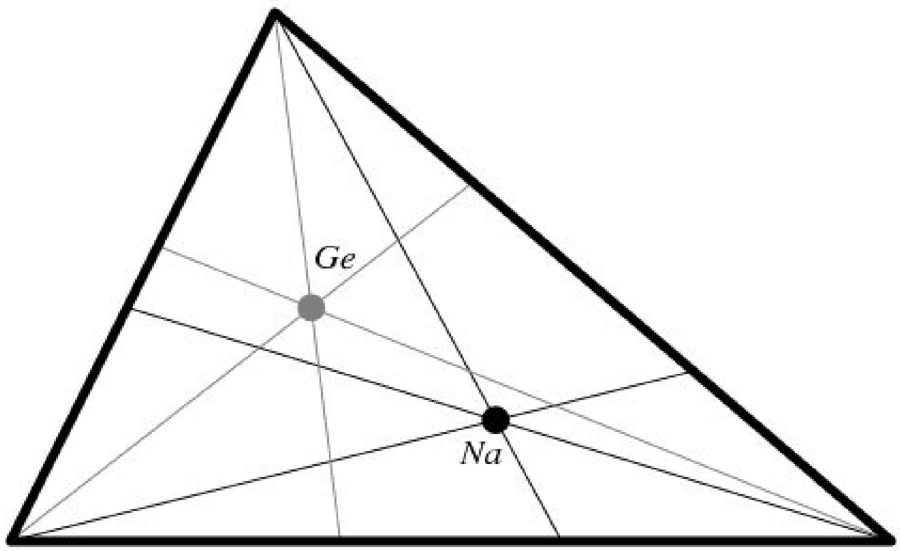
ABOUT NAGEL'S AND GERGONNE'S CEVIANS (II)

By Bogdan Fuștei-Romania



Note by Editor: ***The article is written as a story of discovery triangle inequalities. The author give us a detailed mind process of these discoveries. I consider it an innovative and outstanding method to show results to readers.***

Let ΔABC be any triangle. We've proved that: $b^2 + c^2 - n_a^2 - {g_a}^2 = 2r_a r$ (and the analogs), so we have:

$$b^2 + c^2 = n_a^2 + g_a^2 + 2r_a r \text{ (and the analogs);}$$

But from $m_a \geq \frac{b^2+c^2}{4R}$ (Tereshin's inequality) we will obtain: $m_a \geq \frac{n_a^2+g_a^2+2r_a r}{4R}$ (and the analogs);

We will prove the identity:

$$2(a^2 + b^2 + c^2) = n_a^2 + n_b^2 + n_c^2 + g_a^2 + g_b^2 + g_c^2 + 2r(r_a + r_b + r_c)$$

$$a^2 = 2R\frac{h_b h_c}{h_a} \text{ (and the analogs);}$$

$$\sum \frac{h_b h_c}{h_a} = \frac{n_a^2+n_b^2+n_c^2+g_a^2+g_b^2+g_c^2+2r(4R+r)}{4R};$$

Using the inequality for x, y, z real numbers, we have: $x^2 + y^2 + z^2 \geq xy + xz + yz$ and we will obtain:

$$a^2+b^2+c^2\ge\sum(n_an_b+g_ag_b+rr_a)$$

We know that: $4R^2+16Rr-3r^2-4(R-2r)\sqrt{R(R-2r)}\le a^2+b^2+c^2\le$

$\le 4R^2+16Rr-3r^2+4(R-2r)\sqrt{R(R-2r)}$. Taking into account all the above:

$$8R^2+2r(12R-r)-6r^2-8(R-2r)\sqrt{R(R-2r)}\le\sum(n_a^2+g_a^2)\le$$

$$\le 8R^2+2r(12R-r)-6r^2+8(R-2r)\sqrt{R(R-2r)};$$

$n_a^2=s(s-a)+\frac{(b-c)^2}{a}s$ (and the analogs); $a^2-4rr_a=(b-c)^2$ (and the analogs);

$n_a^2=s^2-\frac{4r_ar}{a}p$; $\frac{s}{a}=\frac{h_a}{2r}$ (and the analogs) because $2S=h_a\times a=2sr$

$n_a^2=s^2-2r_ah_a$ (and the analogs);

$\frac{b+c}{a}=\frac{r_a+h_a}{r_a}$ (and the analogs);

$h_a=\frac{2sr}{a}=\frac{(a+b+c)}{a}r=\left(1+\frac{b+c}{a}\right)r$ (and the analogs);

$\frac{h_a}{r}=2+\frac{h_a}{r_a}$ (and the analogs)$\Rightarrow$ $r_ah_a=(2r_a+h_a)r$ (and the analogs);

$$r_ar_br_c=Ss=s^2r$$

$r_br_c=\frac{h_a(r_b+r_c)}{2}$ (and the analogs);

$s^2=\frac{r_a}{r}\frac{h_a(r_b+r_c)}{2}=\frac{(2r_a+h_a)r}{2r}(r_b+r_c)=\frac{1}{2}(2r_a+h_a)(r_b+r_c)$ (and the analogs);

So we will obtain: $\frac{1}{2}(2r_a+h_a)(r_b+r_c)=n_a^2+2r(2r_a+h_a)$.

Finally we will remember that: $n_a^2=(2r_a+h_a)\left(\frac{r_b+r_c}{2}-2r\right)$ (and the analogs)

$$\sum\frac{n_a^2}{2r_a+h_a}=4R-5r$$

In any acute-angled triangle we have: $\frac{r_b+r_c}{2}\ge m_a$ (and the analogs) because

$$2R\cos^2\frac{A}{2}\ge m_a$$

if the triangle is acute-angled and $\cos^2\frac{A}{2}=\frac{r_b+r_c}{4R}$ (and the analogs)

So, we will have: $n_a^2\ge(2r_a+h_a)(m_a-2r)$ if triangle ABC is acute-angled.

$\frac{n_a^2}{r_ah_a}\ge\frac{m_a-2r}{r}$ if triangle ABC is acute-angled.

$\sum \frac{n_a^2}{2r_a+h_a} \ge m_a + m_a + m_a - 6r$ for any acute-angled ABC triangle.

$\prod \frac{n_a^2}{r_a h_a} \ge \prod \frac{m_a-2r}{r}$ for any acute-angled ABC triangle.

$$\prod \frac{r_a}{h_a} = \frac{R}{2r} \Rightarrow r_a r_b r_c = \frac{R}{2r} h_a h_b h_c$$

So, if ΔABC is acute – angled triangle we have:

$$\prod \frac{n_a^2}{h_a^2} \ge \frac{R}{2r} \prod \frac{m_a - 2r}{r} ;$$

$\frac{R}{2r} \ge \frac{m_a}{h_a}$ (Panaitopol inequality);

$\prod \frac{n_a^2}{h_a^2} \ge \frac{m_a}{h_a} \prod \frac{m_a-2r}{r}$ (and the analogs).

Summing we have the following:

$\prod \frac{n_a^2}{h_a^2} \ge \frac{1}{3} \prod \frac{m_a-2r}{r} \sum \frac{m_a}{h_a}$ for any acute-triangle.

