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ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-I

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{1}{R} \le \frac{1}{r_a + r_b} + \frac{1}{r_b + r_c} + \frac{1}{r_c + r_a} \le \frac{R}{4r^2}$$

Proposed by George Apostolopoulos – Greece

Solution

We prove the following identity:

Lemma.

2) In ΔABC the following relatinship holds:

$$\frac{1}{r_a + r_b} + \frac{1}{r_b + r_c} + \frac{1}{r_c + r_a} = \frac{s^2 + (4R + r)^2}{4Rs^2}$$

Proof.

Using the formula $r_a = \frac{S}{s-a}$ we obtain:

$\sum \frac{1}{r_b + r_c} = \sum \frac{1}{\frac{S}{s-b} + \frac{S}{s-c}} = \frac{1}{S} \sum \frac{(s-b)(s-c)}{a} = \frac{1}{rs} \cdot \frac{r[s^2 + (4R+r)^2]}{4Rs} = \frac{s^2 + (4R+r)^2}{4Rrs^2}$, *which follows from the known inequality in triangle:* $\sum \frac{(s-b)(s-c)}{a} = \frac{r[s^2 + (4R+r)^2]}{4Rs}$

Let's get back to the main problem.

LHS inequality.

Using the Lemma can be written:

$\frac{s^2 + (4R+r)^2}{4Rrs^2} \ge \frac{1}{R} \Leftrightarrow (4R + r)^2 \ge 3s^2$, *(Doucet's inequality), which follows from Gerretsen's inequality* $s^2 \le 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$(4R + r)^2 \ge 3(4R^2 + 4Rr + 3r^2) \Leftrightarrow R^2 - Rr - 2r^2 \ge 0 \Leftrightarrow (R - 2r)(R + r) \ge 0$,

obviously from Euler's inequality $R \ge 2r$. Equality holds if and only if the ΔABC is equilateral.

RHS inequality.

Using the Lemma the inequality holds:

$\frac{s^2+(4R+r)^2}{4Rs^2} \le \frac{R}{4r^2} \Leftrightarrow s^2(R^2-r^2) \ge r^2(4R+r)^2$, *which follows from Gerretsen's inequality*

$s^2 \ge 16Rr-5r^2 \ge \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:* $\frac{r(4R+r)^2}{R+r}(R^2-r^2) \ge r^2(4R+r)^2$

$\Leftrightarrow R \ge 2r$. *(Euler's inequality). Equality holds if and only if the triangle is equilateral.*

Remark.

Inequality 1) can be strenghtened:

3) In ΔABC the following relationship holds:

$$\frac{1}{4R}\left(5-\frac{2r}{R}\right) \le \frac{1}{r_a+r_b}+\frac{1}{r_b+r_c}+\frac{1}{r_c+r_a} \le \frac{1}{2r}$$

Proposed by Marin Chirciu – Romania

Solution

Using the Lemma the inequality can be written:

$\frac{s^2+(4R+r)^2}{4Rs^2} \ge \frac{1}{4R}\left(5-\frac{2r}{R}\right) \Leftrightarrow s^2 \le \frac{R(4R+r)^2}{2(2R-r)}$ *(Blundon-Gerretsen's inequality)*

Equality holds if and only if the triangle is equilateral.

RHS inequality.

Using Lemma the inequality can be written:

$\frac{s^2+(4R+r)^2}{4Rs^2} \le \frac{1}{2r} \Leftrightarrow s^2 \ge \frac{r(4R+r)^2}{2R-r}$, *which follows from Gerretsen's inequality*

$s^2 \ge 16Rr-5r^2 \ge \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:* $\frac{r(4R+r)^2}{R+r} \ge \frac{r(4R+r)^2}{2R-r} \Leftrightarrow R \ge 2r$

(Euler's inequality). Equality holds if and only if the triangle is equilateral.

Remark.

Inequality 3) is stronger than inequality 1).

4) In ΔABC the following inequality holds:

$$\frac{1}{R} \le \frac{1}{4R}\left(5-\frac{2r}{R}\right) \le \frac{1}{r_a+r_b}+\frac{1}{r_b+r_c}+\frac{1}{r_c+r_a} \le \frac{1}{2r} \le \frac{R}{4r^2}$$

Solution

See 3) and Euler's inequality $R \ge 2r$. Equality holds if and only the ΔABC is equilateral.

Remark.

If we replace r_a with h_a we propose:

5) In ΔABC the following relationship holds:

$$\frac{1}{R} \le \frac{1}{h_a + h_b} + \frac{1}{h_b + h_c} + \frac{1}{h_c + h_a} \le \frac{1}{2r}$$

Proposed by Marin Chirciu – Romania

Solution

We prove the followign lemma:

Lemma.

