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ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-II

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$2 \leq \sum \frac{1}{\sin^2 A + \sin B \sin C} \leq 2\left(\frac{R}{2r}\right)^2$$

Proposed by George Apostolopoulos – Greece

Solution

LHS inequality. Using Bergström's inequality, we obtain:

$$\sum \frac{1}{\sin^2 A + \sin B + \sin C} \geq \frac{9}{\sum(\sin^2 A + \sin B \sin C)} = \frac{9}{\sum \sin^2 A + \sum \sin B \sin C} =$$

$$= \frac{9}{\frac{s^2-r^2-4Rr}{2R^2} + \frac{s^2+r^2+4Rr}{4R^2}} = \frac{9}{\frac{3s^2-r^2-4Rr}{4R^2}} = \frac{36R^2}{3s^2-r^2-4Rr} \overset{(1)}{\geq} 2, \text{ where (1)} \Leftrightarrow$$

$\Leftrightarrow 18R^2 \geq 3s^2 - r^2 - 4Rr$, *it follows from Gerretsen's inequality* $s^2 \leq 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$\Leftrightarrow 18R^2 \geq 3(4R^2 + 4Rr + 3r^2) - r^2 - 4Rr \Leftrightarrow 3R^2 - 4Rr - 4r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R - 2r)(3R + 2r) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Above we've used the known inequalities in triangle:

$$\sum \sin^2 A = \frac{s^2-r^2-4Rr}{2R^2} \text{ and } \sum \sin B \sin C = \frac{s^2+r^2+4Rr}{4R^2}$$

Equality holds if and only if the triangle is equileral.

RHS inequality.

Using means inequalities: $\sin^2 A + \sin B \sin C \geq 2\sqrt{\sin^2 A \sin B \sin C}$, *we obtain:*

$$\sum \frac{1}{\sin^2 A + \sin B \sin C} \leq \sum \frac{1}{2\sqrt{\sin^2 A \sin B \sin C}} = \frac{1}{2\sqrt{\prod \sin A}} \sum \frac{1}{\sqrt{\sin A}} \overset{(1)}{\leq} 2\left(\frac{R}{2r}\right)^2$$

where (1) $\Leftrightarrow \frac{1}{\sqrt{\prod \sin A}} \sum \frac{1}{\sqrt{\sin A}} \leq \left(\frac{R}{r}\right)^2$, *which follows from:*

CBS inequality: $\sum \frac{1}{\sqrt{\sin A}} \leq \sqrt{3 \sum \frac{1}{\sin A}}$ *and the identites* $\sum \frac{1}{\sin A} = \frac{s^2+r^2+4Rr}{2rs}, \prod \sin A = \frac{rs}{2R^2}$

We obtain $\frac{1}{\sqrt{\prod \sin A}} \sum \frac{1}{\sqrt{\sin A}} \leq \frac{1}{\sqrt{\frac{rs}{2R^2}}} \sqrt{3 \cdot \frac{s^2+r^2+4Rr}{2rs}} = \frac{R}{rs}\sqrt{3(s^2 + r^2 + 4Rr)} \overset{(2)}{\leq} \left(\frac{R}{r}\right)^2$, *where (2)*

$$\Leftrightarrow r\sqrt{3(s^2+r^2+4Rr)} \le sR \Leftrightarrow 3r^2(s^2+r^2+4Rr) \le s^2R^2 \Leftrightarrow$$

$\Leftrightarrow s^2(R^2-3r^2) \ge 3r^3(4R+r)$, *which follows from Gerretsen's inequality*

$s^2 \ge 16Rr-5r^2$. *It remains to prove:*

$$(16Rr-5r^2)(R^2-3r^2) \ge 3r^3(4R+r) \Leftrightarrow 16R^3-5R^2r-60Rr^2+12r^3 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(16R^2+27Rr-6r^2) \ge 0$, *obviously from Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.