$\cos \frac{B-C}{2} \ge \sqrt{\frac{2r}{R}}$ (and the analogs); $\cos \frac{B-C}{2} = \frac{h_a}{w_a}$ (and the analogs) we will obtain the following inequality:

$\prod \frac{n_a^2}{h_a^2} \ge \frac{1}{3} \prod \frac{m_a-2r}{r} \sum \frac{w_a^2}{h_a^2}$ for any acute – angled triangle;

We've proved that $n_a^2 = s^2 - 2r_a h_a$ (and the analogs). Using the inequality between squared means and arithmetic means we will have: $s\sqrt{2} \ge n_a + \sqrt{r_a h_a}$ (and the analogs)

$3s\sqrt{2} \ge \sum n_a + \sum \sqrt{2r_a h_a}$; $h_a = \frac{2S}{a}$ (and the analogs); $S = sr$

$\frac{s^2}{h_a^2} = \frac{n_a^2}{h_a^2} + 2\frac{r_a}{h_a} \Rightarrow s^2 \frac{a^2}{4S^2} = \frac{n_a^2}{h_a^2} + 2\frac{r_a}{h_a}$, so we will remember that $\frac{a^2}{4r^2} = \frac{n_a^2}{h_a^2} + 2\frac{r_a}{h_a}$ (and the analogs);

$\sin \frac{A}{2} = \sqrt{\frac{r_a - r}{4R}} = \sqrt{\frac{r r_a}{bc}}$ (and the analogs); $bc = 2Rh_a$ (and the analogs);

$\sum \sin^2 \frac{A}{2} = \frac{4R+r-3r}{4R} = \frac{r}{2R} \sum \frac{r_a}{h_a} \Rightarrow \sum \frac{r_a}{h_a} = \frac{2R}{r} - 1$;

So $\frac{a^2+b^2+c^2}{4r^2} = \sum \frac{n_a^2}{h_a^2} + \frac{4R}{r} - 2$; $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$; after calculating we will have: $\frac{s^2}{2r^2} = \sum \frac{n_a^2}{h_a^2} + \frac{6R}{r} - \frac{3}{2}$;

The triangle's fundamental inequality:

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R(R-2r)}\le s^2\le$$

$$\le 2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}$$

Hence: $1+\left(\frac{R}{r}\right)^2-\frac{R}{r}-\frac{(R-2r)\sqrt{R(R-2r)}}{r^2}\le\sum\frac{n_a^2}{h_a^2}\le 1+\left(\frac{R}{r}\right)^2-\frac{R}{r}+\frac{(R-2r)\sqrt{R(R-2r)}}{r^2}$

$h_a=\frac{2r_br_c}{r_b+r_c}$ (and the analogs); $h_ar_a=\frac{2r_ar_br_c}{r_b+r_c}=\frac{2Ss}{r_b+r_c}=\frac{2s^2r}{r_b+r_c}$ (and the analogs);

$\sqrt{2h_ar_a}=2s\sqrt{\frac{r}{r_b+r_c}}$ (and the analogs);

$s\sqrt{2}\ge n_a+2s\sqrt{\frac{r}{r_b+r_c}}$ (and the analogs); so we will obtain: $s\left(1-\sqrt{\frac{2r}{r_b+r_c}}\right)\ge\frac{n_a}{\sqrt{2}}$ (and the analogs)

$$\prod\left(1-\sqrt{\frac{2r}{r_b+r_c}}\right)\ge\frac{n_an_bn_c}{2\sqrt{2}s^3};$$

$m_a+w_b+w_c\le s\sqrt{3}$ (Lessel – Pelling inequality) (and the analogs);

$s\sqrt{2}\ge n_a+\sqrt{2r_ah_a}$ (and the analogs). Summing we will obtain:

$\sqrt{2}+\sqrt{3}\ge\frac{n_a+m_a+w_b+w_c+\sqrt{2r_ah_a}}{s}$ (and the analogs);

So $m_a\le n_a$ (and the analogs) $\Rightarrow\sqrt{2}+\sqrt{3}\ge\frac{n_a+m_a+w_b+w_c+\sqrt{2r_ah_a}}{s}$ (and the analogs);

But $m_a\le n_a$ (and the analogs) $\Rightarrow\sqrt{2}+\sqrt{3}\ge\frac{2m_a+w_b+w_c+\sqrt{2r_ah_a}}{s}$ (and the analogs);

$m_a\ge\frac{b^2+c^2}{4R}$ (Tereshin's inequality) summing we will have the following:

$$m_a+m_b+m_c\ge\frac{a^2+b^2+c^2}{2R}$$

$$\frac{R}{2r^2}(m_a+m_b+m_c)\ge\frac{4R}{r}+\sum\frac{n_a^2}{h_a^2}-2;$$

$$\frac{R}{r}\left(\frac{m_a+m_b+m_c}{2r}-4\right)\ge\sum\frac{n_a^2}{h_a^2}-2;$$

$s\sqrt{2}\ge n_a+\sqrt{2r_ah_a}$ (and the analogs); $\sum\frac{1}{h_a}=\sum\frac{1}{r_a}=\frac{1}{r}$;

$\frac{s\sqrt{2}}{h_a}\ge\frac{n_a}{h_a}+\sqrt{\frac{2r_a}{h_a}}$ (and the analogs); Summing we will have the following:

$$\frac{s}{r} \geq \frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a} + \sum\sqrt{\frac{r_a}{h_a}};$$

$\frac{s\sqrt{2}}{r_a} \geq \frac{n_a}{r_a} + \sqrt{\frac{2h_a}{r_a}}$ (and the analogs). Summing we will have the following:

$$\frac{s}{r} \geq \frac{1}{\sqrt{2}}\sum\frac{n_a}{r_a} + \sum\sqrt{\frac{h_a}{r_a}};$$

$\frac{a^2}{4r^2} = \frac{n_a^2}{h_a^2} + 2\frac{r_a}{h_a}$ (and the analogs); We apply the inequality between squared means and arithmetic means and we will obtain:

$$\frac{a}{r} \geq \sqrt{2}\left(\frac{n_a}{h_a} + \sqrt{\frac{2r_a}{h_a}}\right) \text{ (and the analogs);}$$

$$\prod\frac{a}{r} \geq 2\sqrt{2}\prod\left(\frac{n_a}{h_a} + \sqrt{\frac{2r_a}{h_a}}\right)$$

$s \leq 2R + (3\sqrt{3} - 4)r$ (Blundon – Klamkin's inequality). So we will obtain:

$$3\sqrt{3} + \frac{2R}{r} \geq 4 + \frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a} + \sum\sqrt{\frac{r_a}{h_a}};$$

$$3\sqrt{3} + \frac{2R}{r} \geq 4 + \frac{1}{\sqrt{2}}\sum\frac{n_a}{r_a} + \sum\sqrt{\frac{h_a}{r_a}};$$

We've proved that:

$s^2 = n_a^2 + 2r_a h_a$ (and the analogs);

$2r_a h_a = 2r(r_a + h_a)$ (and the analogs);

$$2(r_a + r_b + r_c) = 8R + 2r \Rightarrow 3s^2 = n_a^2 + n_b^2 + n_c^2 + 2r(8R + 2r + h_a + h_b + h_c);$$

$$h_a + h_b + h_c = \frac{s^2 + 4Rr + r^2}{2R};\ 8R + 2r = \frac{4R(4R + r)}{2R};$$

After calculating we will obtain the following:

$$n_a^2 + n_b^2 + n_c^2 = \frac{s^2(3R - r) - r(4R + r)^2}{R};$$

We will prove that: $\frac{ab + bc + ac}{4\sqrt{3}S} \geq \sqrt{\frac{R}{2r}}$; We know that: $ab + bc + ac = s^2 + 4Rr + r^2; S = sr$.

