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SOME APPLICATIONS ABOUT COMPLEX NUMBERS IN GEOMETRY

By Marian Ursărescu-Romania

Let be $z \in \mathbb{C}, z = a + bi, M(z)$ –corresponding point.

1. $A(z_A), B(z_B) \Rightarrow AB = |Z_B - z_A|$.

2. If $z \in \mathbb{C}, |z| = r$ **then the point** $M(z) \in C(O, r)$.

3. If M **–is the middle point of** AB**, then** $z_M = \frac{z_A + z_B}{2}$.

4. If G **– is the centroid of the triangle** ABC**, then** $z_G = \frac{z_A + z_B + z_C}{3}$.

5. If H **– is the orthocentre of the triangle** ABC**, then** $z_H = z_A + z_B + z_c$.

6. If I **– is the inradii of the triangle** ABC**, then** $z_I = \frac{az_A + bz_B + cz_C}{a + b + c}$.

7. If K **–is the Lemoine's point, then** $z_K = \frac{a^2 z_A + b^2 z_B + c^2 z_C}{a^2 + b^2 + c^2}$.

8. If Ω **–is the center of the Euler's circle, then** $z_\Omega = \frac{z_1 + z_2 + z_3}{3}$.

9. If $\Delta ABC \subset C(0,1)$**, then** $\begin{cases} \sin^2 A = -\frac{(z_B - z_C)^2}{4z_B z_C} \\ \cos^2 A = \frac{(z_B + z_C)^2}{4z_B z_C} \end{cases}$.

Applications:

1. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$**, different in pairs such that** $|z_1| = |z_2| = |z_3| = 1$.

If $\frac{1}{2 + |z_1 + z_2|} + \frac{1}{2 + |z_2 + z_3|} + \frac{1}{2 + |z_3 + z_1|} = 1$, $A(z_1), B(z_2), C(z_3)$ **then** $AB = BC = CA$.

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Solution: Let be $A(z_1), B(z_2), C(z_3), \Delta ABC \subset C(0,1)$

$$|z_1 + z_2| = |z_1 + z_2 + z_3 - z_3| = |z_H - z_C| = HC \text{ then } \frac{1}{2 + AH} + \frac{1}{2 + BH} + \frac{1}{2 + CH} = 1; (1)$$

$$\text{But } AH = 2R\cos A = 2\cos A, (R = 1); (2)$$

$$\text{From } (1), (2) \text{ we have: } \frac{1}{2 + 2\cos A} + \frac{1}{2 + 2\cos B} + \frac{1}{2 + 2\cos C} = 1 \Leftrightarrow \frac{1}{1 + \cos A} + \frac{1}{1 + \cos B} + \frac{1}{1 + \cos C} = 2 \Leftrightarrow$$

$$\frac{1}{2\cos^2 \frac{A}{2}} + \frac{1}{2\cos^2 \frac{B}{2}} + \frac{1}{2\cos^2 \frac{C}{2}} = 2 \Leftrightarrow \frac{1}{\cos^2 \frac{A}{2}} + \frac{1}{\cos^2 \frac{B}{2}} + \frac{1}{\cos^2 \frac{C}{2}} = 4; (3)$$

But $\frac{1}{cos^2\frac{A}{2}}+\frac{1}{cos^2\frac{B}{2}}+\frac{1}{cos^2\frac{C}{2}}=1+\left(\frac{4R+r}{s}\right)^2$; (4)

From $(3),(4)$ we have $\left(\frac{4R+r}{s}\right)^2=3\Leftrightarrow(4R+r)^2=3s^2$ and with $(4R+r)^2\geq 3s^2$ (Doucet Inequality) we obtain equality when the triangle ABC is equilateral.

2. Let be $z_1,z_2,z_3\in\mathbb{C}^*$ different in pairs $|z_1|=|z_2|=|z_3|=1$.

If $\left|z_1^2-(z_2-z_3)^2\right|+\left|z_2^2-(z_3-z_1)^2\right|+\left|z_3^2-(z_1-z_2)^2\right|=12$,

$A(z_1),B(z_2),C(z_3)$ then $AB=BC=CA$.

