ON THE SOLUTION TO NAREN-SERGIO'S PROPOSAL AND ITS CONTINUATION

Ekpo, Samuel O.
ekpoduduabasi@gmail.com

Here, I present evaluations to some amazing identities involving $\prod_{k=1}^{n} k^{k^s}$ proposed by Naren Bhandari and Sergio Esteban in the Romanian Mathematical magazine.

1.1

$$\textit{prove that}$$

$$\prod_{s=0}^{\infty}\left(\lim_{n\to\infty}\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}}}\right)=\frac{1}{e^{\zeta(2)}}$$

$$\textit{also for } s=1 \textit{ prove the following special case}$$

$$\lim_{n\to\infty}\left(\frac{\sqrt[4]{e^{n^2}}}{n^{\frac{6n^2+6n+1}{12}}}\left((n!)^{n+1}\prod_{k=1}^{n-1}\binom{n}{k}\right)^{\frac{1}{2}}\right)=\lim_{n\to\infty}\left(\frac{\sqrt[4]{e^{n^2}}}{n^{\frac{6n^2+6n+1}{12}}}\prod_{k=1}^{n}k^k\right)=A$$

$$\textit{where } A \textit{ is Glaisher Kinkelin constant}$$

1.2

$$\textit{prove that for all } m\in\mathbb{R}$$

$$\lim_{s\to\infty}\left(1+\lim_{n\to\infty}\frac{1}{(s+1)^{ms}}\left(\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}}}\right)^{-1}\right)^{s^{ms}}=e^{e^{-m}}$$

1.3

$$\textit{prove that}$$

$$\prod_{m=1}^{\infty}\prod_{s=1}^{m}\left(\lim_{n\to\infty}\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}m^2}}\right)=\frac{e^3}{e^{\zeta(2)+\frac{7}{4}\zeta(4)}}$$

Considering **1.1**

$$let\ J=\prod_{s=0}^{\infty}\left(\lim_{n\to\infty}\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}}}\right)\ and\ A(m)=\lim_{n\to\infty}\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}m^2}}\ \therefore\ A(1)=\lim_{n\to\infty}\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}}}$$

$$\therefore J=\prod_{s=0}^{\infty}A(1)$$

$$considering\ A(m)=\lim_{n\to\infty}\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}m^2}}\quad \log\big(A(m)\big)=\lim_{n\to\infty}log\left(\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}m^2}}\right)$$

$$=\lim_{n\to\infty}\left(log\left(\frac{1}{\sqrt[s+1]{n}}\right)+log\left(\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}m^2}}\right)\right)=\lim_{n\to\infty}\left(-\frac{\log(n)}{s+1}+\frac{log\left(\prod_{k=1}^{n}k^{k^s}\right)}{n^{s+1}m^2}\right)=\lim_{n\to\infty}\left(-\frac{\log(n)}{s+1}+\frac{\sum_{k=1}^{n}k^s log(k)}{n^{s+1}m^2}\right)$$

$$=\lim_{n\to\infty}\left(-\frac{\log(n)}{s+1}+\frac{\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s log\left(\frac{kn}{n}\right)}{nm^2}\right)=\lim_{n\to\infty}\left(-\frac{\log(n)}{s+1}+\frac{\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s\left(log\left(\frac{k}{n}\right)+\log(n)\right)}{nm^2}\right)$$

$$=\lim_{n\to\infty}\left(-\frac{\log(n)}{s+1}+\frac{\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s log\left(\frac{k}{n}\right)+\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s\log(n)}{nm^2}\right)=\lim_{n\to\infty}\left(\log(n)\left(-\frac{1}{s+1}+\frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s\right)+\frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s log\left(\frac{k}{n}\right)\right)$$

$$=\underbrace{\lim_{n\to\infty}\left(\log(n)\left(-\frac{1}{s+1}+\frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s\right)\right)}_{L_1}+\frac{1}{m^2}\underbrace{\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s log\left(\frac{k}{n}\right)\right)}_{L_2}$$

$$\boxed{\therefore \mathbf{log}(\boldsymbol{A}(\boldsymbol{m}))=\boldsymbol{L_1}+\frac{\mathbf{1}}{\boldsymbol{m^2}}\boldsymbol{L_2}}$$

$$recall\ that\ the\ riemann\ sum\ of\ a\ function\ is\ related\ to\ the\ definite\ integral\ as\ follows:$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}f(c_k)\Delta x_k=\int_a^b f(x)dx$$

