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# ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-III

***By Marin Chirciu – Romania***

**Let $a, b, c$ be the lengths of the sides of $ABC$ triangle with inradius $r$ and circumradius $R$. Show that:**

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)}$$

***Proposed by George Apostolopoulos –Messolonghi- Greece***

***Solution***

*We prove the following lemma:*

***Lemma***

***1) In $\Delta ABC$ the following relationship holds:***

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

***Proof.***

*We have* $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=\frac{\sum a^2(a+b)(a+c)}{\prod(b+c)}=\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$*, which follows from the known identities in triangle* $\sum a^2(a+b)(a+c)\ =4s(s^2-3r^2-4Rr)$ *and*

$$\prod(b+c)\ =2s(s^2+r^2+2Rr)$$

*Let's get back to the main problem.*

*Using the Lemma, we write the inequality:* $\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}\le\frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)}$

*Using Mitrinovic's inequality* $s\le\frac{3R\sqrt{3}}{2}$ *it suffices to prove that:*

$$\frac{3R\sqrt{3}(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}\le\frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)}\Leftrightarrow\frac{s^2-3r^2-4Rr}{s^2+r^2+2Rr}\le\frac{1}{4r}\sqrt{2R(R-r)}\Leftrightarrow$$

$$\Leftrightarrow\left(\frac{s^2-3r^2-4Rr}{s^2+r^2+2Rr}\right)^2\le\frac{2R(R-r)}{16r^2}\Leftrightarrow$$

$$\Leftrightarrow s^2[s^2(R^2-Rr-8r^2)+r(4R^3-2R^2r+62Rr^2+48r^3)]+$$

$$+r^2(4R^4-128R^3r-192R^2r^2-192Rr^3-72r^4)\ge0$$

*We distinguish the following cases:*

*Case 1). If* $[s^2(R^2 - Rr - 8r^2) + r(4R^3 - 2R^2r + 62Rr^2 + 48r^3)] \geq 0$, *the inequality is obvious.*

*Case 2). If* $[s^2(R^2 - Rr - 8r^2) + r(4R^3 - 2R^2r + 62Rr^2 + 48r^3)] < 0$, *the inequality rewrites its self:*

$$r^2(4R^4 - 128R^3r - 192R^2r^2 - 192Rr^3 - 72r^4) \geq$$
$$\geq s^2[s^2(8r^2 + Rr - R^2) - r(4R^3 - 2R^2r + 62Rr^2 + 48r^3)]$$

*which follows from Gerretsen's inequality* $s^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$r^2(4R^4 - 128R^3r - 192R^2r^2 - 192Rr^3 - 72r^4) \geq$$
$$\geq (4R^2 + 4Rr + 3r^2)[(4R^2 + 4Rr + 3r^2)(8r^2 + Rr - r^2) - r(4R^3 - 2R^2r + 62Rr^2 + 48r^3)] \Leftrightarrow$$
$$\Leftrightarrow (R - 2r)(4R^4 + 16R^3r + 5R^2r^2 + 5Rr^3 + 2r^5) \geq 0,$$ *obviously with Euler's inequality* $R \geq 2r$. *Equality holds if and only if the triangle is equilateral.*

***Remark.*** *We can strengthen the inequality:*

## *2) In ΔABC the following relationship holds:*

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \leq \frac{3R^2\sqrt{3}}{4r}$$

***Proposed by Marin Chirciu – Romania***

***Solution*** *Using the Lemma the inequality rewrites:*

$$\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr} \leq \frac{3R^2\sqrt{3}}{4r}.$$

*Using Mitrinovic's inequality* $s \leq \frac{3R\sqrt{3}}{2}$ *it suffices to prove that:*

$$\frac{3R\sqrt{3}(s^2 - 3r^2 - 4Rr)}{s^2 + r^2 + 2Rr} \leq \frac{3R^2\sqrt{3}}{4r} \Leftrightarrow \frac{s^2 - 3r^2 - 4Rr}{s^2 + r^2 + 2Rr} \leq \frac{R}{4r} \Leftrightarrow$$
$$\Leftrightarrow s^2(R - 4r) + r(2R^2 + 17Rr + 12r^2) \geq 0$$

*We distinguish the following cases:*

*Case 1) If* $R - 4r \geq 0$, *inequality is obvious.*

*Case 2). If* $R - 4r < 0$, *the inequality rewrites itself:*

$r(2R^2 + 17Rr + 12r^2) \geq s^2(4r - R)$ *which follows from Gerretsen's inequality:*

$s^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$r(2R^2+17Rr+12r^2)\geq(4R^2+4Rr+3r^2)(4r-R)\Leftrightarrow 2R^2-5Rr+2r^2\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(2R-r)\geq 0$, *obviously from Euler's inequality* $r\geq 2r$.

*Equality holds if and only if the triangle is equilateral.*

**Remark.** *Inequality 2) is stronger than the inequality from Problem 4462 from Crux Mathematicorum, Vol 45, Nr. 7.*

### 3) In $\Delta ABC$ the following relationship holds:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\leq\frac{3R^2\sqrt{3}}{4r}\leq\frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)}$$

***Solution***

*See inequality 2) and* $\frac{3R^2\sqrt{3}}{4r}\leq\frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)}\Leftrightarrow R\leq\sqrt{2R(R-r)}\Leftrightarrow R^2\leq 2R(R-r)\Leftrightarrow$

$\Leftrightarrow R\geq 2r$ *(Euler's inequality). Equality holds if and only if the triangle is equilateral.*

**Remark.** *Let's find an inequality having an opposite sense:*

### 4) In $\Delta ABC$ the following relationship holds:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq s$$

***Solution*** *Using Lemma we write the inequality:* $\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}\geq s\Leftrightarrow s^2\geq 10Rr+7r^2$,

*which follows from Gerretsen's inequality* $s^2\geq 16Rr-5r^2$. *It remains to prove that:*

$$16Rr-5r^2\geq 10Rr+7r^2\Leftrightarrow R\geq 2r\ \text{(Euler's inequality)}$$

*Equality holds if and only if the triangle is equilateral.*

**Remark.** *We write the double inequality:*

### 5) In $\Delta ABC$ the following relationship holds:

$$s\leq\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\leq\frac{3R^2\sqrt{3}}{4r}$$

***Solution*** *See inequalities 2) and 4)*

*Equality holds if and only if the triangle is equilateral.*

**Reference:**

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