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ROMANIAN MATHEMATICAL MAGAZINE

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ABOUT AN INEQUALITY BY RAHIM SHAHBAZOV-II

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\cot^2 A + \cot^2 B + \cot^2 C + \frac{2r}{R} \geq 2$$

Proposed by Rahim Shahbazov-Azerbaijan

Solution *We prove the following lemma:*

Lemma.

2) In ΔABC the following relationship holds:

$$\sum \cot^2 A = \frac{s^4 - s^2(8Rr + 10r^2) + r^2(4R+r)^2}{4s^2r^2}$$

Proof.

We have $\sum \cot^2 A = \sum \frac{\cos^2 A}{\sin^2 A} = \sum \frac{1-\sin^2 A}{\sin^2 A} = \sum \left(\frac{1}{\sin^2 A} - 1\right) = \sum \frac{1}{\sin^2 A} - 3 =$

$= \frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R+r)^2}{4s^2r^2} - 3 = \frac{s^4 - s^2(8Rr + 10r^2) + r^2(4R+r)^2}{4s^2r^2}$, *where*

$$\sum \frac{1}{\sin^2 A} = \frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R+r)^2}{4s^2r^2}$$

Let's get back to the main problem. Using the identity

$\sum \cot^2 A = \frac{s^4 - s^2(8Rr + 10r^2) + r^2(4R+r)^2}{4s^2r^2}$ *we write the inequality:*

$$\frac{s^4 - s^2(8Rr + 10r^2) + r^2(4R+r)^2}{4s^2r^2} + \frac{2r}{R} \geq 2 \Leftrightarrow$$

$$\frac{s^4 - s^2(8Rr + 10r^2) + r^2}{4s^2r^2} + \frac{2r}{R} \geq 2 \Leftrightarrow$$

$$Rs^4 - s^2r(8R^2 + 18Rr - 8r^2) + Rr^2(4R+r)^2 \geq 0 \Leftrightarrow$$

$$s^2[Rs^2 - r(8R^2 + 18Rr - 8r^2)] + Rr^2(4R+r)^2 \geq 0$$

We distinguish the following cases:

Case 1). If $[Rs^2 - r(8R^2 + 18Rr - 8r^2)] \geq 0$, *the inequality is obvious.*

Case 2). If $[Rs^2 - r(8R^2 + 18Rr - 8r^2)] < 0$, *we write the inequality:*

$Rr^2(4R+r)^2 \geq s^2[r(8r^2+18Rr-8r^2)-Rs^2]$, *which follows from Blundon-Gerretsen's*

inequality $16Rr-5r^2 \leq s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$

It remains to prove that:

$$Rr^2(4R+r)^2 \geq \frac{R(4R+r)^2}{2(2R-r)}[r(8r^2+18Rr-8r^2)-R(16Rr-5r^2)] \Leftrightarrow$$

$\Leftrightarrow 8R^2-19Rr+6r^2 \geq 0 \Leftrightarrow (R-2r)(8R-3r) \geq 0$, *obviously from Euler* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark. *Let's find and inequality having an opposite sense:*

3) In ΔABC the following relationship holds:

$$\cot^2 A + \cot^2 B + \cot^2 C + 3 \leq \left(\frac{R}{r}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution *Using the identity* $\sum \cot^2 A = \frac{s^4-s^2(8Rr+10r^2)+r^2(4R+r)^2}{4s^2r^2}$ *we write the inequality:*

$\frac{s^4-s^2(8Rr+10r^2)+r^2(4R+r)^2}{4s^2r^2} + 3 \leq \frac{R^2}{r^2}$ *which follows from Gerretsen's inequality:*

$$\frac{r(4R+r)^2}{R+r} \leq 16Rr-5r^2 \leq s^2 \leq 4R^2+4Rr+3r^2$$

We obtain $\frac{s^4-s^2(8Rr+10r^2)+r^2(4R+r)^2}{4s^2r^2} = \frac{1}{4r^2}\left[s^2-(8Rr+10r^2)+\frac{r^2(4R+r)^2}{s^2}\right] \leq$

$$\leq \frac{1}{4r^2}\left[4R^2+4Rr+3r^2-(8Rr+10r^2)+\frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right] = \frac{4R^2-3Rr-6r^2}{4r^2} \overset{R^2-3r^2}{\leq} \frac{R^2}{r^2}-3.$$

Equality holds if and only if the triangle is equilateral.

Remark. *We write the double inequality:*

4) In ΔABC the following relationship holds:

$$2-\frac{2r}{R} \leq \cot^2 A + \cot^2 B + \cot^2 C \leq \left(\frac{R}{r}\right)^2 - 3$$

Solution *See inequalities 2) and 3).Equality holds if and only if the triangle is equilateral.*

Reference:

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