$$where\ \Delta x_k=\frac{b-a}{n}$$

$$consider\ that;for\ n\in\mathbb{N}\ and\ that\ 1<k<n$$

$$if\ \frac{k-1}{n} \le x \le \frac{k}{n}$$

$$then\ \left(\frac{k-1}{n}\right)^s \le x^s \le \left(\frac{k}{n}\right)^s\ and\ \frac{1}{nm^2}\left(\frac{k-1}{n}\right)^s \le \int_{\frac{k-1}{n}}^{\frac{k}{n}} x^s dx \le \frac{1}{nm^2}\left(\frac{k}{n}\right)^s\ hence\ \frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k-1}{n}\right)^s \le \int_0^1 x^s dx \le \frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s$$

$$\frac{1}{nm^2}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^s \le \frac{1}{s+1} \le \frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s = -\frac{1}{nm^2} + \frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s < \frac{1}{s+1} < \frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s = -\frac{1}{nm^2} < \frac{1}{s+1} - \frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s < 0$$

$$\therefore 0 < -\frac{1}{s+1} + \frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s < \frac{1}{nm^2}$$

$$multiplying\ through\ by\ \log(n)$$

$$0 < \log(n)\left(-\frac{1}{s+1} + \frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s\right) < \frac{\log(n)}{nm^2}$$

$$\therefore L_1 = \lim_{n\to\infty}\left(\log(n)\left(-\frac{1}{s+1} + \frac{1}{nm^2}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s\right)\right) = 0$$

$$considering\ L_2 = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s log\left(\frac{k}{n}\right)\right)\ ;using\ that\ \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \int_a^b f(x)dx$$

$$L_2 = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s log\left(\frac{k}{n}\right)\right) = \int_0^1 x^s \log(x)dx = -\frac{1}{(s+1)^2}\ \therefore\ L_2 = -\frac{1}{(s+1)^2}$$

$$recall\ \log(A(m)) = L_1 + \frac{1}{m^2}L_2\ \ ;L_1 = 0\ and\ L_2 = -\frac{1}{(s+1)^2}$$

$$\therefore \log(A(m)) = -\frac{1}{m^2(s+1)^2}\ and\ \boxed{\boldsymbol{A(m) = e^{-\left(\frac{1}{m^2(s+1)^2}\right)}}}$$

$$\boxed{\therefore \boldsymbol{A(m) = \lim_{n\to\infty}\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n} k^{k^s}\right)^{\frac{1}{n^{s+1}m^2}} = e^{-\left(\frac{1}{m^2(s+1)^2}\right)}}}$$

$$recall\ that\ J = \prod_{s=0}^{\infty} A(1) = \prod_{s=0}^{\infty} e^{-\left(\frac{1}{(s+1)^2}\right)} = e^{\sum_{s=0}^{\infty} -\frac{1}{(s+1)^2}} = e^{-\zeta(2)} = \frac{1}{\boldsymbol{e^{\zeta(2)}}}$$

$$\boxed{\boxed{\therefore \boldsymbol{\prod_{s=0}^{\infty}\left(\lim_{n\to\infty}\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n} k^{k^s}\right)^{\frac{1}{n^{s+1}}}\right) = \frac{1}{e^{\zeta(2)}}}}}$$

to prove the following special case

$$\lim_{n\to\infty}\left(\frac{\sqrt[4]{e^{n^2}}}{n^{\frac{6n^2+6n+1}{12}}}\left((n!)^{n+1}\prod_{k=1}^{n-1}\binom{n}{k}\right)^{\frac{1}{2}}\right)=\lim_{n\to\infty}\left(\frac{\sqrt[4]{e^{n^2}}}{n^{\frac{6n^2+6n+1}{12}}}\prod_{k=1}^{n}k^k\right)=A$$

where A is Glaisher Kinkelin constant

$$consider;\ \prod_{k=1}^{n-1}\binom{n}{k}=;\ \prod_{k=1}^{n-1}\frac{n!}{k!\,(n-k)!}=\ e^{\sum_{k=1}^{n-1}log\left(\frac{n!}{k!(n-k)!}\right)}=\ e^{\sum_{k=1}^{n-1}log(n!)-\sum_{k=1}^{n-1}log(k!)-\sum_{k=1}^{n-1}log(n-k)!}$$