We propose in the same way:

2) In ΔABC the following relationship holds:

$$\sum \frac{1}{\cos^2 A+\cos B\cos C} \ge \frac{12r}{R}$$

Proposed by Marin Chirciu – Romania

Solution

Using Bergström's inequality we obtain:

$$\sum \frac{1}{\cos^2 A+\cos B\cos C} \ge \frac{9}{\sum(\cos^2 A+\cos B\cos C)} = \frac{9}{\sum\cos^2 A+\sum\cos B\cos C} =$$

$$= \frac{9}{\frac{6R^2+4Rr+r^2-s^2}{2R^2}+\frac{s^2+r^2-4R^2}{4R^2}} = \frac{9}{\frac{8R^2+8Rr+3r^2-s^2}{4R^2}} = \frac{36R^2}{8R^2+8Rr+3r^2-s^2} \overset{(1)}{\ge} \frac{12r}{R} \text{ where (1)} \Leftrightarrow$$

$\Leftrightarrow 36R^3 \ge 96R^2r+96Rr^2+36r^3-12s^2r$, *which follows from Gerretsen's inequality*

$s^2 \ge 16Rr-5r^2$. *It remains to prove that:*

$$36R^3 \ge 96R^2r+96Rr^2+36r^3-12r(16Rr-5r^2) \Leftrightarrow$$

$\Leftrightarrow 3R^3-8R^2r+8Rr^2-8r^3 \ge 0 \Leftrightarrow (R-2r)(3R^2-2Rr+4r^2) \ge 0$, *obviously Euler's inequality. Above we have used the known inequalities in triangle:*

$\sum\cos^2 A = \frac{6R^2+4Rr+r^2-s^2}{2R^2}$ *and* $\sum\cos B\cos C = \frac{s^2+r^2-4R^2}{4R^2}$.

Equality holds if and only if the triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\frac{1}{2R^2} \le \sum \frac{1}{a^2+bc} \le \frac{1}{8r^2}$$

LHS inequality. Using Bergström's inequality, we obtain:

$$\sum \frac{1}{a^2+bc} \geq \frac{9}{\sum(a^2+bc)} = \frac{9}{\sum a^2+\sum bc} = \frac{9}{2(s^2-r^2-4Rr)+s^2+r^2+4Rr} =$$

$= \frac{9}{3s^2-r^2-4Rr} \overset{(1)}{\geq} \frac{1}{2R^2}$, *where (1)* $\Leftrightarrow 18R^2 \geq 3s^2-r^2-4Rr$, *which follows from Gerrtsen's inequality* $s^2 \leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$18R^2 \geq 3(4R^2+4Rr+3r^2)-r^2-4Rr \Leftrightarrow 3R^2-4Rr-4r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(3R+2r) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Above we have used the known inequalities in triangle:

$\sum a^2 = 2(s^2-r^2-4Rr)$ *and* $\sum bc = s^2+r^2+4Rr$.

The equality holds if and only if the triangle is equilateral.

RHS inequality. Using the means inequality: $a^2+bc \geq 2\sqrt{a^2bc}$, *we obtain:*

$$\sum \frac{1}{a^2+bc} \leq \sum \frac{1}{2\sqrt{a^2bc}} = \frac{1}{2\sqrt{\prod a}} \sum \frac{1}{\sqrt{a}} \overset{(1)}{\leq} \frac{1}{8r^2}$$

where (1) $\Leftrightarrow \frac{1}{\sqrt{\prod a}} \sum \frac{1}{\sqrt{a}} \leq \left(\frac{1}{2r}\right)^2$, *which follows from CBS inequality* $\sum \frac{1}{\sqrt{a}} \leq \sqrt{3\sum \frac{1}{a}}$ *and the identities* $\sum \frac{1}{a} = \frac{s^2+r^2+4Rr}{4Rrs}$, $\prod a = 4Rrs$. *We obtain*

