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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-IV

By Marin Chirciu – Romania

1) Let $\Delta A'B'C'$ be the circumcevian triangle of I – incenter in acute triangle ABC. Prove that:

$$IA' + IB' + IC' \geq r + \frac{5R}{2}$$

Proposed by Marian Ursărescu – Romania

Solution *We prove the following lemma:*

Lemma.

2) Let $\Delta A'B'C'$ be the circumcevian triangle of I – incenter in triangle ABC. Prove that

$$IA' + IB' + IC' = 2R\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right)$$

Proof. *Let's notice that $\Delta A'IB$ is isoscel with the vertex in A', because $\Delta IBA' = \Delta A'IB = \frac{B}{2} + \frac{C}{2}$, wherefrom $IA' = BA'$. Using sine theorem in $\Delta ABA'$ we obtain $\frac{BA'}{\sin\frac{A}{2}} = 2R$, wherefrom $BA' = 2R\sin\frac{A}{2}$. It follows that $IA' = BA' = 2R\sin\frac{A}{2}$. Let's get back to the main problem.*

Using Lemma the inequality can be written:

$$2R\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \geq r + \frac{5R}{2} \Leftrightarrow \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq \frac{5}{4} + \frac{r}{2R},$$ *which follows from:*

3) In acute-angled triangle ABC the following inequality holds:

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq \frac{5}{4} + \frac{r}{2r}$$

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Proof. *Popoviciu's inequality for concave functions:*

If $f:[a,b] \to \mathbb{R}$ is a concave function on $[a,b]$, then the following inequality holds:

$$\frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \leq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right],$$

$\forall x,y,z\in[a,b]$. Using Popoviciu's inequality for the concave function $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$, $f(x)=\cos x$, we obtain:

$$\frac{\cos A+\cos B+\cos C}{3}+\cos\left(\frac{A+B+C}{3}\right)\le\frac{2}{3}\left(\cos\frac{A+B}{2}+\cos\frac{B+C}{2}+\cos\frac{C+A}{2}\right)\Leftrightarrow$$

$$\Leftrightarrow\frac{1+\frac{r}{R}}{3}+\cos\frac{\pi}{3}\le\frac{2}{3}\left(\sin\frac{C}{2}+\sin\frac{A}{2}+\sin\frac{B}{2}\right)\Leftrightarrow\frac{1+\frac{r}{R}}{3}+\frac{1}{2}\le\frac{2}{3}\left(\sin\frac{C}{2}+\sin\frac{A}{2}+\sin\frac{B}{2}\right)\Leftrightarrow$$

$$\Leftrightarrow\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\ge\frac{5}{4}+\frac{r}{2R}$$

Equality holds if and only if the triangle is equilateral.

Remark. *Let's find an inequality having an opposite sense:*

4) Let $\Delta A'B'C'$ be the circumcevian triangle of I – incenter in triangle ΔABC. Prove that

$$IA'+IB'+IC'\le 3R$$

Proposed by Marin Chirciu – Romania

Solution. *Using Lemma the inequality from enunciation can be written:*

$2R\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)\le 3R\Leftrightarrow\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\frac{3}{2}$, *which follows from applying Jensen's inequality to the concave function $f:(0,\pi)\to\mathbb{R}$, $f(x)=\sin x$, wherefrom*

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le 3\sin\frac{A+B+C}{6}=3\sin\frac{\pi}{6}=\frac{\pi}{2}$$

Equality holds if and only if the triangle is equilateral.

Remark. *The double inequality can be written:*

5) Let $\Delta A'B'C'$ be the circumcevian triangle of I – incenter in acute triangle ΔABC. Prove that:

$$r+\frac{5R}{2}\le IA'+IB'+IC'\le 3R$$

Solution *See inequalities 1) and 4). Equality holds if and only if the triangle is equilateral.*

Reference:

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