A GEOMETRICAL APPROACH OF THE CONSTANTS π, e, φ

Mathematical constant $\pi = 3.1415926535...$
Euler's number $e = 2.7182818284...$
Golden ratio $\varphi = \frac{\sqrt{5}+1}{2} = 1.6180339887...$

Gakopoulos Athanasios

Contents

Introduction

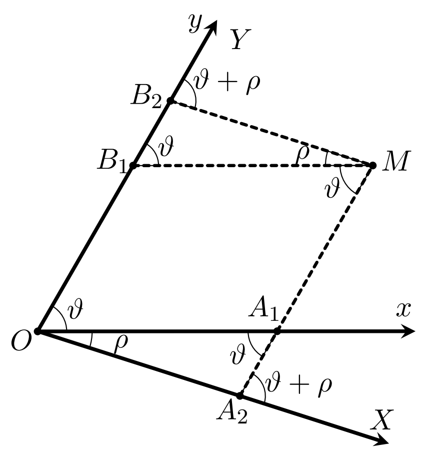
During my research ”Plagiogonal System” it occured to me the idea of calculating the transformation equations of a point’s coordinates from one initial system to another that happens from rotation of one system or another or both of them.

Applications of this idea are the approach of the mathematical constant π , e , ϕ with one single geometrical shape; characteristic of which does not contain circle but ratio of areas of parallelograms. What’s impressive is that the angles of the parallelogram are 30°, 45°, 60° and the approach of the constants is done in such way that the ratio of approach is of order 1.00 to 1.000.

Chapter 1

Plagiogonal canonical system

1.1. Transformation Equations of coordinates of point into plagiogonal canonical system $(x\hat{\mathrm{O}}y, \theta)$ rotating $\mathrm{O}x$ with an angle ρ



$(x\hat{\mathrm{O}}y, \theta)\ :\ \mathrm{M}(x,y)$
$(x\hat{\mathrm{O}}y, \theta+\rho):\ \mathrm{M}(X,Y)$

$\mathrm{OA}_1 = m_1 \quad \mathrm{OA}_2 = M_1$
$\mathrm{OB}_1 = m_2 \quad \mathrm{OB}_2 = M_2$

- $\overset{\triangle}{\mathrm{MB_1B_2}} : \dfrac{\mathrm{B_1B_2}}{\sin\rho} = \dfrac{\mathrm{MB_1}}{\sin(\theta+\rho)} \Rightarrow \dfrac{\mathrm{B_1B_2}}{\sin\rho} = \dfrac{m_1}{\sin(\theta+\rho)}$

$$\Rightarrow \mathrm{B_1B_2} = m_1 \cdot \frac{\sin\rho}{\sin(\theta+\rho)}$$

$$\mathrm{OB_2} = \mathrm{OB_1} + \mathrm{B_1B_2} \Rightarrow M_2 = m_2 + m_1 \cdot \frac{\sin\rho}{\sin(\theta+\rho)}$$

$$\Rightarrow Y = y + x \cdot \frac{\sin\rho}{\sin(\theta+\rho)}$$

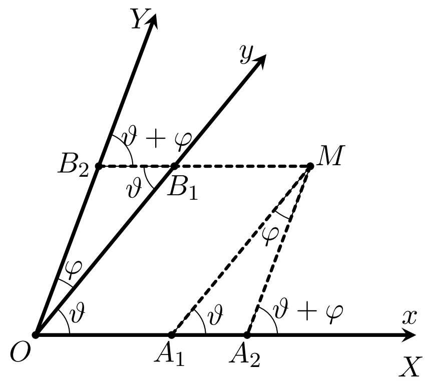
- $\overset{\triangle}{\mathrm{OA_1A_2}} : \dfrac{\mathrm{OA_2}}{\sin\theta} = \dfrac{\mathrm{OA_1}}{\sin(\theta+\rho)} \Rightarrow \dfrac{M_1}{\sin\theta} = \dfrac{m_1}{\sin(\theta+\rho)}$

$$\Rightarrow M_1 = m_1 \cdot \frac{\sin\theta}{\sin(\theta+\rho)}$$

$$\Rightarrow X = x \cdot \frac{\sin\theta}{\sin(\theta+\rho)}$$

Hence $\boxed{\mathbf{X} = \mathbf{x} \cdot \dfrac{\sin\theta}{\sin(\theta+\rho)}}$ and $\boxed{\mathbf{Y} = \mathbf{y} + \mathbf{x} \cdot \dfrac{\sin\rho}{\sin(\theta+\rho)}}$.

