SOME MATRIX RESULTS

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Abstract. In this paper we present some certain results on matrices.

Theorem 1.
If $x \in \mathbb{R}$ and $A(x) = \begin{pmatrix} x+1 & 1 & 1 & 1 \\ 1 & x+1 & 1 & 1 \\ 1 & 1 & x+1 & 1 \\ 1 & 1 & 1 & x+1 \end{pmatrix}$, then:
$A(0) \cdot A(1) \cdot A(2) \cdot A(3) = 210 \cdot A(0)$.

Proof. We have: $A(0) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = E, A(x) = E + x \cdot I_4, E^2 = 4E$, where
$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$A(x) \cdot A(y) = (E + xI_4)(E + yI_4) = E^2 + (x+y)E + xyI_4,$$
$$A(0) \cdot A(1) = 5E, A(2) \cdot A(3) = 9E + 6I_4.$$

Hence,
$A(0)\cdot A(1)\cdot A(2)\cdot A(3) = 5E(9E+6I_4) = 45E^2+30E = 45\cdot 4E+30E = 180E+30E = 210E$

□

Theorem 2.
If $x \in \mathbb{R}$ and $A(x) = \begin{pmatrix} x+1 & 1 & 1 & 1 \\ 1 & x+1 & 1 & 1 \\ 1 & 1 & x+1 & 1 \\ 1 & 1 & 1 & x+1 \end{pmatrix}$, then:

$$A(0) \cdot A(x) \cdot A(y) \cdot A(z) = (x+4)(4(y+z+4)+yz) \cdot A(0), \forall x, y, z \in \mathbb{R}.$$

Proof. We have:
$A(0) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = E, A(x) = E+x\cdot I_4, E^2 = 4E$, where $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$A(0) \cdot A(x) = (x+4)E \text{ and}$$
$$A(y) \cdot A(z) = (E + yI_4)(E + zI_4) = E^2 + (y+z)E + yzI_4 = (y+z+4)E + yzI_4.$$

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Hence,

$$A(0)\cdot A(x)\cdot A(y)\cdot A(z) = (x+4)E((y+z+4)E+yzI_4) = (x+4)(y+z+4)E^2+(x+4)yzE =$$
$$= (x+4)(4(y+z+4)+yz)E$$

□

Theorem 3.

If $x \in \mathbb{R}$ and $A(x) = \begin{pmatrix} x+1 & 1 & \cdots & 1 & 1 \\ 1 & x+1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & x+1 & 1 \\ 1 & 1 & \cdots & 1 & x+1 \end{pmatrix} \in M_n(\mathbb{R})$, then:

$$A(0)\cdot A(1)\cdot A(2)\cdot A(3) = (n+1)(n^2+11n+6)\cdot A(0).$$

Proof.

$$A(0) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} = E \in M_n(\mathbb{R}), \text{ and } A(x) = E + x\cdot I_n, \text{ where}$$

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in M_n(\mathbb{R}).$$

$A(x)\cdot A(y) = (E+x\cdot I_n)(E+y\cdot I_n) = E^2+(x+y)E+xyI_n$, and because $E^2 = nE$, yields that

$$A(x)\cdot A(y) = nE+(x+y)E+xyI_n = (x+y+n)E+xyI_n. \text{ So}$$
$$A(0)\cdot A(1) = (n+1)E \text{ and } A(2)\cdot A(3) = (n+5)E+6I_n.$$

Hence,

$$A(0)\cdot A(1)\cdot A(2)\cdot A(3) = (n+1)E((n+5)E+6I_n) = (n+1)(n+5)E^5+6(n+1)E =$$
$$= n(n+1)(n+5)E+6(n+1)E = (n+1)(n^2+11n+6)E$$

□

Theorem 4.

If $x \in \mathbb{R}$ and $A(x) = \begin{pmatrix} x+1 & 1 & \cdots & 1 & 1 \\ 1 & x+1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & x+1 & 1 \\ 1 & 1 & \cdots & 1 & x+1 \end{pmatrix} \in M_n(\mathbb{R})$, then:

$$A(0)\cdot A(x)\cdot A(y)\cdot A(z) = (x+n)((y+z+n)n+yz)\cdot A(0), \forall x,y,z \in \mathbb{R}.$$

Proof. We have:

$$A(0) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} = E \in M_n(\mathbb{R}), A(x) = E+x\cdot I_n, E^2 = nE, \text{ where}$$

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$A(0) \cdot A(x) = E(E + x \cdot I_n) = E^2 + xE + nE + xE = (x+n)E, \text{ and}$$

$$A(y) \cdot A(z) = (E + yI_n)(E + zI_n) = E^2 + (y+z)E + yzI_n = (y+z+n)E + yzI_n.$$

Hence,

$$A(0) \cdot A(x) \cdot A(y) \cdot A(z) = (x+n)E((y+z+n)E + yzI_n) = (x+n)(y+z+n)E^2 + (x+n)yzE =$$
$$= (x+n)((y+z+n)n + yz)E$$

□

Theorem 5.
If $A \in M_n(\mathbb{R})$ such that $A^2 = O_n \in M_n(\mathbb{R})$, and let $x, y \in \mathbb{R}$ such that $4y \geq x^2$, then $\det(xA + yI_n) \geq 0$.

Proof. We have:

$$xA + yI_n = A^2 + xA + yI_n^2 = \left(A + \frac{x}{2}I_n\right)^2 + yI_n^2 - \frac{x^2}{4}I_n^2 =$$
$$= \left(A + \frac{x}{2}I_n\right)^2 + \frac{4y - x^2}{4}I_n^2 = \left(A + \frac{x}{2}I_n\right)^2 + \left(\frac{\sqrt{4y - x^2}}{2}I_n\right)^2 = U^2 + V^2, \text{ where}$$
$$U = A + \frac{x}{2}I_n, V = \frac{\sqrt{4y - x^2}}{2}I_n.$$

We have:

$$UV = \left(A + \frac{x}{2}I_n\right)\left(\frac{\sqrt{4y - x^2}}{2}I_n\right) = \frac{\sqrt{4y - x^2}}{2}A + \frac{x}{4}\sqrt{4y - x^2}I_n^2 = VU$$

Therefore,

$$\det(xA + yI_n) = \det(U^2 + V^2) = \det((U + iV)(U - iV)) =$$
$$= \det((U + iV)(\overline{U + iV})) = \det(U + iV)\det(\overline{U + iV}) =$$
$$= \det(U + iV) \cdot \overline{\det U + iV} = |\det(U + iV)|^2 \geq 0$$

□

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