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# About Inequality in triangle-907

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***By Marin Chirciu – Romania***

**1) In $\Delta ABC$ the following relationship holds:**

$$\sum \frac{(R-r_a)^2}{h_a} \ge \frac{13r^2-3R^2}{r}$$

***Proposed by Mehmet Şahin – Turkey***

***Solution***

*We prove the following lemma:*

***Lemma.***

**2) In ΔABC the following relationship holds:**

$$\sum \frac{(R-r_a)^2}{h_a} = \frac{5R^2+4Rr-s^2}{r}$$

***Proof.***

*Using* $r_a = \frac{S}{s-a}$ *and* $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum \frac{(R-r_a)^2}{h_a} = \frac{\sum h_b h_c (R-r_a)^2}{\prod h_a} = \frac{5R^2+4Rr-s^2}{r}$$

*Which follows from:*

$$\sum h_b h_c (R-r_a)^2 = \frac{2r}{R}(5R^2+4Rr-s^2)s^2, \sum h_b h_c = \frac{2rs^2}{R}, \sum r_a h_b h_c = \frac{2r}{R}(2R-r)s^2,$$

$$\sum r_a^2 h_b h_c = \frac{2r}{R}(8R^2+2Rr-s^2)s^2 \text{ and } \prod h_a = \frac{2r^2s^2}{R}$$

*Let's get back to the main problem:*

*Using the Lemma we write the inequality:*

$$\frac{5R^2+4Rr-s^2}{r} \ge \frac{13r^2-3R^2}{r} \Leftrightarrow s^2 \le 8R^2+4Rr-13r^2$$

*which follows from Gerretsen's inequality:* $s^2 \le 4R^2+4Rr+3r^2$.

*It remains to prove that:*

$4R^2+4Rr+3r\le 8R^2+4Rr-13r^2\Leftrightarrow R^2\ge 4r^2$, *obviously from Euler's inequality*

$R\ge 2r.$

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*The inequality can be strengthened:*

**3) In ΔABC the following inequality holds:**

$$\sum\frac{(R-r_a)^2}{h_a}\ge\frac{R^2-3r^2}{r}$$

***Solution***

*Using the Lemma we write the inequality:*

$\frac{5R^2+4Rr-s^2}{r}\ge\frac{R^2-3r^2}{r}\Leftrightarrow s^2\le 4R^2+4Rr+3r^2$ *(Gerretsen's inequality)*

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*Inequality 3) is stronger than inequality 1):*

**4) In ΔABC the following relationship holds:**

$$\sum\frac{(R-r_a)^2}{h_a}\ge\frac{R^2-3r^2}{r}\ge\frac{13r^2-3R^2}{r}$$

***Solution***

*See inequality 3) and Euler's inequality* $R\ge 2r$

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*Let's emphasize and inequality having an opposite sense:*

**5) In ΔABC the following relationship holds:**

$$\sum\frac{(R-r_a)^2}{h_a}\le\frac{5R^2-19r^2}{r}$$

***Solution.***

*Using the Lemma we write the inequality:*

$$\frac{5R^2+4Rr-s^2}{r}\le\frac{5R^2-19r^2}{r}\Leftrightarrow s^2\ge 4Rr+19r^2$$

*which follows from Gerretsen's inequality:* $s^2\ge 16Rr-5r^2$

*It remains to prove that:*

$16Rr-5r^2\ge 4Rr+19r^2\Leftrightarrow 12Rr\ge 24r^2$, *obviously from Euler's inequality* $R\ge 2r$.