1.2. Transformation Equations of coordinates of point into plagiogonal canonical system $(x\hat{\mathrm{O}}y, \theta)$ rotating $\mathrm{O}y$ with an angle φ



$(x\hat{\mathrm{O}}y, \theta)$: $\mathrm{M}(x, y)$
$(x\hat{\mathrm{O}}y, \theta + \phi)$: $\mathrm{M}(X, Y)$

$\mathrm{OA}_1 = m_1 \quad \mathrm{OA}_2 = M_1$
$\mathrm{OB}_1 = m_2 \quad \mathrm{OB}_2 = M_2$

- $\overset{\triangle}{\mathrm{MA}_1\mathrm{A}_2} : \dfrac{\mathrm{A}_1\mathrm{A}_2}{\sin\phi} = \dfrac{\mathrm{MA}_1}{\sin(\theta+\phi)} \Rightarrow \dfrac{\mathrm{A}_1\mathrm{A}_2}{\sin\phi} = \dfrac{m_2}{\sin(\theta+\phi)}$

$$\Rightarrow \mathrm{A}_1\mathrm{A}_2 = m_2 \cdot \frac{\sin\phi}{\sin(\theta+\phi)}$$

$$\mathrm{OA}_2 = \mathrm{OA}_1 + \mathrm{A}_1\mathrm{A}_2 \Rightarrow M_1 = m_1 + m_2 \cdot \frac{\sin\phi}{\sin(\theta+\phi)}$$

$$\Rightarrow X = x + y \cdot \frac{\sin\phi}{\sin(\theta+\phi)}$$

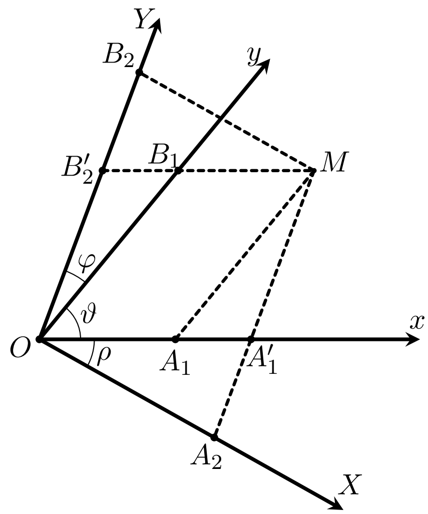
- $\overset{\triangle}{\mathrm{OB}_1\mathrm{B}_2} : \dfrac{\mathrm{OB}_2}{\sin\theta} = \dfrac{\mathrm{OB}_1}{\sin(\theta+\phi)} \Rightarrow \dfrac{M_2}{\sin\theta} = \dfrac{m_2}{\sin(\theta+\phi)}$

$$\Rightarrow M_2 = m_2 \cdot \frac{\sin\theta}{\sin(\theta+\phi)}$$

$$\Rightarrow Y = y \cdot \frac{\sin\theta}{\sin(\theta+\phi)}$$

Hence $\boxed{\mathbf{X} = \mathbf{x} + \mathbf{y} \cdot \dfrac{\sin\phi}{\sin(\theta+\phi)}}$ and $\boxed{\mathbf{Y} = \mathbf{y} \cdot \dfrac{\sin\theta}{\sin(\theta+\phi)}}$.

1.3. Transformation Equations of coordinates of point into plagiogonal canonical system $(x\hat{\mathrm{O}}y, \theta)$ rotating $\mathrm{O}x$ with an angle ρ and $\mathrm{O}y$ with an angle ϕ



$(x\hat{\mathrm{O}}y, \theta)$: $\mathrm{M}(x, y)$
$\left(\mathrm{X\hat{O}Y}, \theta + \phi + \rho\right)$: $\mathrm{M}(X, Y)$
$\mathrm{OA}_1 = m_1$, $\mathrm{OA'}_1 = m_1'$, $\mathrm{OA}_2 = M_1$
$\mathrm{OB}_1 = m_2$, $\mathrm{OB'}_2 = M_2'$, $\mathrm{OB}_2 = M_2$