Squaring we will obtain: $\left(\frac{s^2 + 4Rr + r^2}{4sr\sqrt{3}}\right)^2 \geq \frac{R}{2r} \Rightarrow (s^2 + 4Rr + r^2)^2 \geq 24Rrs^2 \Leftrightarrow$

$$\Leftrightarrow s^2(s^2-16Rr+5r^2)+r^2((4R+r)^2-3s^2)\geq 0;$$

$$s^2\geq 16Rr-5r^2 \text{ (Gerretsen's Inequality)};$$

$$4R+r\geq s\sqrt{3};$$

So the inequality is proved. $bc=2Rh_a$ (and the analogs). After some simplifications from the proved inequality we will obtain: $\sqrt{\frac{R}{2r}}(h_a+h_b+h_c)\geq s\sqrt{3}$;

$m_a+w_b+w_c\leq s\sqrt{3}$ (Lessel Pelling inequality) (and the analogs)

From the above we will obtain: $\sqrt{\frac{R}{2r}}\geq\frac{m_a+w_b+w_c}{h_a+h_b+h_c}$ (and the analogs);

We've proved that: $\frac{s}{r}\geq\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+\sum\sqrt{\frac{r_a}{h_a}}$. Using the inequality between arithmetic means and geometric means we will have: $\sum\sqrt{\frac{r_a}{h_a}}$. Using the inequality between arithmetic means and geometric means we will have: $\sum\sqrt{\frac{r_a}{h_a}}\geq 3\sqrt[3]{\sqrt{\prod\frac{r_a}{h_a}}}=3\sqrt[6]{\frac{R}{2r}}$; taking into account the above we have a new inequality, namely: $\frac{s}{r}\geq\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+3\sqrt[6]{\frac{R}{2r}}$;

$$\frac{s}{r}\geq\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+\sum\sqrt[3]{\frac{m_a+w_b+w_c}{h_a+h_b+h_c}}$$

$\frac{1}{\cos\frac{A}{2}}=\sqrt{\frac{bc}{r_br_c}}$ (and the analogs); $bc=r_br_c+rr_a$ (and the analogs);

$$\frac{1}{\cos^2\frac{A}{2}}=\frac{r_br_c+rr_a}{r_br_c}=1+\frac{rr_a}{r_br_c}\Rightarrow\frac{s^2}{\cos^2\frac{A}{2}}=s^2+s^2\frac{rr_a}{r_br_c}=$$

$=s^2+\frac{r_ar_ar_br_c}{r_br_c}=s^2+r_a^2$ (and the analogs); $r_ar_br_c=Ss=s^2r$

$\frac{s^2}{\cos^2\frac{A}{2}}=s^2+r_a^2$ (and the analogs) $\Rightarrow\cos\frac{A}{2}=\frac{s}{\sqrt{s^2+r_a^2}}$ (and the analogs);

$\cos^2\frac{A}{2}+\sin^2\frac{A}{2}=1\Rightarrow\sin\frac{A}{2}=\frac{r_a}{\sqrt{s^2+r_a^2}}$ (and the analogs); $r_a=s\tan\frac{A}{2}$ (and the analogs);

$\tan\frac{A}{2}=\sin\frac{A}{2}\cdot\frac{1}{\cos\frac{A}{2}}$ (and the analogs); $r_a=s\sin\frac{A}{2}\cdot\frac{1}{\cos\frac{A}{2}}\Rightarrow\frac{r_a}{\sin\frac{A}{2}}=\frac{s}{\cos\frac{A}{2}}$ (and the analogs);

$$\frac{r_a}{\sin\frac{A}{2}} = \sqrt{s^2 + r_a^2} \text{ (and the analogs);}$$

But $s^2 = n_a^2 + 2r_a h_a$ (and the analogs), so we can write the following:

$$\cos\frac{A}{2} = \frac{s}{\sqrt{n_a^2+2r_a h_a+r_a^2}} \text{ (and the analogs);}$$

$$\sin\frac{A}{2} = \frac{r_a}{\sqrt{n_a^2+2r_a h_a+r_a^2}} \text{ (and the analogs);}$$

$$n_a^2 + r_a^2 \geq 2n_a r_a \text{ (and the analogs);}$$

Replacing the above we have the following:

$$\sin\frac{A}{2} \leq \sqrt{\frac{r_a}{2(n_a+h_a)}} \text{ (and the analogs);}$$

$$\cos\frac{A}{2} \leq \frac{s}{\sqrt{2r_a(n_a+h_a)}} \text{ (and the analogs);}$$

From the above we will obtain the following: $\sum \sin\frac{A}{2} \leq \sum \sqrt{\frac{r_a}{2(n_a+h_a)}}$;

$$\sum \cos\frac{A}{2} \leq \sum \frac{s}{\sqrt{2r_a(n_a+h_a)}};$$

From $\sin\frac{A}{2} = \frac{r_a}{\sqrt{n_a^2+2r_a h_a+r_a^2}}$ (and the analogs). Using the inequality between the squared means and the arithmetic means we will have:

$$\sqrt{\frac{n_a^2+2r_a h_a+r_a^2}{3}} \geq \frac{n_a+r_a+\sqrt{2r_a h_a}}{3} \Rightarrow \sin\frac{A}{2} = \frac{r_a}{\sqrt{n_a^2+2r_a h_a+r_a^2}} \leq \frac{r_a\sqrt{3}}{n_a+r_a+\sqrt{2r_a h_a}}.$$

Analogous, we have: $\sin\frac{A}{2} = \frac{r_a}{\sqrt{n_b^2+2r_b h_b+r_a^2}} \leq \frac{r_a\sqrt{3}}{n_b+r_a+\sqrt{2r_b h_b}}$;

$$\sin\frac{A}{2} = \frac{r_a}{\sqrt{n_c^2+2r_c h_c+r_a^2}} \leq \frac{r_a\sqrt{3}}{n_c+r_a+\sqrt{2r_c h_c}} \text{ and the analogous;}$$

$$a = 4R\sin\frac{A}{2}\cos\frac{A}{2} \leq 4Rrs\sqrt{\frac{r_a}{2(n_a+h_a)2r_a(n_a+h_a)}} = 4Rs\frac{1}{2(n_a+h_a)} = \frac{2Rs}{n_a+h_a} \text{ (and the analogous);}$$

$a = 2R\sin A$ (and the analogous) sine theorem $\Rightarrow \sin A \leq \frac{s}{n_a+h_a}$ (and the analogous);

Summing we will obtain: $\sum \frac{1}{\sin A} \geq \frac{n_a+n_b+n_c+h_a+h_b+h_c}{s}$; $a(n_a + h_a) \leq 2Rs$ (and the analogs);

$2S = ah_a = bh_b = ch_c = 2sr \Rightarrow an_a + 2sr \leq 2sR \Rightarrow an_a \leq 2s(R-r)$ (and the analogs);

Summing we will prove a new inequality, namely:

$$an_a + bn_b + cn_c \le 6s(R-r);$$

$$an_a \le 2s(R-r) \Rightarrow \frac{n_a}{a} \le \frac{2s(R-r)}{a^2} \text{ (and the analogs);}$$

$$a^2 = 2R\frac{h_b h_c}{h_a} \text{ (and the analogs);}$$

From the above we will obtain a new inequality:

$$\sum \frac{n_a}{a} \le s\left(1-\frac{r}{R}\right)\sum \frac{h_a}{h_b h_c}$$

But $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \le \frac{1}{4r^2}$, so, we will have a new inequality:

$$\sum \frac{n_a}{a} \le \frac{s}{2r}\left(\frac{R}{r}-1\right);$$

$$s \le 2R + \left(3\sqrt{3}-4\right)r \text{ (Blundon-Klamkin's inequality)} \Rightarrow \sum \frac{n_a}{a} \le \left(\frac{R}{r}-1\right)\left(\frac{R}{r}+\frac{3\sqrt{3}}{2}-2\right)$$