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*We can write the double inequality:*

**6) In ΔABC the following inequality holds:**

$$\frac{R^2-3r^2}{r}\le\sum\frac{(R-r_a)^2}{h_a}\le\frac{5R^2-19r^2}{r}$$

***Proposed by Marin Chirciu – Romania***

***Solution***

*See inequalities 3) and 5).*

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*In the same way we propose:*

**7) In ΔABC the following relationship holds:**

$$r\le\sum\frac{(R-r_a)^2}{r_a}\le\frac{R^2-3r^2}{r}$$

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***Solution***

*We prove the following lemma:*

***Lemma***

**8) In ΔABC the following relationship holds:**

$$\sum\frac{(R-r_a)^2}{r_a}=\frac{(R-r)^2}{r}$$

***Proof.***

*Using* $r_a = \frac{S}{s-a}$*, we obtain:*

$$\sum\frac{(R-r_a)^2}{r_a} = \frac{\sum r_b r_c (R-r_a)^2}{\prod r_a} = \frac{(R-r)^2}{r}, \text{ which follows from:}$$

$$\sum r_b r_c\,(R-r_a)^2 = s^2(R-r)^2, \sum r_b r_c = s^2, \sum r_a = 4R + r \text{ and } \prod r_a = rs^2$$

*Let's get back to the main problem:*

*Using Lemma and Euler's inequality* $R \geq 2r$.

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*Between the sums:* $\sum\frac{(R-r_a)^2}{r_a}$ *and* $\sum\frac{(R-r_a)^2}{h_a}$ *we can write the relationship:*

**9) In ΔABC the following relationship holds:**

$$\sum\frac{(R-r_a)^2}{r_a} \leq \sum\frac{(R-r_a)^2}{h_a}$$

***Solution***

*Using the identities 2) and 8) we write the inequality:*

$$\frac{(R-r)^2}{r} \leq \frac{5R^2+4Rr-s^2}{r} \Leftrightarrow s^2 \leq 4R^2 + 6Rr - r^2, \text{ which follows from Gerretsen's inequality:}$$

$$s^2 \leq 4R^2 + 4Rr + 3r^2 \text{ and Euler's inequality } R \geq 2r.$$

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*We can write the following sequences of inequalities:*

**10) In ΔABC the following relationship holds:**

$$r \leq \sum\frac{(R-r_a)^2}{r_a} \leq \frac{R^2-3r^2}{r} \leq \sum\frac{(R-r_a)^2}{h_a} \leq \frac{5R^2-19r^2}{r}$$

***Proposed by Marin Chirciu – Romania***

***Solution***

*See inequalities 6) and 7)*

*Equality holds if and only if the triangle is equilateral.*

**11) In ΔABC the following inequality holds:**

$$\sum\frac{(R-h_a)^2}{r_a}\le\frac{25R^4-384r^4}{4R^2r}$$

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***Solution***

*We prove the following lemma:*

***Lemma***

**12) In ΔABC the following relationship holds:**

$$\sum\frac{(R-h_a)^2}{r_a}=\frac{s^4+s^2(2r^2-12Rr-4R^2)+(4R^4+32R^3r-4R^2r^2+4Rr^3+r^4)}{4R^2r}$$

***Proof.***

*Using* $h_a=\frac{2S}{a}$ *and* $r_a=\frac{S}{s-a}$, *we obtain:*

$$\sum\frac{(R-h_a)^2}{r_a}=\frac{\sum r_br_c(R-h_a)^2}{\prod r_a}=\frac{s^4+s^2(2r^2-12Rr-4R^2)+(4R^4+32R^3r-4R^2r^2+4Rr^3+r^4)}{4R^2r},$$

*which follows from:*

$$\sum r_br_c\,(R-h_a)^2=\frac{s^2[s^4+s^2(2r^2-12Rr-4R^2)+(4R^4+32R^3r-4R^2r^2+4Rr^3+r^4)]}{4R^2},$$

$$\sum r_br_ch_a=\frac{s^2(s^2+r^2-8Rr)}{2R},\sum h_a^2r_br_c=\frac{s^2[s^4+s^2(2r^2-12Rr)+r^3(4R+r)]}{4R^2}$$

*and* $\prod r_a=rs^2$.