Applyig successively the rotation of $\mathrm{O}y$ with an angle φ and $\mathrm{O}x$ with an angle ρ and using the equations of the paragraphs (1.1) , (1.2) we get:

$$m_1' = m_1 + m_2 \cdot \frac{\sin\varphi}{\sin(\theta+\phi)} \qquad M_2' = m_2 \cdot \frac{\sin\theta}{\sin(\theta+\phi)}$$

$$\begin{aligned} M_1 &= \left(m_1 + m_2 \cdot \frac{\sin\phi}{\sin(\theta+\phi)}\right) \cdot \frac{\sin(\theta+\phi)}{\sin(\theta+\phi+\rho)} \\ &= m_1 \cdot \frac{\sin(\theta+\phi)}{\sin(\theta+\phi+\rho)} + m_2 \cdot \frac{\sin\phi}{\sin(\theta+\phi+\rho)} \end{aligned}$$

Hence:

$$\mathbf{X} = \mathbf{x} \cdot \frac{\sin(\theta+\phi)}{\sin(\theta+\phi+\rho)} + \mathbf{y} \cdot \frac{\sin\phi}{\sin(\theta+\phi+\rho)}$$

Similarly , we get:

$$\mathbf{Y} = \mathbf{x} \cdot \frac{\sin\rho}{\sin(\theta+\phi+\rho)} + \mathbf{y} \cdot \frac{\sin(\theta+\rho)}{\sin(\theta+\phi+\rho)}$$

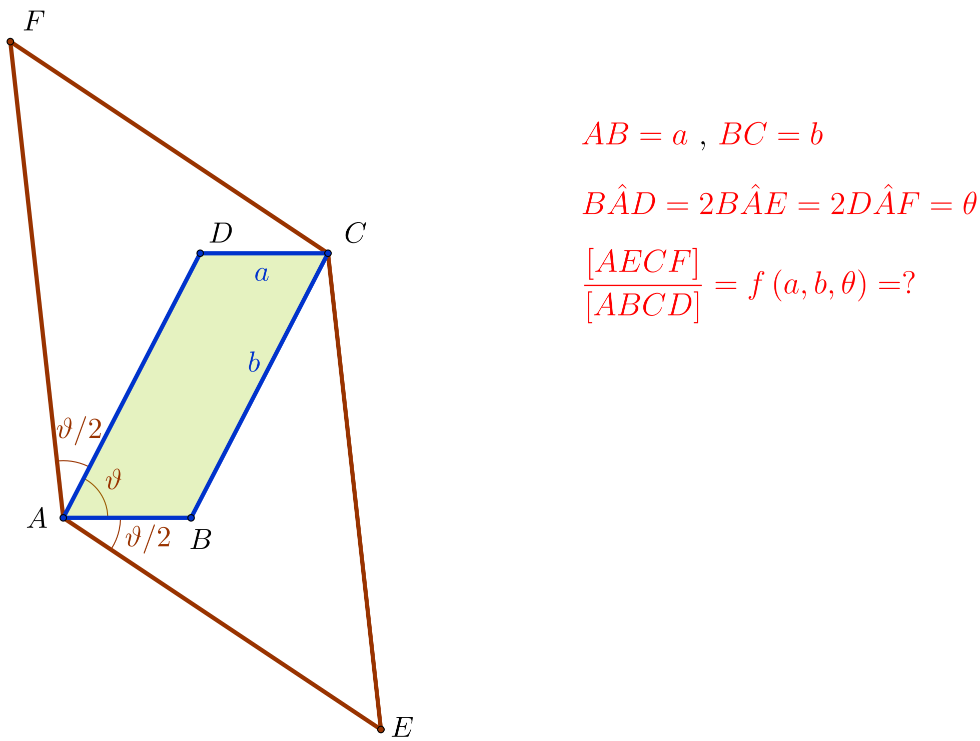
Hence:

$$\mathbf{X} = \frac{\mathbf{x} \cdot \sin(\theta+\phi) + \mathbf{y} \cdot \sin\phi}{\sin(\theta+\phi+\rho)} \quad , \quad \mathbf{Y} = \frac{\mathbf{x} \cdot \sin\rho + \mathbf{y} \cdot \sin(\theta+\rho)}{\sin(\theta+\phi+\rho)}$$

Chapter 2

Ratio of areas of parallelograms

Calculation of ratio of areas of parallelograms $\frac{[AECF]}{[ABCD]}$ of the following figure in relation of the sizes a, b, θ