$\frac{1}{\sqrt{\prod a}} \sum \frac{1}{\sqrt{a}} \leq \frac{1}{\sqrt{4Rrs}} \sqrt{3 \cdot \frac{s^2+r^2+4Rr}{4Rrs}} = \frac{1}{4Rrs}\sqrt{3(s^2+r^2+4Rr)} \overset{(2)}{\leq} \left(\frac{1}{2r}\right)^2$, *where (2)*

$$\Leftrightarrow r\sqrt{3(s^2+r^2+4Rr)} \leq sR \Leftrightarrow 3r^2(s^2+r^2+4Rr) \leq s^2R^2 \Leftrightarrow$$

$\Leftrightarrow s^2(R^2-3r^2) \geq 3r^3(4R+r)$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr-5r^2$. *It remains to prove that:*

$$(16Rr-5r^2)(R^2-3r^2) \geq 3r^3(4R+r) \Leftrightarrow 16R^3-5R^2r-60Rr^2+12r^3 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(16R^2+27Rr-6r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

4) In ΔABC the following inequality holds:

$$\frac{9r}{R} \leq \sum \frac{1}{\tan^2\frac{A}{2}+\tan\frac{B}{2}\tan\frac{C}{2}} \leq \frac{9R}{4r}$$

Proposed by Marin Chirciu – Romania

Solution

LHS inequality. Using Bergström's inequality we obtain:

$$\sum\frac{1}{\tan^2\frac{A}{2}+\tan\frac{B}{2}\tan\frac{C}{2}}\ge\frac{9}{\sum\left(\tan^2\frac{A}{2}+\tan\frac{B}{2}\tan\frac{C}{2}\right)}=\frac{9}{\sum\tan^2\frac{A}{2}+\sum\tan\frac{B}{2}\tan\frac{C}{2}}=$$

$=\frac{9}{\frac{(4R+r)^2-2s^2}{s^2}+1}=\frac{9}{(4R+r)^2-s^2}\overset{(1)}{\ge}\frac{9r}{R}$, *where (1)* $\Leftrightarrow s^2(R+r)\ge r(4R+r)^2$, *which follows*

from Gerretsen's inequality $s^2\le 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$.

Above we've used the known inequaity in triangle:

$$\sum\tan^2\frac{A}{2}=\frac{(4R+r)^2-2s^2}{s^2}\text{ and }\sum\tan\frac{B}{2}\tan\frac{C}{2}=1.$$

Equality holds if and only if the triangle is equilateral.

RHS inequality. Using the means inequality: $\tan^2\frac{A}{2}+\tan\frac{B}{2}\tan\frac{C}{2}\ge 2\sqrt{\tan^2\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}$, *we*

obtain: $\sum\frac{1}{\tan^2\frac{A}{2}+\tan\frac{B}{2}\tan\frac{C}{2}}\le\sum\frac{1}{2\sqrt{\tan^2\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}}=\frac{1}{2\sqrt{\prod\tan\frac{A}{2}}}\sum\frac{1}{\sqrt{\tan\frac{A}{2}}}\overset{(1)}{\le}\frac{9R}{4r}$, *where (1)*

$\Leftrightarrow\frac{1}{\sqrt{\prod\tan\frac{A}{2}}}\sum\frac{1}{\sqrt{\tan\frac{A}{2}}}\le\frac{9R}{2r}$, *which follows from CBS inequalit:* $\sum\frac{1}{\sqrt{\tan\frac{A}{2}}}\le\sqrt{3\sum\frac{1}{\tan\frac{A}{2}}}$ *and the*

identites $\sum\frac{1}{\tan\frac{A}{2}}=\frac{s}{r},\prod\tan\frac{A}{2}=\frac{r}{s}$. *We obtain* $\frac{1}{\sqrt{\prod\tan\frac{A}{2}}}\sum\frac{1}{\sqrt{\tan\frac{A}{2}}}\le\frac{1}{\sqrt{\frac{r}{s}}}\sqrt{3\cdot\frac{s}{r}}=\frac{s}{r}\sqrt{3}\overset{(2)}{\le}\frac{9R}{2r}$, *where*

(2) $\Leftrightarrow s\le\frac{R\sqrt{3}}{2}$ *(Mitrinovic's inequality)*

Equality holds if and only if the triangle is equilateral.