*Let's get back to the main problem:*

*Using the Lemma, we write the inequality:*

$$\frac{s^4+s^2(2r^2-12Rr-4R^2)+(4R^4+32R^3r-4R^2r^2+4Rr^3+r^4)}{4R^2r}\le\frac{25R^4-384r^4}{4R^2r}\Leftrightarrow$$

$$\Leftrightarrow s^2(s^2+2r^2-12Rr-4R^2)+(4R^4+32R^3r-4R^2r^2+4Rr^3+r^4)\le 25R^4-384r^4,$$

*which follows from Gerretsen's inequality* $16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2$ *and the observation that* $(s^2+2r^2-12Rr-4R^2)<0$.

*It remains to prove that:*

$$(16Rr-5r^2)(4R^2+4Rr+3r^2+2r^2-12Rr-4R^2)+$$

$$+(4R^4+32R^3r-4R^2r^2+4Rr^3+r^4)\le 25R^4-384r^4\Leftrightarrow$$

$$\Leftrightarrow 21R^4 - 32R^3r + 132R^2r^2 - 124Rr^3 - 360r^4 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(21R^3 + 10R^2r + 152Rr^2 + 180r^3) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

*Equality holds if and only if the triangle is equilateral.*

**13) In ΔABC the following inequality holds:**

$$\frac{18r^2 - 4R^2}{R} \leq \sum \frac{(R-h_a)^2}{h_a} \leq \frac{R^2 - 3r^2}{r}$$

***Proposed by Marin Chirciu – Romania***

***Solution***

*We prove the following lemma:*

***Lemma:***

**14) In ΔABC the following relationship holds:**

$$\sum \frac{(R-h_a)^2}{h_a} = \frac{rs^2 + 2R^3 - 12R^2r + 4Rr^2 + r^3}{2Rr}$$

***Proof.***

*Using* $h_a = \frac{2S}{a}$, *we obtain:*

$\sum \frac{(R-h_a)^2}{h_a} = \frac{\sum h_b h_c (R-h_a)^2}{\prod h_a} = \frac{rs^2+2R^3-12R^2r+4Rr^2+r^3}{2Rr}$, *which follows from:*

$$\sum h_b h_c (r-h_a)^2 = \frac{rs^2}{R^2}(rs^2 + 2R^3 - 12R^2r + 4Rr^2 + r^3),$$

$\sum h_b h_c = \frac{2rs^2}{R}, \sum h_a = \frac{s^2+r^2+4Rr}{2R}$ *and* $\prod h_a = \frac{2r^2s^2}{R}$.

*Let's get back to the main problem.*

*Using the Lemma, Gerretsen's inequality* $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$ *and Euler's inequality* $R \geq 2r$.

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*Between the sums* $\sum \frac{(R-h_a)^2}{h_a}$ *and* $\sum \frac{(R-r_a)^2}{h_a}$ *we can write the relationship:*

**15) In ΔABC the following relationship holds:**

$$\sum \frac{(R-h_a)^2}{h_a} \le \sum \frac{(R-r_a)^2}{h_a}$$

***Solution***

*Using the identities 8) and 14) we write the inequality:*

$$\frac{rs^2+2R^3-12R^2r+4Rr^2+r^3}{2Rr} \le \frac{5R^2+4Rr-s^2}{r} \Leftrightarrow$$

$\Leftrightarrow s^2(2R+r) \le 8R^3+20R^2r-4Rr^2-r^3$, *which follows from Gerretsen's inequality:*

$s^2 \le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$(4R^2+4Rr+3r^2)(2R+r) \le 8R^3+20R^2r-4Rr^2-r^3 \Leftrightarrow 4R^2-7Rr-2r^2 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(4R+r) \ge 0$, *obviously from Euler's inequality* $R \ge 2r$.

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*We can write the sequence of inequalities:*

**16) In ΔABC the following inequality holds:**

$$\frac{18R^2-4R^2}{R} \le \sum \frac{(R-h_a)^2}{h_a} \le \frac{R^2-3r^2}{r} \le \sum \frac{(R-r_a)^2}{h_a} \le \frac{5R^2-19r^2}{r}$$

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***Solution***

*See inequalities 6) and 13).*

*Equality holds if and only if the triangle is equilateral.*

**Refference:**

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