6) In ΔABC the following relationship holds:

$$\frac{1}{h_a + h_b} + \frac{1}{h_b + h_c} + \frac{1}{h_c + h_a} = \frac{s^4 + s^2(16Rr + 2r^2) + r^2(4R + r)^2}{4rs^2(s^2 + 2r^2 + 2Rr)}$$

Proof.

Using the formula $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum \frac{1}{h_b + h_c} = \sum \frac{1}{\frac{2S}{b} + \frac{2S}{c}} = \frac{1}{2S} \sum \frac{bc}{b + c} = \frac{1}{2rs} \cdot \frac{s^4 + s^2(16Rr + 2r^2) + r^2(4R + r)^2}{2s(s^2 + 2r^2 + 2Rr)} =$$

$$= \frac{s^4 + s^2(16Rr + 2r^2) + r^2(4R + r)^2}{4rs^2(s^2 + 2r^2 + 2Rr)},$$ *which follows from the known identity in triangle:*

$$\sum \frac{bc}{b + c} = \frac{s^4 + s^2(16Rr + 2r^2) + r^2(4R + r)^2}{2s(s^2 + 2r^2 + 2Rr)}$$

Let's get back to the main problem.

LHS

Using the Lemma we write the inequality:

$$\frac{s^4 + s^2(16Rr + 2r^2) + r^2(4R + r)^2}{4rs^2(s^2 + 2r^2 + 2Rr)} \ge \frac{1}{R} \Leftrightarrow$$

$$\Leftrightarrow s^2[s^2(R - 4r) + r(16R^2 - 6Rr - 4r^2)] + Rr^2(4R + r)^2 \ge 0$$

We distinguish the following cases:

Case 1). If $[s^2(R - 4r) + r(16R^2 - 6Rr - 4r^2)] \ge 0$, *the inequality is obviously.*

Case 2). If $[s^2(R - 4r) + r(16R^2 - 6Rr - 4r^2)] < 0$, *we write the inequality:*

$Rr^2(4R+r)^2 \geq s^2[s^2(4r-R)-r(16R^2-6Rr-4r^2)]$, *which follows from Blundon – Gerretsen's inequality* $s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} \leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$Rr^2(4R+r)^2 \geq \frac{R(4R+r)^2}{2(2R-r)}\left[(4R^2+4Rr+3r^2)(4r-R)-r(16R^2-6Rr-4r^2)\right] \Leftrightarrow$$

$\Leftrightarrow 4R^3+4R^2r-15Rr^2-18r^3 \geq 0 \Leftrightarrow (R-2r)(2R+r)^2 \geq 0$, *obviously from Euler's inequality* $R \geq 2r$. *Equality holds if and only if* ΔABC *is equilateral.*

RHS inequality

Using the Lemma the inequality holds:

$\frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{4rs^2(s^2+2r^2+2Rr)} \leq \frac{1}{2r} \Leftrightarrow s^2(s^2-12Rr) \geq r^2(4R+r)^2$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr-5r^2 \geq \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:*

$\frac{r(4R+r)^2}{R+r}(16Rr-5r^2-12Rr) \geq r^2(4R+r)^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality)*

Equality holds if and only if ΔABC *is equilateral.*

Remark.

Between the sums $\sum\frac{1}{r_b+r_c}$ *and* $\sum\frac{1}{h_b+h_c}$ *the following inequality holds:*

7) In ΔABC the following relationship holds:

$$\sum\frac{1}{r_b+r_c} \leq \sum\frac{1}{h_b+h_c}$$

Proposed by Marin Chirciu – Romania

Solution

Using the following Lemma the above inequality holds:

$$\frac{s^2+(4R+r)^2}{4Rs^2} \leq \frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{4rs^2(s^2+2r^2+2Rr)} \Leftrightarrow$$

$\Leftrightarrow s^2[s^2(R-r)-2r^2(4R+r)] \geq r^2(4R+r)^2(R+r)$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr-5r^2 \geq \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:*

$$\frac{r(4R+r)^2}{R+r}\left[(16Rr-5r^2)(R-r)-2r^2(4R+r)\right] \geq r^2(4R+r)^2(R+r) \Leftrightarrow$$

$\Leftrightarrow 15R^2 - 31Rr + 2r^2 \geq 0 \Leftrightarrow (R-2r)(15R-r) \geq 0$, *obviously from Euler's inequality*

$R \geq 2r$. *Equality holds if and only if the triangle is equilateral.*

Remark.

We can write the inequalities:

8) In ΔABC the following relationship holds:

$$\frac{1}{R} \leq \sum \frac{1}{r_b + r_c} \leq \sum \frac{1}{h_b + h_c} \leq \frac{1}{2r}$$

Solution

See inequalities 1), 7) and 5).

Equality holds if and only if ABC triangle is equilateral.

Reference:

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