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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-III

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$108\sum \sin^2 A\cot B\cot C\le\left[2\left(\frac{R}{r}\right)^2+1\right]^2$$

Proposed by Marian Ursărescu – Romania

Solution

We prove the following lemma:

Lemma.

2) In ΔABC the following relationship holds:

$$\sum \sin^2 A\cot B\cot C=\frac{6R^2+4Rr+r^2-s^2}{2R^2}$$

Proof.

We have $\sum \sin^2 A\cot B\cot C=\sum(1-\cos^2 A)\cot B\cot C=$

$$=\sum \cot B\cot C-\cos^2 A\cot B\cot C=1-\prod\cos A\sum\frac{\cos A}{\sin B\sin C}=$$

$$=1-\prod\cos A\cdot\frac{1}{\prod\sin A}\sum\sin A\cos A=1-\prod\cot A\sum\frac{1}{2}\sin 2A=$$

$$=1-\frac{s^2-(2R+r)^2}{2sr}\cdot\frac{1}{2}\cdot\frac{2sr}{R^2}=1-\frac{s^2-(2R+r)^2}{2R^2}=\frac{2R^2+(2R+r)^2-s^2}{2R^2}=$$

$=\frac{6R^2+4Rr+r^2-s^2}{2R^2}$, *where above we've used the known inequalities in triangle:*

$\sum\cot B\cot C=1,\prod\cot A=\frac{s^2-(2R+r)^2}{2sr}$ *and* $\sum\sin 2A=\frac{2sr}{R^2}$.

Let's get back to the main problem.

Using the Lemma the inequality can be written:

$108\cdot\frac{6R^2+4Rr+r^2-s^2}{2R^2}\le\left[2\left(\frac{R}{r}\right)^2+1\right]^2\Leftrightarrow\frac{54(6R^2+4Rr+r^2-s^2)}{R^2}\le\frac{(2R^2+r^2)^2}{r^4}$, *which follows from Gerretsen's inequality:* $s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$$\frac{54(6R^2+4Rr+r^2-16Rr+5r^2)}{R^2}\le\frac{(2R^2+r^2)^2}{r^4}\Leftrightarrow$$

$$\Leftrightarrow 4R^6+4R^4r^2-323R^2r^4+648R^2r^4-324r^6\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(4R^5+8R^4r+20R^3r^2+40R^2r^3-243Rr^4+162r^5)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$. *Equality holds if and only if the triangle is equilateral.*

Remark.

Inequality 1) can be strengthened:

3) In ΔABC the following relationship holds:

$$\sum \sin^2 A\cot B\cot C\le 3\left(1-\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution

Using the Lemma and Gerretsen's inequality $s^2\ge 16Rr-5r^2$, *we obtain:*

$$\frac{6R^2+4Rr+r^2-s^2}{2R^2}\le\frac{6R^2+4Rr+r^2-(16Rr-5r^2)}{2R^2}=\frac{6R^2-12Rr+6r^2}{2R^2}=$$

$=\frac{6(R-r)^2}{2R^2}=3\left(1-\frac{r}{R}\right)^2$. *Equality holds if and only if the triangle is equilateral.*

Remark.

Inequality 3) is stronger than 1).

4) In ΔABC the following relationship holds:

$$\sum \sin^2 A\cot B\cot C\le 3\left(1-\frac{r}{R}\right)^2\le\frac{1}{108}\left[2\left(\frac{R}{r}\right)^2+1\right]^2$$

Solution

See inequality 3) and $3\left(1-\frac{r}{R}\right)^2\le\frac{1}{108}\left[2\left(\frac{R}{r}\right)^2+1\right]^2\Leftrightarrow 324r^4(R-r)^2\le R^2(2R^2+r^2)^2$

$$\Leftrightarrow 18r^2(R-r)\le R(2R^2+r^2)\Leftrightarrow 2R^3-17Rr^2+18r^3\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(2R^2+4Rr-9r^2)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.

Let's find an inequality having an opposite sense:

5) In ΔABC the following inequality holds:

$$\sum \sin^2 A \cot B \cot C \geq 1 - \left(\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution

Using the Lemma and Gerretsen's inequality: $s^2 \leq 4R^2 + 4Rr + 3r^2$ *we obtain:*

$$\frac{6R^2 + 4Rr + r^2 - s^2}{2R^2} \geq \frac{6R^2 + 4Rr + r^2 - (4R^2 + 4Rr + 3r^2)}{2R^2} = \frac{2R^2 - 2r^2}{2R^2} = 1 - \left(\frac{r}{R}\right)^2$$

Equality holds if and only if the triangle is equilateral.

Remark.

We can write the double inequality:

6) In ΔABC the following relationship holds:

$$1 - \left(\frac{r}{R}\right)^2 \leq \sum \sin^2 A \cot B \cot C \leq 3\left(1 - \frac{r}{R}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution

See inequalities 3) and 5).

Equality holds if and only if the triangle is equilateral.

Reference:
Romanian Mathematical Magazine-www.ssmrmh.ro