c}{c} \ge \frac{2r}{R}\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right)$$

Proposed by Marin Chirciu – Romania

Solution:

Using the Lemma and Tereshin's inequality $m_a \ge \frac{b^2+c^2}{4R}$, *we obtain:*

$$\sum \frac{R_a}{a} = \sum \frac{m_b m_c}{3rs} \ge \sum \frac{\frac{c^2+a^2}{4R} \cdot \frac{a^2+b^2}{4R}}{3rs} = \frac{\sum(a^2+b^2)(a^2+c^2)}{48R^2rs} =$$

$= \frac{5s^4-s^2(40Rr+6r^2)+5r^2(4R+r)^2}{48R^2rs}$, *which follows from*

$\sum(a^2+b^2)(a^2+b^2) = 5s^4 - s^2(40Rr + 6r^2) + 5r^2(4R+r)^2$, so

$$\sum \frac{R_a}{a} \ge \frac{5s^4 - s^2(40Rr+6r^2) + 5r^2(4R+r)^2}{48R^2rs}$$

Using the known identity in triangle: $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{4R+r}{s}$.

In order to prove the inequality from enunciation it suffices to prove that:

$$\frac{5s^4 - s^2(40Rr+6r^2) + 5r^2(4R+r)^2}{48R^2rs} \ge \frac{2r}{R} \cdot \frac{4R+r}{s} \Leftrightarrow$$

$\Leftrightarrow s^2(5s^2 - 6r^2 - 40Rr) \ge r^2(4R+r)^2(76R-5r)$, *which follows from Gerretsen's inequality* $s^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}$. *It suffices to prove that:*

$$\frac{r(4R+r)^2}{R+r}\left(5(16Rr-5r^2) - 6r^2 - 40Rr\right) \ge r^2(4R+r)^2(76R-5r) \Leftrightarrow$$

$\Leftrightarrow 84R^2 - 155Rr - 26r^2 \geq 0 \Leftrightarrow (R-2r)(84R+13r) \geq 0$, ***true from Euler's inequality*** $R \geq 2r$***. Equality holds if and only if the triangle is equilateral.***

Remark.

The double inequality can be written:

4) In ΔABC, G – centroid and R_a, R_b, R_c – circumradii $\Delta BCG, \Delta CAG, \Delta ABG$ respectively. Prove that:

$$\frac{2r}{R}\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right) \leq \frac{R_a}{a}+\frac{R_b}{b}+\frac{R_c}{c} \leq \cot A+\cot B+\cot C$$

Solution

See inequalities 1) and 3).

Equality holds if and only if the triangle is equilateral.

Refferences:
Romanian Mathematical Magazine-Interactive Journal-**www.ssmrmh.ro**