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PROBLEMS FOR JUNIORS

JP.316. If $x \in \mathbb{R}_+^* = (0,\infty); a,b,c$ - are the lengths of the sides of ΔABC whith area F, altitudes h_a, h_b, h_c, then:

$$\frac{6x-1}{h_a^2}+\left(\frac{2}{3x}-1\right)\cdot\frac{1}{h_a^2}+\frac{1}{h_c^2}\geq\frac{\sqrt{3}}{F}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

JP.317. In ΔABC the following relationship holds:

$$(a^3+b^3+c^3)\left(\frac{a}{4s^2-a^2}+\frac{b}{4s^2-b^2}+\frac{c}{4s^2-c^2}\right)\geq\frac{27\sqrt{3}}{32}F$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

JP.318. If $x,y,z \in \mathbb{R}_+^* = (0,\infty), a,b,c$ - are the lengths of the sides of ΔABC with area F, altitudes h_a, h_b, h_c, then:

$$\frac{a^2}{(ax+by+cz)h_a}+\frac{b^2}{(bx+cy+az)h_b}+\frac{c^2}{(cx+ay+bz)h_c}\geq\frac{2\sqrt{3}}{x+y+z}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

JP.319. If $a,b,x,y,z \in \mathbb{R}_+^* = (0,\infty)$, then:

$$\frac{x^2+y^2}{a(y+z)^2+bxz}+\frac{y^2+z^2}{a(z+x)^2+byz}+\frac{z^2+x^2}{a(x+y)^2+bzy}\geq\frac{6}{4a+b}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

JP.320. If in $\Delta ABC, D \in (BC), E \in (CA), F \in (AB)$ such that $AD \cap BE \cap CF = \{M\}$, then:

$$\left(\frac{MD^2}{MA^2}+\frac{ME^2}{MB^2}+\frac{MF^2}{MC^2}\right)(a^8+b^8+c^8)\geq 64S^2$$

where S - area of ΔABC.

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

JP.321. Let x,y,z be positive real numbers with $x^2+y^2+z^2\leq 12$. Prove that:

$$\sqrt{(x^3+1)(y^3+1)(z^3+1)}\leq 27$$

Proposed by George Apostolopoulos-Greece

JP.322. Let a, b, c be positive real numbers with $a + b + c = 6$. Prove that:

$$(a^3 + b^3 + c^3 + 12)\left(\frac{a^2}{\sqrt{a^3 + 1}} + \frac{b^2}{\sqrt{b^3 + 1}} + \frac{c^2}{\sqrt{c^3 + 1}}\right) \geq 144$$

Proposed by George Apostolopoulos-Greece

JP.323. Let a, b, c be positive real numbers with $a^2 + b^2 + c^2 = 12$. Prove that:

$$\frac{a^4}{\sqrt{a^3 + 1}} + \frac{b^4}{\sqrt{b^3 + 1}} + \frac{c^4}{\sqrt{c^3 + 1}} \geq 16$$

Proposed by George Apostolopoulos-Greece

JP.324. Let x, y, z be positive real numbers such that $x^4 + y^4 + z^4 = 3$. Find the maximum value of the expression:

$$P = \sqrt{\frac{yz}{7 - 2x}} + \sqrt{\frac{zx}{7 - 2y}} + \sqrt{\frac{xy}{7 - 2z}}$$

Proposed by Hoang Le Nhat Tung - Vietnam

JP.325. Let be the triangle ABC, A', B', C' the middles of the arches $\widehat{BC}, \widehat{AC}, \widehat{AB}$ (made with the circumcircle). Prove that:

$$\frac{AB \cdot BC \cdot AC}{A'B' \cdot B'C' \cdot C'A'} \leq \sqrt{\cos\left(\frac{A - B}{2}\right)\cos\left(\frac{B - C}{2}\right)\cos\left(\frac{C - A}{2}\right)}$$

Proposed by Marian Ursărescu - Romania

JP.326. In $\Delta ABC, AD, BE, CF$- altitudes and H - othocenter. Prove that:

$$\frac{HA}{HD} + \frac{HB}{HE} + \frac{HC}{HF} \geq 2\left(\left(\frac{R}{r}\right)^2 - 1\right)$$