- We have:

$$\begin{aligned} AE &= \frac{a\sin\left(\theta+\frac{\theta}{2}\right)+b\sin\frac{\theta}{2}}{\sin\left(\theta+\frac{\theta}{2}+\frac{\theta}{2}\right)} = \frac{a\sin\frac{3\theta}{2}+b\sin\frac{\theta}{2}}{\sin 2\theta} \\ AF &= \frac{a\sin\frac{\theta}{2}+b\sin\left(\theta+\frac{\theta}{2}\right)}{\sin\left(\theta+\frac{\theta}{2}+\frac{\theta}{2}\right)} = \frac{a\sin\frac{\theta}{2}+b\sin\frac{3\theta}{2}}{\sin 2\theta} \end{aligned} \tag{1}$$

- We have:

$$[AECF] = AE \cdot AF \cdot \sin 2\theta \overset{(1)}{=} \frac{(a^2+b^2)\sin\frac{\theta}{2}\sin\frac{3\theta}{2}}{\sin 2\theta} + \frac{ab\left(\sin^2\frac{\theta}{2}+\sin^2\frac{3\theta}{2}\right)}{\sin 2\theta}$$

$$[ABCD] = AB \cdot AD \cdot \sin\theta = ab\sin\theta$$

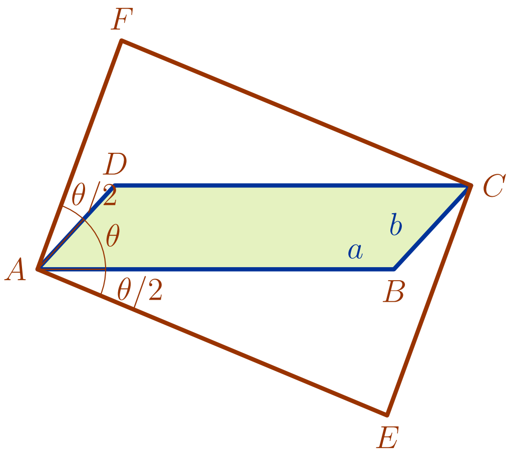
Hence:

$$\boxed{\frac{[\mathbf{AECF}]}{[\mathbf{ABCD}]} = \frac{\left(\frac{\mathbf{a}}{\mathbf{b}}+\frac{\mathbf{b}}{\mathbf{a}}\right)\sin\frac{\theta}{\mathbf{2}}\sin\frac{\mathbf{3}\theta}{\mathbf{2}}+\sin^{\mathbf{2}}\frac{\theta}{\mathbf{2}}\sin^{\mathbf{2}}\frac{\mathbf{3}\theta}{\mathbf{2}}}{\sin\theta\sin\mathbf{2}\theta}} \tag{B-1}$$

Chapter 3

Applications

3.1. Approach of π



$ABCD$ parallelogram

$AECF$ parallelogram

$B\hat{A}D = 2B\hat{A}E = 2D\hat{A}F = \theta = 45°$

$\dfrac{[AECF]}{[ABCD]} = \dfrac{AB}{AD} = m$

Calculate the ratio. $\dfrac{m}{\pi}$.

From equation (B − 1) we have $\dfrac{a}{b} = \dfrac{[AECF]}{[ABCD]} = m$, $\theta = 45°$ hence:

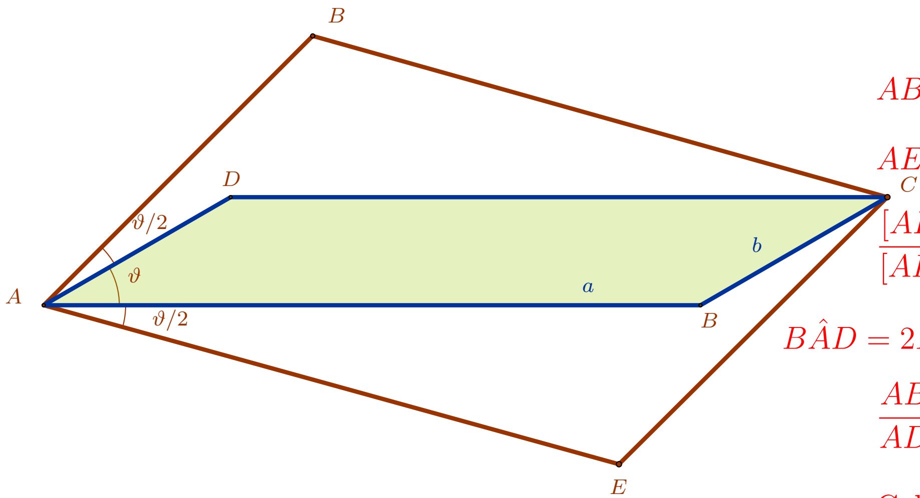
$$m = \frac{\left(m + \frac{1}{m}\right)\sin 22.5° \sin 67.5° + \sin^2 67.5° + \sin^2 22.5°}{\sin 45° \sin 90°} \xrightarrow{m>0} m = \sqrt{3} + \sqrt{2} \approx 3.14$$

