R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

ABOUT AN INEQUALITY BY ERTAN YILDIRIM-I

Proposed by Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{b+c}{r_a}+\frac{c+a}{r_b}+\frac{a+b}{r_c}\geq\frac{4}{s}(m_a+m_b+m_c)$$

Proposed by Ertan Yidirim-Turkey

Solution *We prove the following lemma:*

Lemma. ***2) In ΔABC the following relationship holds:***

$$\frac{b+c}{r_a}+\frac{c+a}{r_b}+\frac{a+b}{r_c}=\frac{2(s^2-r^2-4Rr)}{sr}$$

Proof.*Using the formula $r_a=\frac{S}{s-a}$ we obtain:*

$\sum\frac{b+c}{r_a}=\sum\frac{b+c}{\frac{S}{s-a}}=\frac{1}{S}\sum(b+c)(s-a)=\frac{1}{sr}\cdot 2(s^2-r^2-4Rr)=\frac{2(s^2-r^2-4Rr)}{sr}$, *which follows from the known identity in triangle* $\sum(b+c)\,(s-a)=2(s^2-r^2-4Rr)$

Let's get back to the main problem.Using the lemma the inequality can be written:

$$\frac{2(s^2-r^2-4Rr)}{sr}\geq\frac{4}{s}(m_a+m_b+m_c)$$

Because $m_a+m_b+m_c\leq 4R+r$ *it suffices to prove that:*

$\frac{2(s^2-r^2-4Rr)}{sr}\geq\frac{4}{s}(4R+r)\Leftrightarrow s^2\geq 3r(4R+r)$, *which follows from Gerretsen's inequality* $s^2\geq 16Rr-5r^2$. *It remains to prove that:* $16Rr-5r^2\geq 3r(4R+r)\Leftrightarrow R\geq 2r$ *(Euler's inequality). We deduce the inequality from enunciation holds, with equality if and only if the triangle is equilateral.*

Remark.*Let's find an inequality having an opposite sense:*

3) In ΔABC the following relationship holds:

$$\frac{b+c}{r_a}+\frac{c+a}{r_b}+\frac{a+b}{r_c}\leq\frac{2R(4R+r)}{rs}$$

Proposed by Marin Chirciu – Romania

Solution *Using the Lemma we write the inequality:*

$\frac{2(s^2-r^2-4Rr)}{sr} \le \frac{2R(4R+r)}{rs} \Leftrightarrow s^2 \le (4R+r)(R+r)$, *which follows from Gerretsen's inequality* $s^2 \le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$4R^2+4Rr+3r^2 \le (4R+r)(R+r) \Leftrightarrow R \ge 2r$ *(Euler's inequality)*

We deduce that the inequality from enunciation holds, if and only if the triangle is equilateral.

Remark. *We can write the double inequalities:*

4) In ΔABC the following relationship holds:

$$\frac{4}{s}(m_a+m_b+m_c) \le \frac{4}{s}(4R+r) \le \frac{b+c}{r_a}+\frac{c+a}{r_b}+\frac{a+b}{r_c} \le \frac{2R(4R+r)}{rs}$$

Solution *See inequalities 1) and 3).*

Equality holds if and only if the triangle is equilateral.

Remark. *We prove the following lemma:*

Lemma.

6) In ΔABC the following relationship holds:

$$\frac{b+c}{h_a}+\frac{c+a}{h_b}+\frac{a+b}{h_c} = \frac{s^2+r^2+4Rr}{sr}$$

Proof.

Using the formula, $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum \frac{b+c}{h_a} = \sum \frac{b+c}{\frac{2S}{a}} = \frac{1}{2S}\sum a(b+c) = \frac{1}{2sr}\cdot 2\sum bc = \frac{1}{sr}\cdot \sum bc =$$

$= \frac{1}{sr}(s^2+r^2+4Rr) = \frac{s^2+r^2+4Rr}{sr}$, *which follows from the known identity in triangle* $\sum bc = s^2+r^2+4Rr$. *Let's get back to the main problem*

Using the Lemma, we write the inequality

$$\frac{2(s^2-r^2-4Rr)}{sr} \ge \frac{4}{s}(m_a+m_b+m_c)$$

Because $m_a+m_b+m_c \le 4R+r$ *it suffices to prove that:*

$\frac{2(s^2-r^2-4Rr)}{sr} \geq \frac{4}{s}(4R+r) \Leftrightarrow s^2 \geq 3r(4R+r)$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2$. *It remains to prove that:* $16Rr - 5r^2 \geq 3r(4R+r) \Leftrightarrow R \geq 2r$ *(Euler's inequality).We deduce that the inequality from enunciation holds, if and only if the triangle is equilateral.*

Remark. *Let's find an inequality having an opposite sense:*

7) In ΔABC the following relationship holds:

$$\frac{b+c}{h_a} + \frac{c+a}{h_b} + \frac{a+b}{h_c} \leq \frac{2R(4R+r)}{rs}$$

Proposed by Marin Chirciu – Romania

Solution *Using the Lemma the inequality can be written*

$\frac{s^2+r^2+4Rr}{sr} \leq \frac{2R(4R+r)}{rs} \Leftrightarrow s^2 \leq (4R+r)(2R-r)$, *which follows from Gerretsen's inequality* $s^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$4R^2 + 4Rr + 3r^2 \leq (4R+r)(2R-r) \Leftrightarrow 2R^2 - 3R - 2r^2 \geq 0 \Leftrightarrow (R-2r)(2R+r) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

We deduce that the inequality from enunciation holds, if and only if the triangle is equilateral.

Remark.*We can write the following inequalities:*

8) In ΔABC the following relationship holds:

$$\frac{4}{s}(m_a + m_b + m_c) \leq \frac{4}{s}(4R+r) \leq \frac{b+c}{h_a} + \frac{c+a}{h_b} + \frac{a+b}{h_c} \leq \frac{2R(4R+r)}{rs}$$

Solution *See inequalities 1) and 3). Equality holds if and only if the triangle is equilateral.*

Remark *Between the sums* $\sum \frac{b+c}{h_a}$ *and* $\sum \frac{b+c}{r_a}$ *the inequality holds. We can write the inequalities:*

9) In ΔABC the following relationship holds:

$$\sum \frac{b+c}{h_a} \leq \sum \frac{b+c}{r_a}$$

Proposed by Marin Chirciu – Romania

Solution *Using the above Lemmas we write the inequality:*

$\frac{s^2+r^2+4Rr}{sr} \le \frac{2(s^2-r^2-4Rr)}{sr} \Leftrightarrow s^2 \ge 3r(4R+r)$, *which follows from Gerretsen's inequality*

$s^2 \ge 16Rr - 5r^2$. *It remains to prove that:*

$16Rr - 5r^2 \ge 3r(4R+r) \Leftrightarrow R \ge 2r$ *(Euler's inequality)*

We deduce that the inequality from enunciation holds, if and only if the triangle is equilateral.

Remark. *We can write the sequence of inequalities.*

10) In ΔABC the following relationship holds:

$$\frac{12r}{s} \le \frac{4}{s}(m_a+m_b+m_c) \le \frac{4}{s}(4R+r) \le \sum \frac{b+c}{h_a} \le \sum \frac{b+c}{r_a} \le \frac{2R(4R+r)}{rs} \le \frac{9R^2}{S}$$

Proposed by Marin Chirciu – Romania

Solution *See inequalities 8), 9),* $m_a+m_b+m_c \ge 3r$ *and Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Reference:

Romanian Mathematical Magazine-www.ssmrmh.ro