We know that $(s-b)(s-c) = rr_a$ (and the analogs). Using the inequality between arithmetic means and geometric means we will obtain:

$$\sqrt{(s-b)(s-c)} \le \frac{s-b+s-c}{2} = \frac{a}{2} \Rightarrow \sqrt{rr_a} \le \frac{a}{2} \Rightarrow rr_a \le \frac{a^2}{4} \text{ (and the analogs)} \Rightarrow \frac{r_a}{a} \le \frac{a}{4r}$$

Summing we will obtain: $\sum \frac{r_a}{a} \le \frac{s}{2r}$

Taking into account the above inequality we have: $\sum \frac{n_a+r_a}{a} \le \frac{R}{2r}\cdot\frac{s}{r};$

$$an_a\frac{r_a}{a} \le 2s(R-2r)\frac{a}{4r} \Rightarrow n_a r_a \le \frac{as(R-r)}{2r} \text{ (and the analogs);}$$

Summing we have a new inequality, namely: $\sum n_a r_a \le s^2\left(\frac{R}{r}-1\right)$

$\sum \frac{n_a r_a}{a} \le \frac{3}{2}s\left(\frac{R}{r}-1\right)$. We know that $\sum r_b r_c = s^2 \Rightarrow \frac{n_a r_a + n_b r_b + n_c r_c}{r_a r_b + r_b r_c + r_a r_c} \le \frac{R}{r}-1;$

$$an_a \le 2s(R-r) \Rightarrow \frac{n_a}{h_a}a \le \frac{2s(R-r)}{h_a} \Rightarrow \frac{n_a}{h_a} \le \frac{R}{r}-1;$$

$$an_a \le 2s(R-r) \Rightarrow \frac{n_a}{h_a}a \le \frac{2s(R-r)}{h_a} \Rightarrow \frac{n_a}{h_a} \le \frac{R}{r}-1$$

$$\left(\frac{R}{r}-1\right)^3 \ge \frac{n_a n_b n_c}{h_a h_b h_c} \Rightarrow \frac{R}{r} \ge 1 + \sqrt[3]{\frac{n_a n_b n_c}{h_a h_b h_c}} \text{ (Euler's inequality refinement)}$$

$$\frac{R-r}{r} \ge \frac{n_a}{h_a} \Rightarrow \frac{R-r}{n_a} \ge \frac{r}{h_a} \text{ (and the analogs) } \sum \frac{1}{h_a} = \frac{1}{r};$$

Summing we will obtain the following $(R-r)\sum \frac{1}{n_a} \ge \frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c} = 1 \Rightarrow \sum \frac{1}{n_a} \ge \frac{1}{R-r}$

$$\frac{R}{r} \geq \frac{n_a+h_a}{h_a} \Rightarrow \frac{R}{n_a+h_a} \geq \frac{r}{h_a} \text{ (and the analogs)}$$

Summing we will obtain a new inequality: $\sum \frac{1}{n_a+h_a} \geq \frac{1}{R}$;

$$n_a \geq h_a \text{ (and the analogs)} \Rightarrow \frac{1}{n_a+h_a} \leq \frac{1}{2h_a} \Rightarrow \sum \frac{1}{n_a+h_a} \leq \frac{1}{2r}$$

So finally we have a new inequality: $2r \leq \left(\sum \frac{1}{n_a+h_a}\right)^{-1} \leq R$

We've proved that $\sin\frac{A}{2} = \frac{r_a}{\sqrt{n_a^2+2r_ah_a+r_a^2}} = \sqrt{\frac{r}{2R}}\sqrt{\frac{r_a}{h_a}} \Rightarrow \frac{R}{r} = \frac{n_a^2+2r_ah_a+r_a^2}{2r_ah_a}$ and the analogs

$$\frac{R}{r} - 1 = \frac{n_a^2+r_a^2}{2r_ah_a} \text{ (and the analogs)}$$

We will prove a new identity: $8\left(\frac{R}{r}-1\right)^3 = \frac{(n_a^2+r_a^2)(n_b^2+r_b^2)(n_c^2+r_c^2)}{r_ar_br_ch_ah_bh_c}$;

$$r_ar_br_c = \frac{R}{2r}h_ah_bh_c$$

Taking into account the above we have the following:

$$\frac{4R}{r}\left(\frac{R}{r}-1\right)^3 = \frac{(n_a^2+r_a^2)(n_b^2+r_b^2)(n_c^2+r_c^2)}{h_a^2h_b^2h_c^2}$$

$$\frac{r}{R}\left(\frac{R}{r}-1\right)^3 = \frac{(n_a^2+r_a^2)(n_b^2+r_b^2)(n_c^2+r_c^2)}{16r_a^2r_b^2r_c^2}$$

$$\frac{R}{r} - 1 = \frac{n_a^2+r_a^2}{2r_ah_a} \text{ (and the analogs); } n_a^2 + r_a^2 \geq 2n_ar_a \Rightarrow \frac{R}{r} \geq \frac{n_a+h_a}{h_a} \Rightarrow$$

$$\frac{R}{r} \geq \sqrt[3]{\frac{(n_a+h_a)(n_b+h_b)(n_c+h_c)}{h_ah_bh_c}};$$

$$a^2 = 2R\frac{h_bh_c}{h_a} \text{ (and the analogs); } 2S = h_aa = h_bb = h_bc$$

$$\frac{a^2h_a}{2R} = h_bh_c \Rightarrow h_bh_c = \frac{Sa}{R} \text{ (and the analogs); } r_ar_br_c = Ss$$

$\sum h_bh_c = \frac{2Ss}{R} = \frac{2}{R}r_ar_br_c; \frac{R}{2r}h_ah_bh_c = r_ar_br_c$ so we will obtain the following:

$$h_ah_bh_c = r(h_ah_b + h_bh_c + h_ah_c)$$

$$r_ar_br_c = \frac{R}{2}(h_ah_b + h_bh_c + h_ah_c)$$

Taking into account the above we have the following:

$$\frac{R^3}{r^2} \geq \frac{(n_a + h_a)(n_b + h_b)(n_c + h_c)}{h_a h_b + h_b h_c + h_a h_c}$$

But $h_a^2 + h_b^2 + h_c^2 \geq h_a h_b + h_b h_c + h_a h_c$ we will have the following inequality:

$$\frac{R^3}{r^2} \geq \frac{(n_a + h_a)(n_b + h_b)(n_c + h_c)}{h_a^2 + h_b^2 + h_c^2}$$

But $s_a \geq h_a$ and the analogs we will obtain a weaker inequality:

$$\frac{R^3}{r^2} \geq \frac{(n_a + h_a)(n_b + h_b)(n_c + h_c)}{s_a s_b + s_b s_c + s_a s_c}$$

$$\frac{R^3}{r^2} \geq \frac{(n_a + h_a)(n_b + h_b)(n_c + h_c)}{s_a^2 + s_b^2 + s_c^2}$$