$$=e^{(n-1)log(n!)-\log G(n+1)-\log G(n+1)}=e^{(n-1)log(n!)-2\log G(n+1)}$$

$$=e^{log(n!)^{(n-1)}-\log G(n+1)^2}=e^{log\left(\frac{(n!)^{(n-1)}}{G(n+1)^2}\right)}=\frac{(n!)^{(n-1)}}{G(n+1)^2}$$

$$hence\ (n!)^{n+1}\prod_{k=1}^{n-1}\binom{n}{k}=\frac{(n!)^{(2n)}}{G(n+1)^2}=\left(\frac{(n!)^{(n)}}{G(n+1)}\right)^2\ and\ \left((n!)^{n+1}\prod_{k=1}^{n-1}\binom{n}{k}\right)^{\frac{1}{2}}=\frac{(n!)^{(n)}}{G(n+1)}$$

$$recall\ that\ H(n)=\prod_{k=1}^{n}k^k=\frac{(n!)^{(n)}}{G(n+1)}$$

$$\therefore\left((n!)^{n+1}\prod_{k=1}^{n-1}\binom{n}{k}\right)^{\frac{1}{2}}=\frac{(n!)^{(n)}}{G(n+1)}=\ H(n)=\prod_{k=1}^{n}k^k$$

$$hence\ \lim_{n\to\infty}\left(\frac{\sqrt[4]{e^{n^2}}}{n^{\frac{6n^2+6n+1}{12}}}\left((n!)^{n+1}\prod_{k=1}^{n-1}\binom{n}{k}\right)^{\frac{1}{2}}\right)=\lim_{n\to\infty}\left(\frac{\sqrt[4]{e^{n^2}}}{n^{\frac{6n^2+6n+1}{12}}}\prod_{k=1}^{n}k^k\right)$$

$$recall\ that\ A=\lim_{n\to\infty}A_n=\lim_{n\to\infty}\frac{H(n)}{n^{\frac{6n^2+6n+1}{12}}e^{-\frac{n^2}{4}}}$$

$$\boxed{\therefore\lim_{n\to\infty}\left(\frac{\sqrt[4]{e^{n^2}}}{n^{\frac{6n^2+6n+1}{12}}}\left((n!)^{n+1}\prod_{k=1}^{n-1}\binom{n}{k}\right)^{\frac{1}{2}}\right)=\lim_{n\to\infty}\left(\frac{\sqrt[4]{e^{n^2}}}{n^{\frac{6n^2+6n+1}{12}}}\prod_{k=1}^{n}k^k\right)=\lim_{n\to\infty}\left(\frac{\sqrt[4]{e^{n^2}}}{n^{\frac{6n^2+6n+1}{12}}}H(n)\right)=A}$$

where A is Glaisher Kinkelin constant

$$G(n+1)=\prod_{k=1}^{n-1}k!\ \ is\ the\ Barnes\ function\ \ ,H(n)=\prod_{k=1}^{n}k^k\ \ is\ the\ hyperfactorial\ function$$

Considering **1.2**

$$let\ \alpha = \lim_{s\to\infty}\left(1+\lim_{n\to\infty}\frac{1}{(s+1)^{ms}}\left(\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}}}\right)^{-1}\right)^{s^{ms}}$$

$$and\ recall\ that\ A(m) = \lim_{n\to\infty}\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}m^2}} = e^{-\left(\frac{1}{m^2(s+1)^2}\right)}$$

$$\therefore\ \alpha = \lim_{s\to\infty}\left(1+\frac{1}{(s+1)^{ms}}\left(A(1)\right)^{-1}\right)^{s^{ms}} = \lim_{s\to\infty}\left(1+\frac{1}{(s+1)^{ms}}\left(e^{-\left(\frac{1}{(s+1)^2}\right)}\right)^{-1}\right)^{s^{ms}} = \lim_{s\to\infty}\left(1+\frac{e^{\frac{1}{(s+1)^2}}}{(s+1)^{ms}}\right)^{s^{ms}}$$

$$\log(\alpha) = \lim_{s\to\infty}s^{ms}log\left(1+\frac{e^{\frac{1}{(s+1)^2}}}{(s+1)^{ms}}\right)\sim\lim_{s\to\infty}s^{ms}\left(\frac{e^{\frac{1}{(s+1)^2}}}{(s+1)^{ms}}\right)$$