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Solution: Let be $A(z_1),B(z_2),C(z_3),\Delta ABC\subset C(0,1)$. We have:

$$|z_1^2-(z_2-z_3)^2|=|z_1+z_2-z_3|\cdot|z_1-z_2+z_3|;(1)$$

$$|z_1+z_2-z_3|=|z_3-z_1-z_2|=|2z_3-z_1-z_2-z_3|=2\left|z_3-\frac{z_1+z_2+z_3}{2}\right|=2|z_3-z_\Omega|$$
$$=2C\Omega;(2)$$

From $(1),(2)$ we have: $|z_1^2-(z_2-z_3)^2|=4CH\cdot BH$ and analogs.

The hypothesis of the problem, becomes: $A\Omega\cdot B\Omega+B\Omega\cdot C\Omega+C\Omega\cdot A\Omega=3;(3)$

From medians theorem, we have: $A\Omega^2=\frac{2(AO^2+AH^2)-OH^2}{4}=\frac{2(R^2+4R^2-a^2)-9R^2+a^2+b^2+c^2}{4}$ then

$$A\Omega^2=\frac{4b^2+c^2-a^2}{4};(4)$$

From (4) we get: $A\Omega^2+B\Omega^2+C\Omega^2=\frac{3+a^2+b^2+c^2}{4};(5)$.

But $a^2+b^2+c^2\leq 9R^2=9,(R=1)$ with equality if the triangle ABC is equilateral (6).

From $(5),(6)$ we have: $A\Omega^2+B\Omega^2+C\Omega^2\leq 3;(7)$.

But $A\Omega\cdot B\Omega+B\Omega\cdot C\Omega+C\Omega\cdot A\Omega\leq A\Omega^2+B\Omega^2+C\Omega^2;(8)$

From $(3),(7),(8)$ the inequality holds with equality when the triangle ABC is equilateral.

3. Let be $z_A,z_B,z_C\in\mathbb{C}^*$ different in pairs, $|z_A|=|z_B|=|z_C|$.

If $z_Bz_C(b+c)=z_Cz_A(c+a)+z_Az_B(a+b)=0$, $A(z_1),B(z_2),C(z_3)$ then

$AB=BC=CA$.

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Solution: Let $|z_A| = |z_B| = |z_C| = R$.

$$z_B z_C(b+c) = z_C z_A(c+a) + z_A z_B(a+b) = 0 \Leftrightarrow \frac{b+c}{z_A} + \frac{c+a}{z_B} + \frac{a+b}{z_C} = 0 \Leftrightarrow$$

$$(b+c)\frac{\overline{z_A}}{R^2} + (c+a)\frac{\overline{z_B}}{R^2} + (a+b)\frac{\overline{z_C}}{R^2} = 0 \Leftrightarrow (b+c)\overline{z_A} + (c+a)\overline{z_B} + (a+b)\overline{z_C} = 0 \Leftrightarrow$$

$$(b+c)z_A + (c+a)z_B + (a+b)z_C = 0 \Leftrightarrow (a+b+c)(z_A+z_B+z_C) = az_A + bz_B + cz_C$$

$$\Leftrightarrow z_A + z_B + z_C = \frac{az_A+bz_B+cz_C}{a+b+c} \Leftrightarrow z_H = z_I \Leftrightarrow H = I \Leftrightarrow \Delta ABC \text{ is equilateral.}$$

4. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs $|z_1| = |z_2| = |z_3| = 1$.

