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ABOUT INEQUALITY IN TRIANGLE-978

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By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$4\sum m_a \le \sum \frac{r_a}{\sin^2\frac{A}{2}}$$

Proposed by Bogdan Fuștei – Romania

Solution

We prove the following lemma:

Lemma.

2) In ΔABC the following relationship holds:

$$\sum \frac{r_a}{\sin^2\frac{A}{2}} = \frac{s^2+r^2+4Rr}{r}$$

Proof.

$$Using\ r_a = \frac{S}{s-a} = \sum \frac{\frac{S}{s-a}}{\frac{(s-b)(s-c)}{bc}} = \frac{S}{\prod(s-a)}\sum bc = \frac{rs}{r^2 s}(s^2+r^2+4Rr) = \frac{s^2+r^2+4Rr}{r}$$

Let's get back to the main problem.

Using the known inequality $\sum m_a \le 4R+r$ and the Lema it suffices to prove that:

$4(4R+r) \le \frac{s^2+r^2+4Rr}{r} \Leftrightarrow s^2 \ge 12Rr+3r^2$, *which follows from Gerretsen's inequality*

$s^2 \ge 16Rr-5r^2$ *and Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.

For the sum $\sum \frac{r_a}{\sin^2\frac{A}{2}}$ we can write the relationship:

3) In ΔABC the following relationship holds:

$$4(5R-r)\le\sum\frac{r_a}{\sin^2\frac{A}{2}}\le\frac{4(R+r)^2}{r}$$

Solution

Using Lemma, the inequality can be written: $4(5R-r)\le\frac{s^2+r^2+4Rr}{r}\le\frac{4(R+r)^2}{r}$

which follows from Gerretsen's inequality: $16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2$

Equality holds if and only if the triangle is equilateral.

Remark.

If we replace r_a *with* h_a *we propose:*

4) In ΔABC the following relationship holds:

$$4\sum m_a\le\sum\frac{h_a}{\sin^2\frac{A}{2}}$$

Proposed by Marin Chirciu – Romania

Solution

We prove the following lemma:

Lemma.

5) In ΔABC the following relationship holds:

$$\sum\frac{h_a}{\sin^2\frac{A}{2}}=\frac{s^4+s^2(2r^2-12Rr)+r^3(4Rr+r)}{2Rr^2}$$

Proof.

Using $h_a=\frac{2S}{a}$ *and* $\sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}$ *we obtain:*

$$\sum\frac{h_a}{\sin^2\frac{A}{2}}=\sum\frac{\frac{2S}{a}}{\frac{(s-b)(s-c)}{bc}}=2S\sum\frac{bc}{a(s-b)(s-c)}=$$

$$=2rs\cdot\frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{4Rr^3s}=\frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{2Rr^2}$$

Let's get back to the main problem.

Using $\sum m_a \le 4R + r$ *and Lemma it suffices to prove that:*

$$4(4R+r) \le \frac{s^4 + s^2(2r^2 - 12Rr) + r^3(4R+r)}{2Rr^2} \Leftrightarrow$$

$$\Leftrightarrow s^2(s^2 + 2r^2 - 12Rr) \ge r^2(32R^2 + 4Rr - r^2),$$ *which follows from Gerretsen's inequality* $s^2 \ge 16Rr - 5r^2$*. It remains to prove that:*

$$(16Rr - 5r^2)(16Rr - 5r^2 + 2r^2 - 12Rr) \ge r^2(32R^2 + 4Rr - r^2) \Leftrightarrow$$

$$\Leftrightarrow 4R^2 - 9Rr + 2r^2 \ge 0 \Leftrightarrow (R - 2r)(4R - r) \ge 0,$$ *obiviously from Euler's inequality*

$R \ge 2r$*. Equality holds if and only if the triangle is equilateral.*

Remark.

For the sum $\sum \frac{h_a}{\sin^2\frac{A}{2}}$ *the following relationship can be written:*

6) In ΔABC the following relationship holds:

$$\frac{8(2R-r)^2}{R} \le \sum \frac{h_a}{\sin^2\frac{A}{2}} \le \frac{8(R^4 - 7r^4)}{Rr^2}$$

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Solution

Using the Lemma the inequality can be written:

$$\frac{8(2R-r)^2}{R} \le \frac{s^4 + s^2(2r^2 - 12Rr) + r^3(4R+r)}{2Rr^2} \le \frac{8(R^4 - 7r^4)}{Rr^2}$$

Which follows from Gerretsen's inequality: $16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2$.

Equality holds if and only if the triangle is equilateral.

Remark.

Between the sums $\sum \frac{h_a}{\sin^2\frac{A}{2}}$ *and* $\sum \frac{r_a}{\sin^2\frac{A}{2}}$ *the following relationship holds:*

7) In ΔABC the following relationship holds:

$$\sum \frac{h_a}{\sin^2\frac{A}{2}} \ge \sum \frac{r_a}{\sin^2\frac{A}{2}}$$

Solution

Using the following identities $\sum \frac{h_a}{\sin^2\frac{A}{2}} = \frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{2Rr^2}$ *and* $\sum \frac{r_a}{\sin^2\frac{A}{2}} = \frac{s^2+r^2+4Rr}{r}$

the inequality can be written:

$$\frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{2Rr^2} \geq \frac{s^2+r^2+4Rr}{r} \Leftrightarrow$$

$\Leftrightarrow s^2(s^2+2r^2-14Rr) \geq r^2(8R^2-2Rr-r^2)$, *which follows from Gerretsen's inequality*

$s^2 \geq 16Rr-5r^2$. *It remains to prove that:*

$$(16Rr-5r^2)(16Rr-5r^2+2r^2-14Rr) \geq r^2(8R^2-2Rr-r^2) \Leftrightarrow$$

$3R^2-7Rr+2r^2 \geq 0 \Leftrightarrow (R-2r)(3R-r) \geq 0$, *obviously form Euler's inequalty* $R \geq 2r$.

Remark.

We can write the following sequence of inequalities:

8) In ΔABC the following relationship holds:

$$36r \leq 4(5R-r) \leq \sum \frac{r_a}{\sin^2\frac{A}{2}} \leq \sum \frac{h_a}{\sin^2\frac{A}{2}} \leq \frac{8(R^4-7r^4)}{Rr^2}$$

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Solution

See inequalities 7), 6) and 3)

Equality holds if and only if the triangle is equilateral.

Refference:

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