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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-II

By Marin Chirciu – Romania

1) Prove that in any acute-angled triangle the following inequality holds:

$$\frac{a^6}{w_b w_c}+\frac{b^6}{w_c w_a}+\frac{c^6}{w_a w_b}\geq [4R(R+r)]^2$$

Proposed by Marian Ursărescu – Romania

Solution. *Using Bergström's inequality we obtain:*

$\sum\frac{a^6}{w_b w_c}=\sum\frac{(a^3)^2}{w_b w_c}\geq\frac{\left(\sum a^3\right)^2}{\sum w_b w_c}\geq\frac{\left(\sum a^3\right)^2}{s^2}$, *which follows from* $\sum w_b w_c\leq\sum w_a^2\leq\sum s(s-a)=s^2$

It suffices to prove that: $\frac{\left(\sum a^3\right)^2}{s^2}\geq[4R(R+r)]^2\Leftrightarrow\frac{\sum a^3}{s}\geq 4R(4R+r)\Leftrightarrow$

$\Leftrightarrow\frac{2s(s^2+r^2+4Rr)}{s}\geq 4R(4R+r)\Leftrightarrow s^2\geq 2R^2+8Rr+3r^2$ *(Walker's inequality, acute)*

Equality holds if and only if the triangle is equilateral.

Remark.

In the same way we propose:

2) Prove that in any acute-angled triangle the following inequality holds:

$$\frac{a^6}{r_b r_c}+\frac{b^6}{r_c r_a}+\frac{c^6}{r_a r_b}\geq [4R(R+r)]^2$$

Proposed by Marin Chirciu – Romania

Solution. *Using Bergström's inequality we obtain:*

$\sum\frac{a^6}{r_b r_c}=\sum\frac{(a^3)^2}{r_b r_c}\geq\frac{\left(\sum a^3\right)^2}{\sum r_b r_c}=\frac{\left(\sum a^3\right)^2}{s^2}$, *which follows from* $\sum r_b r_c=s^2$.

It suffices to prove that: $\frac{\left(\sum a^3\right)^2}{s^2}\geq[4R(R+r)]^2\Leftrightarrow\frac{\sum a^3}{s}\geq 4R(4R+r)\Leftrightarrow$

$\Leftrightarrow\frac{2s(s^2+r^2+4Rr)}{s}\geq 4R(4R+r)\Leftrightarrow s^2\geq 2R^2+8Rr+3r^2$ *(Walker's inequality, acute)*

Equality holds if and only if the triangle is equilateral.

3) Prove that in any acute-angled triangle the following inequality holds:

$$\frac{a^6}{h_bh_c}+\frac{b^6}{h_ch_a}+\frac{c^6}{h_ah_b}\ge[4R(R+r)]^2$$

Solution. *Using Bergström's inequality we obtain:*

$$\sum\frac{a^6}{h_bh_c}=\sum\frac{(a^3)^2}{h_bh_c}\ge\frac{(\sum a^3)^2}{\sum h_bh_c}\ge\frac{(\sum a^3)^2}{s^2}, \text{ which follows from}$$

$$\sum h_bh_c\le\sum w_bw_c\le\sum w_a^2\le\sum s(s-a)=s^2$$

It suffices to prove that: $\frac{(\sum a^3)^2}{s^2}\ge[4R(R+r)]^2\Leftrightarrow\frac{\sum a^3}{s}\ge 4R(4R+r)\Leftrightarrow$

$\Leftrightarrow\frac{2s(s^2+r^2+4Rr)}{s}\ge 4R(4R+r)\Leftrightarrow s^2\ge 2R^2+8Rr+3r^2$ *(Walker's inequality, acute)*

Equality holds if and only if the triangle is equilateral.

4) In ΔABC the following inequality holds:

$$\frac{a^6}{m_bm_c}+\frac{b^6}{m_cm_a}+\frac{c^6}{m_am_b}\ge\frac{1}{3}(8S)^2$$

Proposed by Marin Chirciu – Romania

Solution. *Using Bergström's inequality we obtain:*

$$\sum\frac{a^6}{m_bm_c}=\sum\frac{(a^3)^2}{m_bm_c}\ge\frac{(\sum a^3)^2}{\sum m_bm_c}\ge\frac{(\sum a^3)^2}{27R^2}, \text{ which follows from}$$

$$\sum m_bm_c\le\sum m_a^2=\frac{3}{4}\sum a^2\le\frac{3}{4}\cdot 9R^2=\frac{27R^2}{4}$$

It suffices to prove that: $\frac{(\sum a^3)^2}{\frac{27R^2}{4}}\ge\frac{1}{3}(8S)^2\Leftrightarrow\frac{\sum a^3}{\frac{3R}{2}}\ge 8S\Leftrightarrow\sum a^3\ge 12SR\Leftrightarrow$

$\Leftrightarrow 2s(s^2-3r^2-6Rr)\ge 12Rrs\Leftrightarrow s^2\ge 12Rr+3r^2$, *which follows from Gerretsen's inequality* $s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$$16Rr-5r^2\ge 12Rr+3r^2\Leftrightarrow R\ge 2r \text{ (Euler's inequality)}$$

Equality holds if and only if the triangle is equilateral.

Reference:

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