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ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS

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By Marin Chirciu – Romania

1) Let $x, y, z > 0$, such that $x + y + z = 48$. Prove that:

$$\frac{1}{8-\sqrt{x}} + \frac{1}{8-\sqrt{y}} + \frac{1}{8-\sqrt{z}} \geq \frac{3}{4}$$

Proposed by George Apostolopoulos – Greece – RMM 2019

Solution

We prove the following lemma:

Lemma.

$$\text{If } 0 < x < 64 \text{ then } \frac{1}{8-\sqrt{x}} \geq \frac{x+16}{128}$$

Proof.

We are looking for an inequality having the form $\frac{1}{8-\sqrt{x}} \geq ax + b$, *having the property that the polynomial attached equation to the double root* $x = 16$.

We obtain $\begin{cases} 64a + 4b = 1 \\ 16a - b = 0 \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{128} \\ b = \frac{16}{128} \end{cases}$. *It follows* $\frac{1}{8-\sqrt{x}} \geq \frac{x+16}{128} \Leftrightarrow \sqrt{x}\left(\sqrt{x}-4\right)^2 \geq 0$, *with equality for* $x = 16$. *Let's get back to the main problem. Using the Lemma it follows:*

$$\sum \frac{1}{8-\sqrt{x}} \leq \sum \frac{x+16}{128} = \frac{x+y+z+16\cdot 3}{128} = \frac{48+48}{128} = \frac{96}{128} = \frac{3}{4}$$

Equality holds if and only if $x = y = z = 16$.

Remark.

Inequality 1) can be developed.

2) Let $x, y, z, n > 0$, such that $x + y + z = 3n^2$. Prove that:

$$\frac{1}{2n-\sqrt{x}} + \frac{1}{2n-\sqrt{y}} + \frac{1}{2n-\sqrt{z}} \geq \frac{3}{n}$$

Proposed by Marin Chirciu – Romania

Solution

We prove the following lemma.

Lemma.

$$\boldsymbol{If\ 0 < x < 4n^2\ then\ \frac{1}{2n-\sqrt{x}} \geq \frac{x+16}{2n^3}}$$

Proof.

We are looking an inequality having the form $\frac{1}{2n-\sqrt{x}} \geq ax+b$, *having the property that the polynomial attached equation to have double root* $x = n^2$. *We obtain*

$$\begin{cases} n^3a + nb = 1 \\ n^2a - b = 0 \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{2n^3} \\ b = \frac{n^2}{2n^3} \end{cases}. \text{ It follows from } \frac{1}{2n-\sqrt{x}} \geq \frac{x+n^2}{2n^3} \Leftrightarrow \sqrt{x}\left(\sqrt{x}-n\right)^2 \geq 0, \text{ with}$$

equality for $x = n^2$. *Let's get back to the main problem.*

Using the Lemma it follows:

$$\sum \frac{1}{2n-\sqrt{x}} \leq \sum \frac{x+n^2}{2n^3} = \frac{x+y+z+n^2\cdot 3}{2n^3} = \frac{3n^2+3n^2}{2n^3} = \frac{6n^2}{2n^3} = \frac{3}{n}$$

Equality if and only if $x = y = z = n^2$.

Note.

For $n = 4$ *we obtain the proposed by problem by George Apostolopoulos, RMM 8/2019.*

Remark.

Inequality 2) can be generalized.

3) Let $x_1, x_2, \dots, x_k, n > 0$, such that $x_1 + x_2 + \dots + x_k = kn^2$. Prove that:

$$\frac{1}{2n-\sqrt{x_1}} + \frac{1}{2n-\sqrt{x_2}} + \dots + \frac{1}{2n-\sqrt{x_k}} \geq \frac{k}{n}$$

Proposed by Marin Chirciu – Romania

Solution

We prove the following lemma:

Lemma.

$$\boldsymbol{If\ 0 < x < 4n^2\ then\ \frac{1}{2n-\sqrt{x}} \geq \frac{x+16}{2n^3}}$$

Proof.

We are looking an inequality having the form $\frac{1}{2n-\sqrt{x}} \geq ax+b$*, having the property that the polynomial attached equation to have double root* $x=n^2$.

We obtain $\begin{cases} n^3a+nb=1 \\ n^2a-b=0 \end{cases} \Leftrightarrow \begin{cases} a=\frac{1}{2n^3} \\ b=\frac{n^2}{2n^3} \end{cases}$. *It follows* $\frac{1}{2n-\sqrt{x}} \geq \frac{x+n^2}{2n^3} \Leftrightarrow \sqrt{x}\left(\sqrt{x}-n\right)^2 \geq 0$*, with equality for* $x=n^2$*. Let's get back to the main problem. Using the Lemma it follows:*

$$\sum \frac{1}{2n-\sqrt{x_1}} \leq \sum \frac{x_1+n^2}{2n^3} = \frac{x_1+x_2+\dots+x_k+n^2k}{2n^3} = \frac{kn^2+kn^2}{2n^3} = \frac{2kn^2}{2n^3} = \frac{k}{n}$$

Equality holds if and only if $x_1=x_2=\dots=x_k=n^2$.

Note.

For $n=4$ *and* $k=3$ *we obtain the proposed problem by George Apostolopoulos – Greece – RMM 8/2019*

Refference:

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