$$\lim_{s\to\infty}s^{ms}\left(\frac{e^{\frac{1}{(s+1)^2}}}{(s+1)^{ms}}\right) = \lim_{s\to\infty}e^{\frac{1}{(s+1)^2}}\left(\frac{s}{s+1}\right)^{ms}$$

$$let\ \beta = \lim_{s\to\infty}e^{\frac{1}{(s+1)^2}}\left(\frac{s}{s+1}\right)^{ms}\ \therefore \log(\alpha) = \beta$$

$$log(\beta) = \lim_{s\to\infty}log\left(e^{\frac{1}{(s+1)^2}}\left(\frac{s}{s+1}\right)^{ms}\right) = \lim_{s\to\infty}\left(\frac{1}{(s+1)^2}+log\left(\frac{s}{s+1}\right)^{ms}\right) = \lim_{s\to\infty}\left(log\left(\frac{s}{s+1}\right)^{ms}\right) = \lim_{s\to\infty}ms\left(log\left(1-\frac{1}{s+1}\right)\right)$$

$$\lim_{s\to\infty}ms\left(log\left(1-\frac{1}{s+1}\right)\right)\sim\lim_{s\to\infty}-\frac{ms}{s+1}$$

$$\lim_{s\to\infty}-\frac{ms}{s+1} = -m\lim_{s\to\infty}\left(1-\frac{1}{s+1}\right) = -m$$

$$\therefore log(\beta) = -m\ , \beta = e^{-m}$$

$$recall\ that\ \log(\alpha) = \beta\ \therefore \alpha = e^{\beta} = e^{e^{-m}}$$

$$\boldsymbol{\therefore\ \alpha = \lim_{s\to\infty}\left(1+\lim_{n\to\infty}\frac{1}{(s+1)^{ms}}\left(\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}}}\right)^{-1}\right)^{s^{ms}} = e^{e^{-m}}}$$

Considering **1.3**

$$let\ P = \prod_{m=1}^{\infty}\prod_{s=1}^{m}\left(\lim_{n\to\infty}\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}m^2}}\right)\ and\ recall\ A(m) = \lim_{n\to\infty}\frac{1}{\sqrt[s+1]{n}}\left(\prod_{k=1}^{n}k^{k^s}\right)^{\frac{1}{n^{s+1}m^2}} = e^{-\left(\frac{1}{m^2(s+1)^2}\right)}$$

$$hence\ P = \prod_{m=1}^{\infty}\prod_{s=1}^{m}A(m) = \prod_{m=1}^{\infty}\prod_{s=1}^{m}e^{-\left(\frac{1}{m^2(s+1)^2}\right)} = e^{\sum_{m=1}^{\infty}\sum_{s=1}^{m}-\frac{1}{m^2(s+1)^2}} = e^{-\sum_{m=1}^{\infty}\sum_{s=1}^{m}\frac{1}{m^2(s+1)^2}}$$

$$considering\ the\ sum\ \sum_{m=1}^{\infty}\sum_{s=1}^{m}\frac{1}{m^2(s+1)^2} = \sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{s=1}^{m}\frac{1}{(s+1)^2}$$

$$using\ that\ \sum_{s=1}^{m}\frac{1}{(s+1)^n} = H_{m+1}^{(n)} - 1\ then\ \sum_{s=1}^{m}\frac{1}{(s+1)^2} = H_{m+1}^{(2)} - 1$$

$$\therefore \sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{s=1}^{m}\frac{1}{(s+1)^2} = \sum_{m=1}^{\infty}\frac{H_{m+1}^{(2)}-1}{m^2} = \sum_{m=1}^{\infty}\frac{H_{m+1}^{(2)}}{m^2} - \sum_{m=1}^{\infty}\frac{1}{m^2} = \sum_{m=1}^{\infty}\frac{H_{m+1}^{(2)}}{m^2} - \zeta(2)$$

$$we\ know\ that;\ H_m^{(r)} = H_{m+1}^{(r)} - \frac{1}{(m+1)^r}$$

$$\therefore H_{m+1}^{(2)} = H_m^{(2)} - \frac{1}{(m+1)^2}$$

$$\therefore \sum_{m=1}^{\infty}\frac{H_{m+1}^{(2)}}{m^2} - \zeta(2) = \underbrace{\sum_{m=1}^{\infty}\frac{H_m^{(2)}}{m^2}}_{S_1} - \underbrace{\sum_{m=1}^{\infty}\frac{1}{m^2(m+1)^2}}_{S_2} - \zeta(2)$$