If $\frac{z_1(z_2+z_3)^2}{|z_2+z_3|} + \frac{z_2(z_3+z_1)^2}{|z_3+z_1|} + \frac{z_3(z_1+z_2)^2}{|z_1+z_2|} = 3z_1z_2z_3$, $A(z_1), B(z_2), C(z_3)$ then:

$$AB = BC = CA.$$

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Solution: Let be $A(z_1), B(z_2), C(z_3), \Delta ABC \subset C(0,1)$. We have:

$$\frac{(z_2+z_3)^2}{4z_2z_3|z_2+z_3|} + \frac{(z_1+z_3)^2}{4z_1z_3|z_1+z_3|} + \frac{(z_1+z_2)^2}{4z_1z_2|z_1+z_2|} = \frac{3}{4}; (1)$$

$$|z_2+z_3| = |z_1+z_2+z_3-z_1| = |z_H - z_A| = AH = 2R\cos A = 2\cos A; (2) \text{ and}$$

$$\frac{(z_2+z_3)^2}{4z_2z_3} = \cos^2 A; (3).$$

From (1), (2), (3) we have: $\sum \frac{\cos^2 A}{2\cos A} = \frac{3}{4} \Leftrightarrow \sum \cos A = \frac{3}{2}; (6)$.

But in any triangle $ABC, \sum \cos A \le \frac{3}{2}$ with equality when the triangle ABC is equilateral.

5. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs $|z_1| = |z_2| = |z_3|$.

If $\sum z_2z_3(|z_1-z_2|^2 + |z_1-z_3|^2) = 0$, $A(z_1), B(z_2), C(z_3)$ then $AB = BC = CA$.

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Solution: Let $|z_1| = |z_2| = |z_3| = R$ then $A(z_1), B(z_2), C(z_3), \Delta ABC \subset C(0,R)$,

$|z_1 - z_2| = AB = c$ and analogs.

The hypothesis, becomes: $\sum z_2z_3(b^2+c^2)=0 \Leftrightarrow \sum \frac{b^2+c^2}{z_1}=0;(1)$

But $|z_1|^2=R^2 \Leftrightarrow z_1\cdot\overline{z_1}=R^2;(2)$.

From $(1),(2) \Rightarrow \sum(b^2+c^2)\overline{z_1}=0 \Leftrightarrow \sum(b^2+c^2)z_1=0 \Leftrightarrow \sum(a^2+b^2+c^2-a^2)z_1=0$

$\Leftrightarrow (a^2+b^2+c^2)(z_1+z_2+z_3)=\frac{a^2z_1+b^2z_2+c^2z_3}{a^2+b^2+c^2} \Leftrightarrow z_H=z_K \Leftrightarrow H=K \Leftrightarrow \Delta ABC$ is equilateral.

6. Let be $z_1,z_2,z_3\in\mathbb{C}^*$ different in pairs $|z_1|=|z_2|=|z_3|=1$.

If $\sum\frac{z_1(z_2-z_3)^2}{|z_1-z_2|\cdot|z_1-z_3|}+3z_1z_2z_3=0$, $A(z_1),B(z_2),C(z_3)$ then $AB=BC=CA$.

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Solution: Let be $A(z_1),B(z_2),C(z_3),\Delta ABC\subset C(0,1)$. We have:

$$\sum\frac{|z_2-z_3|(z_2-z_3)^2}{z_2z_3}+3|z_1-z_2|\cdot|z_1-z_3|\cdot|z_2-z_3|=0;(1).$$

$|z_2-z_3|=b;(2)$ and $sin^2A=-\frac{(z_2-z_3)^2}{4z_2z_3}$; (3)

From $(1),(2),(3)$ we have: $-\sum asin^2A+\frac{3}{4}abc=0, abc=4RS=4S,(R=1)$ then

$\sum asin^2A=3S;(4)$

We have: $\sum asin^2A=\frac{s(s^2-3r^2-6Rr)}{2R^2};(5)$.

From $(4),(5)$ we have: $\frac{s(s^2-3r^2-6Rr)}{2R^2}=4S;(6)$.

From Carlitz Inequality we have: $s^2\geq 12Rr+3r^2;(7)$.

From $(5),(7)$ we get: $\sum asin^2A\geq s\cdot\frac{6Rr}{2R^2}=3sr=3S;(8)$.

