A new approach to prove bijections on the real line

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1 Introduction

We know that the infamous Schrőder-Bernstein theorem, can be used to show that some set $\mathcal{D}$ is bijective to some set $\mathcal{E}$, by proving that there is a one-one function from $\mathcal{D}$ to $\mathcal{E}$, and proving the existence of a one-one function from $\mathcal{E}$ to $\mathcal{D}$.

In this math note, we will discuss a new method to prove that certain subsets of $\mathbb{R}$ are bijective by using this simple property of countability of rational numbers, and without needing to use the Schrőder-Bernstein theorem, or the even more sophisticated theories of cardinal arithmetic.

Let us adopt a notation as below for our convenience.

Notation : For sets $\mathcal{A}, \mathcal{B}$ we will write $\mathcal{A} \sim \mathcal{B}$ to denote that *there is a one-one function from $\mathcal{A}$ onto $\mathcal{B}$.*

2 The method

We will demonstrate our method through an easy example :

Example 2.1. Prove that $[0, 1] \sim (0, 1]$.

Solution. Let $\mathcal{A} = \{x \in \mathbb{Q} \ : 0 \leq x \leq 1\}$, and let $\mathcal{B} = \{x \in \mathbb{Q} \ : 0 < x \leq 1\}$. Then, we know that both $\mathcal{A}$, and $\mathcal{B}$ are infinite subsets of $\mathbb{Q}$, and hence by countability of rational numbers, it follows that $\mathcal{A} \sim \mathbb{N} \sim \mathcal{B}$, and hence $\mathcal{A} \sim \mathcal{B}$. Now, let $f : \mathcal{A} \to \mathcal{B}$ be a bijection.
Then, we consider the function $g : [0, 1] \to (0, 1]$ defined by :

$$g(x) = \begin{cases} f(x) & x \in \mathcal{A} \\ x & \text{otherwise} \end{cases}$$

Clearly, we have $g : [0, 1] \to (0, 1]$ is a bijection.
This completes the solution. $\square$

This simple method, can be used with a bigger class of sets. Let us record the method.

Proposition 2.2 (<u>Common subset elimination method</u>)**.** Let $\mathcal{A} \subseteq \mathcal{B}$, and let $\exists$ a set $\mathcal{C}$, such that $\mathcal{B} \setminus \mathcal{A} \subseteq \mathcal{B} \cap \mathcal{C}$, and $\mathcal{B} \cap \mathcal{C} \sim \mathcal{A} \cap \mathcal{C}$. Then, we must have $\mathcal{A} \sim \mathcal{B}$.

Proof. Let $f : \mathcal{A} \cap \mathcal{C} \to \mathcal{B} \cap \mathcal{C}$ be a bijection.
We define $g : \mathcal{A} \to \mathcal{B}$, by :

$$g(x) = \begin{cases} f(x) & x \in \mathcal{A} \cap \mathcal{C} \\ x & \text{otherwise} \end{cases}$$

Then, it is easy to see that $g : \mathcal{A} \to \mathcal{B}$ is a bijection.
This implies that $\mathcal{A} \sim \mathcal{B}$.
This completes the proof of the proposition. $\square$

3 Problems to try

Here, we leave a few exercises for the reader to try solving using the method demonstrated by us on the previous page :

1. If $\mathcal{A} = \left\{ x \in \mathbb{Q} \ : 0 < x < 1, \& \ x = \dfrac{a}{2020^n} \text{ for } a \in \mathbb{Z}, n \in \mathbb{Z}, \ \& \ \mathsf{gcd}(a, 2020^n) = 1 \right\}$, then prove that : $[0,1] \sim [0,1] \setminus \mathcal{A}$.

2. Prove that the set $\mathcal{I}$ of all irrational numbers in $[0,1]$ is bijective to $[0,1]$.

3. A number t is called a *transcendental number*, if and only if there is no integer coefficient polynomial $p(x) \in \mathbb{Z}[x]$, such that t is a root of p. Prove that the set of all and only the transcendental numbers is bijective to $\mathbb{R}$.

4. Let a, b, c be three real numbers with $a < b < c$. Prove that $[a,b] \sim (a,c) \sim [a,c]$.

4 Hints to selected problems

1. Consider the common subset to be given by

$$\mathcal{C} = \left\{ \frac{a}{3^n} \ : a \in \mathbb{Z}, n \in \mathbb{N}, \ \& \ \mathsf{gcd}(a, 3^n) = 1 \right\} \cup \left\{ \frac{a}{2020^n} \ : a \in \mathbb{Z}, n \in \mathbb{N}, \ \& \ \mathsf{gcd}(a, 2020^n) = 1 \right\}.$$

3. Consider the common subset to be given by

$$\mathcal{C} = \mathbb{Q} \cup \left\{ \frac{\pi}{a} \ : a \in \mathbb{N} \right\}.$$

Comment

To appreciate the significance of the method demonstrated in proposition 2.2, try to give alternate solutions of the above problems, using the Schrőder-Bernstein theorem.