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ABOUT AN INEQUALITY BY RAHIM SHAHBAZOV-I

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$8\cos A\cos B\cos C\le\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2$$

Proposed by Rahim Shahbazov-Azerbaijan

Solution

We prove the following inequality:

2) In ΔABC the following relationship holds:

$$8\cos A\cos B\cos C\le\left(\frac{2r}{R}\right)^2$$

Proof.

Using the identity $\prod\cos A=\frac{s^2-(2R+r)^2}{4R^2}$ *the inequality can be written:*

$$8\cdot\frac{s^2-(2R+r)^2}{4R^2}\le\left(\frac{2r}{R}\right)^2\Leftrightarrow s^2\le 4R^2+4Rr+3r^2 \text{ (Gerretsen's inequality)}$$

It suffices to prove that $\left(\frac{2r}{R}\right)^2\le\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2\Leftrightarrow\frac{2r}{R}\le\frac{ab+bc+ca}{a^2+b^2+c^2}$

3) In ΔABC the following relationship holds:

$$\frac{2r}{R}\le\frac{ab+bc+ca}{a^2+b^2+c^2}$$

Proof.

Using the identities $\sum bc=s^2+r^2+4Rr$ *and* $\sum a^2=2(s^2-r^2-4Rr)$, *the inequality can be written:* $\frac{2r}{R}\le\frac{s^2+r^2+4Rr}{2(s^2-r^2-4Rr)}\Leftrightarrow s^2(R-4r)+r(4R^2+17Rr+4r^2)\ge 0$

We distinguish the following cases:

Case 1) If $(R-4r)\ge 0$, *the inequality is obvious.*

Case 2) If $(R-4r)<0$, *the inequality can be written:*

$r(4R^2+17Rr+4r^2)\ge s^2(4r-R)$, *which follows from Gerretsen's inequality*

$s^2\le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$r(4R^2+17Rr+4r^2)\geq(4R^2+4Rr+3r^2)(4r-R)\Leftrightarrow R^3-2R^2r+Rr^2-2r^3\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(R^2+2r^2)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.

We can write the sequence of inequalities:

4) In ΔABC the following relationship holds:

$$8\cos A\cos B\cos C\leq\left(\frac{2r}{R}\right)^2\leq\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2\leq 1$$

Solution

See inequalities 2), 3) and $\frac{ab+bc+ca}{a^2+b^2+c^2}\leq 1\Leftrightarrow a^2+b^2+c^2\geq ab+bc+ca$, *obviously with equality for* $a=b=c$.

Equality holds if and only if the triangle is equilateral.

Remark.

In the same way we propose the following problems:

5) In ΔABC the following relationship holds:

$$8\cos A\cos B\cos C\leq\left(\frac{R}{2r}\right)^2\left(\frac{3abc}{a^3+b^3+c^3}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution

We prove the inequality:

6) In ΔABC the following relationship holds:

$$8\cos A\cos B\cos C\leq\left(\frac{2r}{R}\right)^2$$

Proof.

Using the identity $\prod\cos A=\frac{s^2-(2R+r)^2}{4R^2}$ *we write the inequality:*

$$8\cdot\frac{s^2-(2R+r)^2}{4R^2}\leq\left(\frac{2r}{R}\right)^2\Leftrightarrow s^2\leq 4R^2+4Rr+3r^2\ \ (Gerretsen's\ inequality)$$

Equality holds if and only if the triangle is equilateral.

It suffices to prove that: $\left(\frac{2r}{R}\right)^2 \le \left(\frac{R}{2r}\right)^2\left(\frac{3abc}{a^3+b^3+c^3}\right) \Leftrightarrow \frac{2r}{R} \le \frac{R}{2r}\cdot\frac{3abc}{a^3+b^3+c^3}$

7) In ΔABC the following relationship holds:

$$\frac{2r}{R} \le \frac{R}{2r}\cdot\frac{3abc}{a^3+b^3+c^3}$$

Proof.

Using the identities $\sum a^3 = 2s(s^2-3r^2-6Rr)$ *and* $abc = 4Rrs$, *the inequality can be written:* $\frac{2r}{R} \le \frac{R}{2r}\cdot\frac{3\cdot 4Rrs}{2s(s^2-3r^2-6Rr)} \Leftrightarrow 2r(s^2-3r^2-6Rr) \le 3R^3$

which follows from Gerretsen's inequality $s^2 \le 4R^2+4Rr+3r^2$

It remains to prove that: $2r(4R^2+4Rr+3r^2-3r^2-6Rr) \le 3R^3 \Leftrightarrow$

$\Leftrightarrow 3R^3-8R^2r+4Rr^2-2r^3 \ge 0 \Leftrightarrow R(R-2r)(3R-2r) \ge 0$, *obviously from Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.

We can write the sequence of inequalities:

8) In ΔABC the following relationship holds:

$$8\cos A\cos B\cos C \le \left(\frac{2r}{R}\right)^2 \le \left(\frac{R}{2r}\right)^2\left(\frac{3abc}{a^3+b^3+c^3}\right)^2 \le \left(\frac{R}{2r}\right)^2$$

Solution

See inequalities 6), 7) and $\frac{3abc}{a^3+b^3+c^3} \le 1 \Leftrightarrow a^3+b^3+c^3 \ge 3abc$, *obviously with equality for* $a=b=c$.

Equality holds if and only if the triangle is equilateral.

Reference:
Romanian Mathematical Magazine-www.ssmrmh.ro