$n_a \geq m_a$ (and the analogs) we will have the following:

$$\frac{R^3}{r^2} \geq \frac{(m_a + h_a)(m_b + h_b)(m_c + h_c)}{h_a h_b + h_b h_c + h_a h_c}$$

$$\frac{R^3}{r^2} \geq \frac{(m_a + h_a)(m_b + h_b)(m_c + h_c)}{h_a^2 + h_b^2 + h_c^2}$$

$$\frac{R^3}{r^2} \geq \frac{(m_a + h_a)(m_b + h_b)(m_c + h_c)}{s_a s_b + s_b s_c + s_a s_c}$$

$$\frac{R^3}{r^2} \geq \frac{(m_a + h_a)(m_b + h_b)(m_c + h_c)}{s_a^2 + s_b^2 + s_c^2};$$

We've prove that

$$\frac{R}{2r} \geq \sqrt[3]{\frac{(n_a + h_a)(n_b + h_b)(n_c + h_c)}{8h_a h_b h_c}};$$

$$\frac{s}{r} \geq \frac{1}{\sqrt{2}} \sum \frac{n_a}{h_a} + 3\sqrt[3]{\frac{R}{2r}}$$

We know that $\cos\frac{B-C}{2} \geq \sqrt{\frac{2r}{R}}$ (and the analogs). But $\cos\frac{B-C}{2} = \frac{h_a}{w_a}$ (and the analogs) so we will have: $\sqrt[6]{\frac{R}{2r}} \geq \sqrt[3]{\frac{w_a}{h_a}}$ (and the analogs);

$\frac{R}{2r} \geq \frac{m_a}{h_a}$ (Panaitopol inequality)

So we will have the following:

1) $\frac{s}{r} \geq \frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a} + \sqrt[3]{\frac{w_a}{h_a}} + 3\sqrt[3]{\frac{w_b}{h_b}} + \sqrt[3]{\frac{w_c}{h_c}}$

2) $\frac{s}{r} \geq \frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a} + \sqrt[6]{\frac{m_a}{h_a}} + \sqrt[6]{\frac{m_b}{h_b}} + \sqrt[6]{\frac{m_c}{h_c}}$

3) $\frac{s}{R} \geq \frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a} + \sqrt[18]{\frac{(n_a+h_a)(n_b+h_b)(n_c+h_c)}{8h_ah_bh_c}} + \sqrt[6]{\frac{m_a}{h_a}} + \sqrt[3]{\frac{w_a}{h_a}}$ (and the analogs)

4) $\frac{s}{r} \geq \frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a} + \sqrt[3]{\frac{m_a+w_b+w_c}{h_a+h_b+h_c}} + \sqrt[6]{\frac{m_a}{h_a}} + \sqrt[3]{\frac{w_a}{h_a}}$ (and the analogs)

5) $\frac{s}{r} \geq \frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a} + \sqrt[18]{\frac{(n_a+h_a)(n_b+h_b)(n_c+h_c)}{8s_as_bs_c}} + \sqrt[6]{\frac{m_a}{h_a}} + \sqrt[3]{\frac{w_a}{h_a}}$ (and the analogs)

We've proved that $\sqrt{\frac{R}{2r}}(h_a + h_b + h_c) \geq s\sqrt{3}$ so we have $\sqrt[6]{\frac{R}{2r}} \geq \sqrt[3]{\frac{s\sqrt{3}}{h_a+h_b+h_c}}$

We will have the following:

6) $\frac{s}{r} \geq \frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a} + \sqrt[18]{\frac{(n_a+h_a)(n_b+h_b)(n_c+h_c)}{8h_ah_bh_c}} + \sqrt[3]{\frac{s\sqrt{3}}{h_a+h_b+h_c}} + \sqrt[3]{\frac{w_a}{h_a}}$ (and the analogs)

7) $\frac{s}{r} \geq \frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a} + \sqrt[18]{\frac{(n_a+h_a)(n_b+h_b)(n_c+h_c)}{8h_ah_bh_c}} + \sqrt[3]{\frac{s\sqrt{3}}{h_a+h_b+h_c}} + \sqrt[6]{\frac{m_a}{h_a}}$ (and the analogs)

$\sqrt{\frac{R}{2r}} \geq \frac{w_a}{h_a}$ (and the analogs); $w_a \geq \frac{b+c}{2}\cos\frac{A}{2}$ (and the analogs); $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$ (and the analogous); $s(s-a) = r_br_c$ (and the analogs); $h_a = \frac{2r_br_c}{r_b+r_c}$ (and the analogs). Taking into account the above we can write that:

$$m_aw_a \geq s(s-a) = r_br_c \text{ (and the analogs) (Panaitopol)}$$

So we will have $\frac{m_aw_a}{h_a} \geq \frac{r_b+r_c}{2}$ (and the analogs); $\sqrt{\frac{R}{2r}} \geq \frac{w_a}{h_a}$ (and the analogs)

$\Rightarrow m_a\sqrt{\frac{R}{2r}} \geq \frac{m_aw_a}{h_a}$ (and the analogs)

$$m_a\sqrt{\frac{R}{2r}} \geq \frac{m_aw_a}{h_a} \geq \frac{r_b+r_c}{2} \text{ (and the analogs)}$$

So $(m_a + m_b + m_c)\sqrt{\frac{R}{2r}} \geq \sum\frac{m_aw_a}{h_a} \geq r_a + r_b + r_c = 4R + r$

Finally: $\sqrt{\frac{R}{2r}} \geq \frac{r_a+r_b+r_c}{m_a+m_b+m_c}$;

So we can write:

8) $\frac{s}{r}\geq\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+3\sqrt[3]{\frac{r_a+r_b+r_c}{m_a+m_b+m_c}}$

9) $\frac{s}{r}\geq\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+\sqrt[3]{\frac{r_a+r_b+r_c}{m_a+m_b+m_c}}+\sqrt[3]{\frac{w_a}{h_a}}+\sqrt[6]{\frac{m_a}{h_a}}$

10) $\frac{s}{r}\geq\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+\sqrt[3]{\frac{s\sqrt{3}}{h_a+h_b+h_c}}+\sqrt[18]{\frac{(n_a+h_a)(n_b+h_b)(n_c+h_c)}{8h_ah_bh_c}}+\sqrt[3]{\frac{r_a+r_b+r_c}{m_a+m_b+m_c}}$

We've showed that $\sqrt{\frac{R}{2r}}\geq\frac{r_a+r_b+r_c}{m_a+m_b+m_c}$ and $\sqrt{\frac{R}{2r}}(h_a+h_b+h_c)\geq s\sqrt{3}\Rightarrow$

$$\Rightarrow\sqrt{\frac{R}{2r}}(m_a+m_b+m_c+h_a+h_b+h_c)\geq r_a+r_b+r_c+s\sqrt{3}$$

so finally we will have the following inequality:

$$\sqrt{\frac{R}{2r}}\geq\frac{r_a+r_b+r_c+s\sqrt{3}}{m_a+m_b+m_c+h_a+h_b++h_c}$$

11) $\frac{s}{r}\geq\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+\sqrt[3]{\frac{r_a+r_b+r_c+s\sqrt{3}}{m_a+m_b+m_c+h_a+h_b++h_c}}+\sqrt[18]{\frac{(n_a+h_a)(n_b+h_b)(n_c+h_c)}{8h_ah_bh_c}}+\sqrt[3]{\frac{m_a+w_b+w_c}{h_a+h_b+h_c}}$ (and the analogs)