Proposed by Marian Ursărescu - Romania

JP.327. Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\sin^2 A \cdot \cos\frac{B}{2} + \sin^2 B \cdot \cos\frac{C}{2} + \sin^2 C \cdot \cos\frac{A}{2} \leq 3\sqrt{3}\left(\frac{1}{2} - \frac{r^3}{R^3}\right)$$

Proposed by George Apostolopoulos-Greece

JP.328. Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$4 \leq \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} \leq \frac{2R}{r}$$

Proposed by George Apostolopoulos-Greece

JP.329. Suppose that a triangle in the plane has inradius r, circumradius R, angles A, B, C, and corresponding medians m_A, m_B, m_C. Prove:

$$6\sqrt{3}r \le \frac{m_A}{\cos\frac{A}{2}} + \frac{m_B}{\cos\frac{B}{2}} + \frac{m_c}{\cos\frac{C}{2}} \le \frac{3\sqrt{6}}{2}R\sqrt{\frac{R}{r}}$$

Proposed by George Apostolopoulos-Greece

JP.330. Let a, b, c be positive real numbers such that $abc = 1$. Find the maximum value of the expression:

$$P = \sqrt{\frac{ab}{a^5+b^3-2a+6}} + \sqrt{\frac{bc}{b^5+c^3-2b+6}} + \sqrt{\frac{ca}{c^5+a^3-2c+6}}$$

Proposed by Hoang Le Nhat Tung - Vietnam

PROBLEMS FOR SENIORS

SP.316. If $m \in \mathbb{N}$, s - is semiperimeter of ΔABC, then:

$$\sqrt{\left(\frac{a}{s-a}\right)^{m+1}} + \sqrt{\left(\frac{b}{s-b}\right)^{m+1}} + \sqrt{\left(\frac{c}{s-c}\right)^{m+1}} + 3m \ge 3(m+1)\sqrt{2}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

SP.317. If $a, b, c, d, e \in \mathbb{R}_+^* = (0, \infty)$ and $a^2+b^2+c^2+d^2 = e^2$, then:

$$(a+c)(b+d) \le e^2$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

SP.318. If $x, y \in \mathbb{R}_+^* = (0, \infty)$ and in ΔABC - are the lengths of the sides, h_a, h_b, h_c - are the lengths of the altitudes, then:

$$\frac{(2x-y)xa}{h_a} + \frac{(2y-x)yb}{h_b} + \frac{xyc}{h_c} \ge 2\sqrt{3}xy$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru - Romania

SP.319. If $(H_n)_{n\ge1}$, $H_n = \sum_{k=1}^{n} \frac{1}{k}$, then find:

$$\lim_{n\to\infty} e^{-H_n} \cdot \sum_{k=1}^{n} \frac{e^{H_k}}{\sqrt[k]{k!}}$$

Proposed by D.M. Bătineţu-Giurgiu - Romania

SP.320. If $x \in \mathbb{R}_+^* = (0,\infty)$ and in $\Delta ABC, a, b, c$ -are lengths of the sides, h_a, h_b, h_c - are lengths of the altitudes, then:

$$\frac{(6x-1)a}{h_a} + \frac{(\frac{2}{3x}-1)b}{h_b} + \frac{c}{h_c} \ge 2\sqrt{3}$$

Proposed by D.M. Bătineţu-Giurgiu - Romania

SP.321. Let a, b, c be the lengths of the sides of a triangle with circumradius R and iradius r. Prove that:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} - 4\left(\frac{a^2+b^2}{b^2+c^2} + \frac{b^2+c^2}{c^2+a^2} + \frac{c^2+a^2}{a^2+b^2}\right) + 12\left(\frac{R}{2r}\right)^2 \ge 3$$

Proposed by George Apostolopoulos-Greece

SP.322. Let a, b, c be the lengths of the sides of a triangle with circumradius R and iradius r. Prove that:

$$\frac{2r}{R} \le \frac{a^2}{b^2+bc+c^2} + \frac{b^2}{c^2+ca+a^2} + \frac{c^2}{a^2+ab+b^2} \le \frac{R^2}{2r^2} - 1$$

Proposed by George Apostolopoulos-Greece

SP.323. Let be $z_A, z_B, z_C \in \mathbb{C}^*$ different in pairs such that $|z_A| = |z_B| = |z_C| = 1$. If $|z_A - z_B - z_C| + |z_B - z_C - z_A| +$ $+ |z_C - z_A - z_B| = 6$, then ΔABC is an equilateral triangle.