and $\boxed{\dfrac{\mathbf{m}}{\boldsymbol{\pi}} \approx \mathbf{1.00}}$

Conversely , if , $\dfrac{[AECF]}{[ABCD]} = \dfrac{a}{b} = \pi$, hence:

$$(\mathrm{B}-1) \Rightarrow \pi = \frac{\left(\pi + \frac{1}{\pi}\right)\sin\frac{\theta}{2}\sin\frac{3\theta}{2} + \sin^2\frac{\theta}{2} + \sin^2\frac{3\theta}{2}}{\sin\theta \sin 2\theta} \Rightarrow \boxed{\boldsymbol{\theta \approx 44.92688122°}}$$

3.2. Approach of e (Euler)



$ABCD$ parallelogram

$AECF$ parallelogram

$\frac{[AECF]}{[ABCD]} = m$

$B\hat{A}D = 2B\hat{A}E = 2D\hat{A}E = \theta = 30^\circ$

$\frac{AB}{AD} = \frac{a}{b} = \frac{3}{1}$

Calculate the ratio $\frac{e}{m}$.

From equation (B − 1) we have:

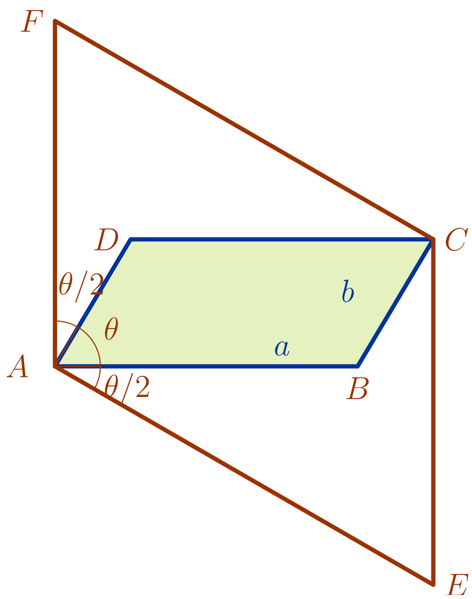
$$\frac{[AECF]}{[ABCD]} = \frac{\left(\frac{3}{1}+\frac{1}{3}\right)\sin 15^\circ \sin 45^\circ + \sin^2 15^\circ + \sin^2 45^\circ}{\sin 30^\circ \sin 60^\circ} \Rightarrow m = \frac{21+2\sqrt{3}}{9}$$

and $\boxed{\frac{\mathbf{e}}{\mathbf{m}} \approx \mathbf{1.0000}}$.

Conversely , if $m = e$ then:

$$e = \frac{\left(\frac{3}{1}+\frac{1}{3}\right)\sin\frac{\theta}{2}\sin\frac{3\theta}{2} + \sin^2\frac{\theta}{2} + \sin^2\frac{3\theta}{2}}{\sin\theta\sin 2\theta} \Rightarrow \boxed{\boldsymbol{\theta} = \mathbf{30.00296497^\circ}}$$

3.3. Approach of φ (golden ratio)



$ABCD$ parallelogram

$AECF$ parallelogram

$B\hat{A}D = 2B\hat{A}E = 2D\hat{A}F = \theta = 60°$

$$\frac{[AECF]}{[ABCD]} = m = \frac{5+2\sqrt{5}}{3}$$

Calculate the ratio. $\dfrac{a}{b}$.

From equation (B − 1) we have:

$$\frac{[AECF]}{[ABCD]} = \frac{5+2\sqrt{5}}{3} = \frac{\left(\frac{a}{b}+\frac{b}{a}\right)\sin 30° \sin 90° + \sin^2 30 + \sin^2 90°}{\sin 60° \sin 120°} \Rightarrow \frac{a}{b} = \frac{\sqrt{5} \pm 1}{2}$$

Hence $\boxed{\mathbf{\frac{a}{b}} = \varphi}$ or $\boxed{\mathbf{\frac{a}{b}} = \frac{\mathbf{1}}{\varphi}}$.