$\sqrt{\frac{R}{2r}}\geq\frac{m_a+w_b+w_c}{h_a+h_b+h_c}$ (and the analogs). Summing we will have the following:

$$3\sqrt{\frac{R}{2r}}(h_a+h_b+h_c)\geq m_a+m_b+m_c+2(w_a+w_b+w_c)$$

$$\sqrt{\frac{R}{2r}}\geq\frac{1}{3}\cdot\frac{m_a+m_b+m_c+2(w_a+w_b+w_c)}{h_a+h_b++h_c}$$

We've proved that $\frac{s}{r}\geq\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+\sum\sqrt{\frac{r_a}{h_a}}$ and $\frac{s}{r}\geq\frac{1}{\sqrt{2}}\sum\frac{n_a}{r_a}+\sum\sqrt{\frac{h_a}{r_a}}$ and summing we will prove a new inequality:

$$\frac{2s}{r}\geq\frac{1}{2\sqrt{2}}\sum\left(\frac{n_a}{h_a}+\frac{n_a}{r_a}\right)+\sum\left(\sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{h_a}{r_a}}\right)$$

$$\sqrt{\frac{R}{2r}}(h_a+h_b+h_c)\ge s\sqrt{3}\Rightarrow\sqrt{\frac{R}{6r}}(h_a+h_b+h_c)\ge s$$

$$\sqrt{2}+\sqrt{3}\ge\frac{n_a+m_a+w_b+w_c+\sqrt{2r_ah_a}}{s}\text{ (and the analogs)}\Rightarrow$$

$$s\left(\sqrt{2}+\sqrt{3}\right)\ge n_a+m_a+w_b+w_c+\sqrt{2r_ah_a}\text{ (and the analogs)}$$

We will obtain that $\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)\sqrt{\frac{R}{r}}\ge\frac{s\left(\sqrt{2}+\sqrt{3}\right)}{(h_a+h_b+h_c)}\Rightarrow\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)\sqrt{\frac{R}{r}}\ge$

$$\frac{n_a+m_a+w_b+w_c+\sqrt{2r_ah_a}}{h_a+h_b+h_c}\text{ (and the analogs)}$$

$$\sqrt{\frac{R}{2r}}\ge\frac{r_a+r_b+r_c}{m_a+m_b+m_c}$$

Summing the two inequalities we will obtain a new result.

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)\frac{R}{r}\ge\frac{(r_a+r_b+r_c)(n_a+m_a+w_b+w_c+\sqrt{2r_ah_a})}{(m_a+m_b+m_c)(h_a+h_b+h_c)}\text{ (and the analogs)}$$

$$m_a\sqrt{\frac{R}{2r}}\ge\frac{r_b+r_c}{2}\text{ (and the analogs)}\Rightarrow\sqrt{\frac{R}{2r}}\ge\frac{r_b+r_c}{2m_a}\text{ (and the analogs)}$$

12) $\frac{s}{r}\ge\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+\sqrt[3]{\frac{r_b+r_c}{2m_a}}+\sqrt[3]{\frac{r_a+r_c}{2m_b}}+\sqrt[3]{\frac{r_a+r_b}{2m_c}}$

$$3\sqrt{\frac{R}{2r}}\ge\frac{1}{2}\sum\frac{r_b+r_c}{m_a}=>\sqrt{\frac{R}{2r}}\ge\frac{1}{6}\sum\frac{r_b+r_c}{m_a}$$

13) $\frac{s}{r}\ge\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+\sqrt[3]{\frac{9}{2}\sum\frac{r_b+r_c}{m_a}}$

14) $\frac{s}{r}\ge\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+\sqrt[3]{\frac{r_b+r_c}{2m_a}}+\sqrt[3]{\frac{m_a+w_b+w_c}{h_a+h_b+h_c}}+\sqrt[3]{\frac{w_a}{h_a}}$ (and the analogs)

Practically any expression smaller than $\frac{R}{2r}$ can be used in the following inequality:

$\frac{s}{r}\ge\frac{1}{\sqrt{2}}\sum\frac{n_a}{h_a}+\sqrt[6]{\frac{R}{2r}}$ in order to obtain a new inequality.

There are limitless possibilities, but $\frac{s}{r}=\sum\cot\frac{A}{2}$, replacing in the above obtained inequalities it will follow a new series of inequalities.

$$\cos\frac{A}{2}=\cos\frac{A}{2}\cdot\frac{1}{\sin\frac{A}{2}}\text{ (and the analogs)}$$

$a = \sqrt{(r_b + r_c)(r_a - r)}$ (and the analogs)

$\sin\frac{A}{2} = \sqrt{\frac{r_a - r}{4R}}$ (and the analogs)

$\cos\frac{A}{2} = \sqrt{\frac{r_b + r_c}{4R}}$ (and the analogs)

From the above we have the following: $\cot\frac{A}{2} = \sqrt{\frac{r_b + r_c}{r_a - r}}$ (and the analogs);

$\Rightarrow \sum\sqrt{\frac{r_b + r_c}{r_a - r}} = \frac{s}{r}$, replacing in this expression in the above inequities we will obtain a series of equivalent inequalities.

$\cot\frac{A}{2} = \sqrt{\frac{r_b + r_c}{r_a - r}} = \sqrt{\frac{(r_b + r_c)(r_b + r_c)}{(r_a - r)(r_b + r_c)}} = \frac{r_b + r_c}{a}$ (and the analogs)

So we will obtain a new identity and namely: $\sum\frac{r_b + r_c}{a} = \frac{s}{r}$ (This identity can be found as a proposed problem by Prof. Mehmet Sahin)

We will remember that $\sum\frac{r_b + r_c}{a} = \sum\sqrt{\frac{r_b + r_c}{r_a - r}} = \sum\cot\frac{A}{2} = \frac{s}{r}$

We've proved that $\frac{R}{r} \geq 1 + \frac{n_a}{h_a}$ (and the analogs); $1 + \frac{n_a}{h_a} \geq 2\sqrt{1\frac{n_a}{h_a}} = 2\sqrt{\frac{n_a}{h_a}}$

(the inequalities between arithmetic means and geometric means) $\Rightarrow \frac{R}{2r} \geq \sqrt{\frac{n_a}{h_a}}$ (and the analogs)

We've proved that $\frac{ab + bc + ac}{4\sqrt{3}S} \geq \sqrt{\frac{R}{2r}} \Rightarrow \frac{ab + bc + ac}{4\sqrt{3}S} \geq \sqrt[4]{\frac{n_a}{h_a}}$ (and the analogs)

$\Rightarrow 3(ab + bc + ac) \geq 4\sqrt{3}S\sum\sqrt[4]{\frac{n_a}{h_a}}$; $bc = 2Rh_a$ (and the analogs), so we will have

$$h_a + h_b + h_c \geq \frac{2\sqrt{3}}{3}\frac{S}{R}\sum\sqrt[4]{\frac{n_a}{h_a}}$$

$h_a = \left(1 + \frac{b + c}{a}\right)r$ (and the analogs)

$$h_a + h_b + h_c = \left(3 + \frac{b + c}{a} + \frac{a + c}{b} + \frac{b + a}{c}\right)r \Rightarrow$$

$$\Rightarrow 3+\frac{b+c}{a}+\frac{a+c}{b}+\frac{b+a}{c}\ge\frac{2\sqrt{3}}{3}\cdot\frac{s}{R}\sum\sqrt[4]{\frac{n_a}{h_a}}$$