5) In ΔABC the following inequality holds:

$$2\left(\frac{r}{R}\right)^2\le\sum\frac{1}{\cot^2\frac{A}{2}+\cot\frac{B}{2}\cot\frac{C}{2}}\le\frac{R}{4r}$$

Proposed by Marin Chirciu – Romania

Solution

LHS inequality. Using Bergström's inequality obtain:

$$\sum \frac{1}{\cot^2\frac{A}{2}+\cot\frac{B}{2}\cot\frac{C}{2}} \ge \frac{9}{\sum\left(\cot^2\frac{A}{2}+\cot\frac{B}{2}\cot\frac{C}{2}\right)} = \frac{9}{\sum\cot^2\frac{A}{2}+\sum\cot\frac{B}{2}\cot\frac{C}{2}} =$$

$= \frac{9}{\frac{s^2-2r^2-8Rr}{r^2}+\frac{4R+r}{r}} = \frac{9r^2}{s^2-r^2-4Rr} \overset{(1)}{\ge} \frac{2r^2}{R^2}$, *where (1)* $\Leftrightarrow 2s^2 \le 9R^2+8Rr+2r^2$, *which follows from Gerretsen's inequality* $s^2 \le 4R^2+4Rr+3r^2$. *It remains to prove:*

$2(4R^2+4Rr+3r^2) \le 9R^2+8Rr+2r^2 \Leftrightarrow R^2 \ge 4r^2$, *obviously from Euler's inequality* $R \ge 2r$. *Above we've used the known inequalities in triangle:*

$\sum\cot^2\frac{A}{2} = \frac{s^2-2r^2-8Rr}{r^2}$ *and* $\sum\cot\frac{B}{2}\cot\frac{C}{2} = \frac{4R+r}{r}$. *Equality holds if and only if the triangle is equilateral.*

RHS inequality. Using the means inequality $\cot^2\frac{A}{2}+\cot\frac{B}{2}\cot\frac{C}{2} \ge 2\sqrt{\cot^2\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}$ *we obtain that:* $\sum\frac{1}{\cot^2\frac{A}{2}+\cot\frac{B}{2}\cot\frac{C}{2}} \le \sum\frac{1}{2\sqrt{\cot^2\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}} = \frac{1}{2\sqrt{\prod\cot\frac{A}{2}}}\sum\frac{1}{\sqrt{\cot\frac{A}{2}}} \overset{(1)}{\le} \frac{R}{4r}$ *where (1)*

$\Leftrightarrow \frac{1}{\sqrt{\prod\tan\frac{A}{2}}}\sum\frac{1}{\sqrt{\tan\frac{A}{2}}} \le \frac{R}{2r}$, *which follows from CBS inequality* $\sum\frac{1}{\sqrt{\cot\frac{A}{2}}} \le \sqrt{3\sum\frac{1}{\cot\frac{A}{2}}}$ *and the identites* $\sum\frac{1}{\cot\frac{A}{2}} = \frac{4R+r}{s}$ $\prod\cot\frac{A}{2} = \frac{s}{r}$. *We obtain*

$$\frac{1}{\sqrt{\prod\cot\frac{A}{2}}}\sum\frac{1}{\sqrt{\cot\frac{A}{2}}} \le \frac{1}{\sqrt{\frac{s}{r}}}\sqrt{3\cdot\frac{4R+r}{s}} = \sqrt{3\cdot\frac{r(4R+r)}{s^2}} \overset{(2)}{\le} \frac{R}{2r},\ \textit{where (2)}$$

$\Leftrightarrow s^2R^2 \ge 12r^3(4R+r)$, *which follows from Gerretsen's inequality* $s^2 \ge 16Rr-5r^2$. *It remains to prove that:*

$$(16Rr-5r^2)R^2 \ge 12r^3(4R+r) \Leftrightarrow 16R^3-5R^2r-48Rr^2-12r^3 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(16R^2+27Rr+6r^2) \ge 0$, *obviously from Euler's inequality* $R \ge 2r$. *Equality holds if and only if the triangle is equilateral.*

Reference:

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