Proposed by Marian Ursărescu - Romania

SP.324. Find all functions $f : (0,\infty) \to \mathbb{R}$ such that:

$$f(xy) \le xf(x) + yf(y) \le \log(xy), \forall x, y > 0$$

Proposed by Marian Ursărescu - Romania

SP.325. If $A, B \in M_2(\mathbb{C})$ are such that:
$\det[(I_2 - B)A + (A - I_2)B] = \det(A - B)$, then find:
$\Omega = (AB - BA)^n, n \in \mathbb{N}^*$

Proposed by Florică Anastase - Romania

SP.326. Let x, y, z be positive real numbers such that $xyz = 1$. Find the minimum value of:

$$P = \frac{x^3}{(2y^2-yz+2z^2)^2} + \frac{y^3}{(2z^2-zx+2x^2)^2} + \frac{z^3}{(2x^2-xy+2y^2)^2} + \frac{xy+yz+zx}{3}$$

Proposed by Hoang Le Nhat Tung - Vietnam

SP.327. Let x, y, z be positive real numbers such that $ab + bc + ca = 3$. Find the minimum value of expression:

$$P = \frac{1}{(a+b)^5} + \frac{1}{(b+c)^5} + \frac{1}{(c+a)^5}$$

Proposed by Hoang Le Nhat Tung - Vietnam

SP.328. Let $a, b, c \in [1, 3]$ and such that $a + b + c = 6$. Find the maximum value of the expression:

$$P = a^6 + b^6 + c^6$$

Proposed by Hoang Le Nhat Tung - Vietnam

SP.329. Find:

$$\lim_{n\to\infty} \frac{e^{\sum_{k=1}^{n} \frac{(-1)^k \binom{n}{k}}{k}}}{\sqrt[n]{n!}}$$

Proposed by Marian Ursărescu - Romania

SP.330. Let ABC be a triangle with iradius r and circumradius R. Prove that:

$$\frac{48r}{R} \le \frac{(\sec\frac{A}{2} + \sec\frac{B}{2} + \sec\frac{C}{2})^3}{\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}} \le \frac{12R}{r}$$

Proposed by George Apostolopoulos-Greece

UNDERGRADUATE PROBLEMS

UP.316. If $(H_n)_{n\ge1}, H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the armonic sequence, find:

$$\lim_{n\to\infty} e^{-2H_n} \cdot \sum_{k=2}^{n} \sqrt[k]{(2k-1)!!}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.317. If $a, b \in \mathbb{R}$, find:

$$\lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)^a \cdot ((2n+1)!!)^b} - \sqrt[n]{n^a \cdot ((2n-1)!!)^b} \right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.318. Find:

$$\lim_{n\to\infty}\sqrt{n}\cdot\left(\frac{n+1}{\sqrt[2n+2]{(n+1)!}}-\frac{n}{\sqrt[2n]{n!}}\right)$$

Proposed by D.M. Bătineţu-Giurgiu - Romania

UP.319. If $(H_n)_{n\geq 1}, H_n=\sum_{k=1}^{n}\frac{1}{k}, (a_n)_{n\geq 1}$ is sequence of real numbers strictly positive such that:

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}=a\in\mathbb{R}_+^*=(0,\infty),$$

then find:

$$\lim_{n\to\infty}e^{-3H_n}\cdot\sum_{k=2}^{n}\sqrt[k]{a_k}$$

Proposed by D.M. Bătineţu-Giurgiu - Romania

UP.320. If $a,b,c\in\mathbb{R}, x_n=n!, y_n=(2n-1)!!, \forall n\in\mathbb{N}^*$, then find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)^a\cdot x_{n+1}^b\cdot y_{n+1}^c}-\sqrt[n]{n^a\cdot x_n^b\cdot y_n^c}\right)$$