$$\therefore \boxed{\boldsymbol{P = e^{-(S_1 - S_2 - \zeta(2))}}}$$

$$consider\ S_1 = \sum_{m=1}^{\infty}\frac{H_m^{(2)}}{m^2}$$

$$from\ euler\ sums\ we\ know\ that\ s_h(m,n) = \sum_{k=1}^{\infty}\frac{H_k^{(m)}}{k^n} = \sigma_h(m,n) + \zeta(m+n)$$

$$when\ m = n\ ;\ \sigma_h(n,n) = \frac{1}{2}[(\zeta(n))^2 - \zeta(2n)]\ \therefore \sum_{k=1}^{\infty}\frac{H_k^{(n)}}{k^n} = \frac{1}{2}[(\zeta(n))^2 - \zeta(2n)] + \zeta(2n)$$

$$\therefore \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2} = \frac{1}{2}[(\zeta(2))^2 - \zeta(4)] + \zeta(4) = \frac{1}{2}\left[\left(\frac{\pi^2}{2}\right)^2 - \frac{\pi^4}{90}\right] + \frac{\pi^4}{90} = \frac{1}{2}\left[\frac{\pi^4}{36} - \frac{\pi^4}{90}\right] + \frac{\pi^4}{90} = \frac{\pi^4}{120} + \frac{\pi^4}{90} = \frac{7\pi^4}{360}\frac{90}{90} = \frac{7}{4}\frac{\pi^4}{90} = \frac{7}{4}\zeta(4)$$

$$\therefore \boldsymbol{S_1 = \sum_{m=1}^{\infty}\frac{H_m^{(2)}}{m^2} = \frac{7\pi^4}{360} = \frac{7}{4}\zeta(4)}$$

$$\text{consider } S_2=\sum_{m=1}^{\infty}\frac{1}{m^2(m+1)^2}=\sum_{m=1}^{\infty}\left(\frac{1}{m^2}+\frac{1}{(m+1)^2}+\frac{2}{(m+1)}-\frac{2}{m}\right)=\sum_{m=1}^{\infty}\frac{1}{m^2}+\sum_{m=1}^{\infty}\frac{1}{(m+1)^2}-2\sum_{m=1}^{\infty}\left(\frac{1}{m}-\frac{1}{m+1}\right)$$

$$=\zeta(2)+(\zeta(2)-1)-2=2\,\zeta(2)-3$$

$$\therefore \boldsymbol{S_2}=\sum_{m=1}^{\infty}\frac{\mathbf{1}}{\boldsymbol{m^2(m+1)^2}}=\mathbf{2}\,\boldsymbol{\zeta}(\mathbf{2})-\mathbf{3}$$

$$\text{recall that } P=e^{-(S_1-S_2-\zeta(2))}$$

$$\therefore P=e^{-(S_1-S_2-\zeta(2))}=e^{-\left(2\,\zeta(2)-3+\frac{7}{4}\zeta(4)-\zeta(2)\right)}=e^{-\left(-3+\zeta(2)+\frac{7}{4}\zeta(4)\right)}=e^{\left(3-\zeta(2)-\frac{7}{4}\zeta(4)\right)}=e^3e^{-\left(\zeta(2)+\frac{7}{4}\zeta(4)\right)}=\frac{e^3}{e^{\zeta(2)+\frac{7}{4}\zeta(4)}}$$

$$\boxed{\boldsymbol{P}=\prod_{\boldsymbol{m}=\mathbf{1}}^{\infty}\prod_{\boldsymbol{s}=\mathbf{1}}^{\boldsymbol{m}}\left(\lim_{\boldsymbol{n}\to\infty}\frac{\mathbf{1}}{\sqrt[s+1]{\boldsymbol{n}}}\left(\prod_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{n}}\boldsymbol{k}^{\boldsymbol{k}^{\boldsymbol{s}}}\right)^{\frac{\mathbf{1}}{\boldsymbol{n}^{s+1}\boldsymbol{m}^{2}}}\right)=\frac{\boldsymbol{e}^{\mathbf{3}}}{\boldsymbol{e}^{\zeta(\mathbf{2})+\frac{\mathbf{7}}{\mathbf{4}}\zeta(\mathbf{4})}}}$$