But $\frac{b+c}{a}=\frac{r_a+r}{r_a-r}$ (and the analogs) so we will obtain a new inequality:

$$3+\sum\frac{r_a+r}{r_a-r}\ge\frac{2\sqrt{3}}{3}\cdot\frac{s}{R}\sum\sqrt[4]{\frac{n_a}{h_a}};$$

$$\frac{b+c}{a}=\frac{r_a+h_a}{r_c}=1+\frac{h_a}{r_a}\ \text{(and the analogs)}$$

So we will obtain a new inequality, namely:

$$6+\sum\frac{h_a}{r_a}\ge\frac{2\sqrt{3}}{3}\cdot\frac{s}{R}\sum\sqrt[4]{\frac{n_a}{h_a}};$$

But $\sum\frac{b+c}{a}=\sum\frac{h_b+h_c}{h_a}\Rightarrow 3+\sum\frac{h_b+h_c}{h_a}\ge\frac{2\sqrt{3}}{3}\cdot\frac{s}{R}\sum\sqrt[4]{\frac{n_a}{h_a}};$

$$m_a\ge\frac{b^2+c^2}{4R}\ \text{(Tereshin's inequality)}$$

$$\frac{m_a}{h_a}\ge\frac{b^2+c^2}{4Rh_a}=\frac{b^2+c^2}{2bc}=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)\ \text{(and the analogs)}$$

$\sum\frac{m_a}{h_a}\ge\frac{1}{2}\sum\frac{b+c}{a}\Rightarrow 2\sum\frac{m_a}{h_a}\ge\sum\frac{b+c}{a}$ / +3 so we can write that:

$$3+2\sum\frac{m_a}{h_a}\ge\frac{2\sqrt{3}}{3}\,\frac{s}{R}\sum\sqrt[4]{\frac{n_a}{h_a}};$$

We know that $w_a=\frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}$ (and the analogs); $\sqrt{s(s-a)}=\sqrt{r_br_c}$ (and the analogs);

$\frac{b+c}{2\sqrt{bc}}=\frac{\sqrt{r_br_c}}{w_a}$ (and the analogs); squaring we will obtain the following:

$\frac{r_br_c}{w_a^2}=\frac{1}{2}+\frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}\right)$ (and the analogs) $\Rightarrow\sum\frac{r_br_c}{w_a^2}=\frac{3}{2}+\frac{1}{4}\sum\frac{b+c}{a}$; we can write the following:

$$\sum\frac{r_br_c}{w_a^2}\ge\frac{3}{4}+\frac{\sqrt{3}}{6}\cdot\frac{s}{R}\sum\sqrt[4]{\frac{n_a}{h_a}}$$

$$\cos\frac{B-C}{2}=\left(\frac{b+c}{a}\right)\sin\frac{A}{2}\ \text{(and and analogs);}$$

$$\cos\frac{B-C}{2}=\frac{h_a}{w_a}\ \text{(and the analogs);}$$

$$\sin\frac{A}{2}=\sqrt{\frac{r}{2R}}\sqrt{\frac{r_a}{h_a}}\ \text{(and the analogs);}$$

$$\frac{h_a}{w_a}=\sqrt{\frac{r}{2R}}\sqrt{\frac{r_a}{h_a}}\left(\frac{b+c}{a}\right)$$ (and the analogs);

We will obtain: $\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\cdot\frac{h_a}{w_c}\sqrt{\frac{h_a}{r_a}}$ (and the analogs); taking into account the above we obtain the following:

$$\sum\frac{h_a}{w_a}\sqrt{\frac{h_a}{r_a}}\geq\sqrt{\frac{3r}{2R}}\left(\frac{2}{3}\cdot\frac{s}{R}\sum\sqrt[4]{\frac{n_a}{h_a}}-\sqrt{3}\right);$$

We proved that: $4m_a^2=n_a^2+g_a^2+2r_br_c$ (and the analogs);

$b^2+c^2=n_a^2+g_a^2+2r_ar$ (and the analogs);

$bc=r_br_c+rr_a$ (and the analogs);

We will obtain the following identities, namely:

$(b+c)^2=4(m_a^2+r_ar)$ (and the analogs);

$(b+c)^2=n_a^2+g_a^2+2r_br_c+4r_ar$ (and the analogs);

Using the inequality between the squared means and arithmetic means:

$b+c\geq\frac{1}{2}\left(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}\right)$ (and the analogs);

We also know that: $2\sqrt{3}m_a\geq n_a+g_a+\sqrt{2r_br_c}$ (and the analogs);

Summing we obtain: $b+c+m_a\sqrt{3}\geq n_a+g_a+\sqrt{2r_br_c}+\sqrt{r_ar}$ (and the analogs);

Taking into account the above we will have the inequality:

$$s\geq\frac{1}{8}\sum\left(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}\right)$$

$a(b+c)=2R(h_b+h_c)$ (and the analogs); $a=2R\sin A$ (sine theorem)

$$2R(h_b+h_c)\geq\frac{2R\sin A}{2}\left(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}\right)\Rightarrow 2(h_b+h_c)\geq$$

$\geq\sin\left(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}\right)$ (and the analogs); summing we will obtain the following:

$$h_a+h_b+h_c\geq\frac{1}{4}\sum\sin A\left(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}\right)$$

$$\frac{2}{\sin A}\geq\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{h_b+h_c}$$ (and the analogs);

Summing we have the following: $\sum\frac{2}{\sin A}\geq\sum\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{h_b+h_c}$ (and the analogs)

Summing we have the following: $\sum\frac{2}{\sin A}\ge\sum\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{h_b+h_c}$;

We know that $a(b+c)=(r_a+r)(r_b+r_c)$ (and the analogs)

$$(r_a+r)(r_b+r_c)\ge\frac{1}{2}a\left(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}\right)$$

$$\frac{r_b+r_c}{a}\ge\frac{1}{2}\cdot\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{r_a+r};$$

$$\sum\frac{r_b+r_c}{a}\ge\frac{1}{2}\sum\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{r_a+r};$$

$\sum\frac{r_b+r_c}{a}=\frac{s}{r};\frac{r_b+r_c}{a}=\cot\frac{A}{2}$ (and the analogs) $\Rightarrow\cot\frac{A}{2}\ge\frac{1}{2}\cdot\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{r_a+r}$ (and the analogs)

$\sum\cot\frac{A}{2}\ge\frac{1}{2}\sum\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{r_a+r};r_a=s\tan\frac{A}{2}$ (and the analogs);

$\tan\frac{A}{2}=\frac{1}{\cot\frac{A}{2}}$; so we will have:

$r_a=\frac{s}{\cot\frac{A}{2}}$ (and the analogs) $\Rightarrow\frac{r_a+r}{r_a}\ge\frac{1}{2}\cdot\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{s}$ (and the analogs);

$$3+\frac{b+c}{a}+\frac{a+c}{b}+\frac{b+a}{c}=\frac{a+b+c}{a}+\frac{a+b+c}{b}+\frac{a+b+c}{c}=$$
$$=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right);$$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge\frac{2\sqrt{3}}{3}\cdot\frac{s}{R}\sum\sqrt[4]{\frac{n_a}{h_a}};$$

$$\frac{b+c}{a}\ge\frac{1}{2}\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{a}$$

$$\frac{1}{2}\sum\frac{b+c}{a}\ge\frac{1}{4}\sum\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{a}$$

$$\sum\frac{m_a}{h_a}\ge\frac{1}{2}\sum\frac{b+c}{a}\Rightarrow\sum\frac{m_a}{h_a}\ge\sum\frac{b+c}{a}\ge\frac{1}{4}\sum\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{a};$$