Proposed by D.M. Bătineţu-Giurgiu - Romania

UP.321. Let $A_0A_1\ldots A_n$ be an Euclidean n-simplex. We will use the following notations:
- O,V,R,r the centre if its circumscribed hypersphere, its volume, its circumradius and its inradius, respectively.
-O_i,R_i the centre and the radius of the hypersphere tangent to the circumscribed sphere of $A_0,A_1,\ldots A_n$ in the vertex A_i and to the hyperplane $A_0,A_1\ldots A_{i-1}A_{i+1}\ldots A_n$ simultaneously. With the above notations, the following identity holds:

$$\sum_{i=0}^{n}\frac{1}{R_i}=\frac{n}{R}+\frac{1}{r}$$

Proposed by Vasile Jiglău - Romania

UP.322. Find:

$$\lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}k\left(\binom{k}{k}+\binom{k+1}{k}+\ldots+\binom{n}{k}\right)\binom{n}{k}}$$

Proposed by Marian Ursărescu - Romania

UP.323. If $S_n=\sum_{k=1}^{n}\log(\cos\frac{\pi}{2^{k+2}})$, then find:

$$\Omega=\lim_{n\to\infty}(\sqrt[n]{n\cdot S_n})^{\sum_{k=3}^{n}\tan\frac{\pi}{k}}$$

Proposed by Florică Anastase - Romania

UP.324. For $n \in \mathbb{N}^*, n \geq 2, P_n = \prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$, find:

$$\Omega = \lim_{n\to\infty} \frac{n}{2} \cdot P_n \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos 3x}{\sin^n x} dx$$

Proposed by Florică Anastase - Romania

UP.325. Let be $(a_n)_{n\geq 1}, (f_n(x))_{n>1}; n \in \mathbb{N}, n \geq 7, x > 1$

$$a_n = \left(\prod_{k=1}^{n} \binom{n}{k}\right)^2, f_n(x) = \int_x^{x^2} \frac{1}{\log \sqrt[n]{t}} dt$$

Then find:

$$\Omega_1 = \lim_{x\to\infty} f_n(x) \text{ and } \Omega_2 = \lim_{n\to\infty} \left(\frac{1}{a_n} \lim_{x\to 1} f_n(x)\right)$$

Proposed by Florică Anastase - Romania

UP.326. Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{n}{\log n}\right)^e \cdot e^{\int_0^e \log\left(\frac{\log(x+e)}{x^2+ne}\right) dx}$$

Proposed by Florică Anastase - Romania

UP.327. If $(x_n)_{n\geq 1}, x_n \in \mathbb{R}_+^*, \forall n \in \mathbb{N}^*$ satisfy $\lim_{n\to\infty}(x_{n+1} - x_n) = x \in \mathbb{R}_+^*$, then compute:

$$\lim_{n\to\infty} (x_{n+1} \sqrt[n+1]{n+1} - x_n \sqrt[n]{n})$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.328. Let $\{\gamma_n\}_{n\geq 1}, \gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$, with $\lim_{n\to\infty} \gamma_n = \gamma$ (γ is Euler - Mascheroni constant), then find:

$$\lim_{n\to\infty} (\sin \gamma_n - \sin \gamma) \sqrt[n]{n!}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.329. If $(a_n)_{n\geq 1}$ is a sequence of real positive numbers such that:

$$\lim_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} = a \in \mathbb{R}_+^*,$$

then find:

$$\lim_{n\to\infty} \left(\sqrt[n+1]{\frac{a_{n+1} F_{n+1}}{(2n+1)!!}} - \sqrt[n]{\frac{a_n F_n}{(2n-1)!!}}\right)$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.330. For $n \in \mathbb{N}, n \geq 2, F_n$ - Fibonacci numbers, prove that:

$$\frac{F_1}{3(F_1^2+F_2^2)^2}+\frac{F_2}{4(F_1^2+F_2^2+F_3^2)^2}+\cdots+\frac{F_n}{(n+2)(F_1^2+F_2^2+\ldots+F_{n+1}^2)^2} \geq$$

$$\geq \frac{(F_{n+2}-F_1)^2}{F_{n+2}^2(nF_{n+2}+F_n)}$$

Proposed by Florică Anastase - Romania

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