From $(6),(8)$ we get: ΔABC is equilateral.

Proposed problems:

1. Let be $z_1,z_2,z_3\in\mathbb{C}^*$ different in pairs $|z_1|=|z_2|=|z_3|=1$.

If $|(z_1-z_2-z_3)(z_2-z_1-z_3)(z_3-z_1-z_2|=8$,

$A(z_1),B(z_2),C(z_3)$ then $AB=BC=CA$.

2. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs $|z_1| = |z_2| = |z_3| = 1$.

If $|z_1 - z_2 - z_3| + |z_2 - z_1 - z_3| + |z_3 - z_1 - z_2| = 6$, $A(z_1), B(z_2), C(z_3)$ then $AB = BC = CA$.

3. Let be $z_A, z_B, z_C \in \mathbb{C}^*$ different in pairs $|z_A| = |z_B| = |z_C|$.

If $z_B z_C(b + c) + z_C z_A(c + a) + z_A z_B(a + b) = 2(a z_B z_C + b z_C z_A + c z_A z_B)$,

then the triangle ABC is equilateral.

4. Let be $z_A, z_B, z_C \in \mathbb{C}^*$, different in pairs $|z_A| = |z_B| = |z_C|$.

If $\prod(b(z_A - z_B) + c(z_A - z_C) = (a + b + c)^3$, then $AB = BC = CA$.

5. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs $|z_1| = |z_2| = |z_3| = 1$.

If $z_1(z_2 + z_3)^2|z_2 + z_1|^2 + z_2(z_1 + z_3)^2|z_1 + z_3|^2 + z_3(z_1 + z_2)^2|z_1 + z_2|^2 = 3z_1z_2z_3$,

$A(z_1), B(z_2), C(z_3)$ then $AB = BC = CA$.

6. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs $|z_1| = |z_2| = |z_3| = 1$.

If $\frac{z_1(z_2-z_3)^2}{|z_2-z_3|} + \frac{z_2(z_1-z_3)^2}{|z_1-z_3|} + \frac{z_3(z_1-z_2)^2}{|z_1-z_2|} + 3\sqrt{3}z_1z_2z_3 = 0$,

$A(z_1), B(z_2), C(z_3)$ then $AB = BC = CA$.

7. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs $|z_1| = |z_2| = |z_3| = 1$.

If $\frac{z_1(z_2+z_3)^2}{|z_2+z_3|} + \frac{z_2(z_3+z_1)^2}{|z_3+z_1|} + \frac{z_3(z_1+z_2)^2}{|z_1+z_2|} = z_1z_2z_3$,

$A(z_1), B(z_2), C(z_3)$ then $AB = BC = CA$.

8. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs $|z_1| = |z_2| = |z_3| = 1$.

If $\frac{z_1}{(z_2-z_3)^2} + \frac{z_2}{(z_1-z_3)^2} + \frac{z_3}{(z_1-z_2)^2} + \frac{1}{z_1z_2z_3} = 0$,

$A(z_1), B(z_2), C(z_3)$ then $AB = BC = CA$.

9. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ **different in pairs** $|z_1| = |z_2| = |z_3| = 1$.

$$\text{If } \sum \frac{z_1 z_2}{14 z_1 z_2 - z_1^2 - z_2^2} = \frac{1}{5}, A(z_1), B(z_2), C(z_3) \text{ then } AB = BC = CA.$$

10. be $z_1, z_2, z_3 \in \mathbb{C}^*$ **different in pairs different in pairs** $|z_1| = |z_2| = |z_3| = 1$.

$$\text{If } \sum \frac{z_1 (z_2 - z_3)^2}{|z_2 - z_3|} - 3\sqrt{3} z_1 z_2 z_3 = 0, A(z_1), B(z_2), C(z_3) \text{ then } AB = BC = CA.$$

Note: ***All the problems are published by Marian Ursărescu in Romanian Mathematical Magazine www.ssmrmh.ro***