$\frac{m_a}{h_a}\ge\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$ (and the analogs); $\frac{m_a}{h_a}=\frac{m_a}{w_a}\cdot\frac{w_a}{h_a};\frac{h_a}{w_a}\ge\sqrt{\frac{2r}{R}}$ (and the analogs);

$\Rightarrow\frac{m_a}{h_a}=\frac{m_a}{w_a}\frac{w_a}{h_a}\le\frac{m_a}{w_a}\sqrt{\frac{R}{2r}}$ (and the analogs). From the above we will write:

$$\frac{m_a}{w_a}\sqrt{\frac{R}{2r}} \ge \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right);$$

$$\frac{m_a}{w_a} \ge \sqrt{\frac{r}{2R}}\left(\frac{b}{c}+\frac{c}{b}\right) \text{ (and the analogs);}$$

Summing we have: $\sum\frac{m_a}{w_a} \ge \sqrt{\frac{r}{2R}}\left(\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}\right) \Rightarrow \sum\frac{m_a}{w_a} \ge \frac{1}{2}\sqrt{\frac{r}{2R}}\sum\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{a}$

We've proved that: $\frac{b+c}{a} \ge \frac{1}{2}\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{a}$; $\frac{b+c}{a} = \sqrt{\frac{2R}{r}}\cdot\frac{h_a}{w_a}\sqrt{\frac{h_a}{r_a}}$

We will obtain the following:

$$\sqrt{\frac{2R}{r}} \ge \frac{(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar})w_a}{4S}\cdot\sqrt{\frac{r_a}{h_a}} \text{ (and the analogs)}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{rr_a}{bc}};\ bc = 2Rh_a \text{ (and the analogs);}$$

$\frac{1}{\sin\frac{A}{2}} \ge \frac{(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar})w_a}{4S}$ (and the analogs); $AI = \frac{r}{\sin\frac{A}{2}}$ (and the analogs);

$AI \ge \frac{(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar})w_a}{4s} \Rightarrow \frac{AI}{w_a} \ge \frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{4s}$ (and the analogs);

$$\sum\frac{AI}{w_a} \ge \frac{1}{4s}\sum\left(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}\right)$$

We know that $s_a \le w_a$ (and the analogs) $\Rightarrow \sum\frac{AI}{s_a} \ge \frac{1}{4}\sum\left(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}\right)$;

But $AI = \sqrt{2R(h_a-2r)}$ (and the analogs) $\Rightarrow$

$$\Rightarrow \sum\frac{\sqrt{h_a-2r}}{w_a} \ge \frac{1}{4s\sqrt{2R}}\sum\left(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}\right);$$

We easily prove that: $b+c = 4R\cos\frac{A}{2}\cos\frac{B-C}{2}$ (and the analogs);

$$\Rightarrow 8R\cos\frac{A}{2} \ge \frac{w_a}{h_a}\left(n_a+g_a+\sqrt{2r_br_c}+2\sqrt{2r_ar}\right) \text{ (and the analogs);}$$

$\cos\frac{A}{2} = \sqrt{\frac{r_b+r_c}{4R}}$ (and the analogs) $\Rightarrow 8R\cos\frac{A}{2} = 4\sqrt{R(r_b+r_c)}$;

$$\frac{4\sqrt{R(r_b+r_c)}}{w_a} \ge \frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{h_a} \text{ (and the analogs);}$$

$$\sum\frac{\sqrt{r_b+r_c}}{w_a} \ge \frac{1}{4\sqrt{R}}\sum\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{h_a}$$

It is known that $AI^2 = (r_b - r)(r_c - r)$ (and the analogs); applying the inequality between the arithmetic means and geometric means we obtain:

$$\frac{r_b+r_c-2r}{2} \geq AI \text{ (and the analogs);}$$

$$\frac{AI}{w_a} \geq \frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{4s} \text{ (and the analogs)}$$

We will remind that: $\frac{r_b+r_c-2r}{w_a} \geq \frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{r_ar}}{2s}$ (and the analogs);

Summing we will obtain two new inequalities:

$$\sum \frac{r_b + r_c - 2r}{w_a} \geq \frac{1}{2s} \sum (n_a + g_a + \sqrt{2r_br_c} + 2\sqrt{r_ar})$$

$$\sum \frac{r_b + r_c - 2r}{n_a + g_a + \sqrt{2r_br_c} + 2\sqrt{r_ar}} \geq \frac{w_a + w_b + w_c}{2s}$$

We've prove that:

$$s^2 = n_a^2 + 2r_ah_a \text{ (and the analogs);}$$

$$2r_ah_a = 2r(h_a + h_a) \text{ (and the analogs);}$$

$\frac{R-r}{r} = \frac{n_a^2+r_a^2}{2r_ah_a}$ (and the analogs) $\Rightarrow \frac{R-r}{r} = \frac{n_a^2+r_a^2}{2r(2r_a+h_a)} \Rightarrow R - r = \frac{n_a^2+r_a^2}{2(2r_a+h_a)}$ (and the analogs);

Summing we have:

$$6(R-r) = \sum \frac{n_a^2+r_a^2}{(2r_a+h_a)}; 2(R-r)(2r_a + h_a) = n_a^2 + r_a^2 \geq 2n_ar_a \Rightarrow R - r \geq \frac{n_ar_a}{2r_a+h_a};$$

Summing we have:

$3(R-r) \geq \sum \frac{n_a^2+r_a^2}{2r_a+h_a}$. But $n_a \geq m_a$ (and the analogs) we will obtain:

$$6(R-r) \geq \sum \frac{m_a^2+r_a^2}{2r_a+h_a}; 3(R-r) \geq \sum \frac{m_ar_a}{2r_a+h_a};$$

$\frac{R-r}{r_a} \geq \frac{n_a}{2r_a+h_a}$, $\sum \frac{1}{r_a} = \frac{1}{r}$ we will obtain: $\frac{R-r}{r} \geq \sum \frac{n_a}{2r_a+h_a}$

Using the inequality between squared means and arithmetic means:

$$\sqrt{\frac{n_a^2 + r_a^2}{2}} \geq \frac{n_a + r_a}{2} \Rightarrow \sqrt{2(R-r)(2r_a + h_a)} \geq \frac{\sqrt{2}}{2}(n_a + r_a)$$

$\sqrt{(R-r)(2r_a + h_a)} \geq \frac{1}{2}(n_a + r_a)$ (and the analogs), summing we will obtain a new inequality:

$$\sqrt{R-r}\sum\sqrt{2r_a+h_a}\geq\frac{1}{2}(r_a+r_b+r_c+n_a+n_b+n_c)$$

We will finish with the following:

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R(R-2r)}\leq s^2$$
$$\leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}$$

$$s^2=n_a^2+2r(r_a+h_a)\quad\text{(and the analogs)}$$

We will obtain:

$$2R^2+2r(5R-2r_a-h_a)-r^2-2(R-2r)\sqrt{R(R-2r)}\leq n_a^2\leq$$
$$\leq 2R^2+2r(5R-2r_a-h_a)-r^2+2(R-2r)\sqrt{R(R-2r)}\quad\text{(and as always, the analogs).}$$

References:

1. Themistocles M. RASSIAS, Hari M. SRIVASTAVA – Analytic and Geometric Inequalities and Applications.

2. Bogdan FUȘTEI, Romania – About Nagel and Gergonne's cevians, Romanian Mathematical Magazine, www.ssmrmh.ro