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ABOUT AN INEQUALITY BY VASILE MIRCEA POPA-I

By Marin Chirciu – Romania

1) In acute angled triangle ABC the following relationship holds:

$$\tan A+\tan B+\tan C+3\sqrt{3}\geq 2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution

Using Popoviciu's inequality: If $f:[a,b]\to\mathbb{R}$ is a convexe function on $[a,b]$, then the following inequality holds:

$$\frac{f(x)+f(y)+f(z)}{3}+f\left(\frac{x+y+z}{3}\right)\geq\frac{2}{3}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right],$$

$\forall x,y,z\in[a,b]$. We consider the convexe function $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=\tan x$, for which we apply Popoviciu's inequality. We obtain:

$$\frac{\tan A+\tan B+\tan C}{3}+\tan\left(\frac{A+B+C}{3}\right)\geq\frac{2}{3}\left[\tan\left(\frac{A+B}{2}\right)+f\left(\frac{B+C}{2}\right)+f\left(\frac{C+A}{2}\right)\right], \text{ wherefrom}$$

$$\frac{\tan A+\tan B+\tan C}{3}+\tan\left(\frac{\pi}{3}\right)\geq\frac{2}{3}\left[\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)+\tan\left(\frac{\pi}{2}-\frac{A}{2}\right)+\tan\left(\frac{\pi}{2}-\frac{B}{2}\right)\right]\Leftrightarrow$$

$$\Leftrightarrow\frac{\tan A+\tan B+\tan C}{3}+\sqrt{3}\geq\frac{2}{3}\left[\cot\frac{C}{2}+\cot\frac{A}{2}+\cot\frac{B}{2}\right]\Leftrightarrow$$

$$\Leftrightarrow\tan A+\tan B+\tan C+3\sqrt{3}\geq 2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)$$

Equality holds if and only if the triangle is equilateral.

Remark.

In the same way we can propose:

2) In acute triangle ABC the following relationship holds:

$$\cot A+\cot B+\cot C+\sqrt{3}\geq 2\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

Proposed by Marin Chirciu, Geometric Inequalities, 2015

Solution

We apply Popoviciu's inequality for the convexe function $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=\cot x$.

3) In ΔABC the following relationship holds:

$$\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{\sin^2 C}+4\geq 2\left(\frac{1}{\cos^2\frac{A}{2}}+\frac{1}{\cos^2\frac{B}{2}}+\frac{1}{\cos^2\frac{C}{2}}\right)$$

Proposed by Marin Chirciu, Geometric Inequalities, 2015

Solution

We apply Popoviciu's inequality for the convexe function $f:(0,\pi)\to\mathbb{R}, f(x)=\frac{1}{\sin^2 x}$

4) In ΔABC the following inequality holds:

$$\frac{r_a}{r_a-r}+\frac{r_b}{r_b-r}+\frac{r_c}{r_c-r}+\frac{9}{2}\geq\frac{4s^2}{s^2+r_ar_b}+\frac{4s^2}{s^2+r_br_c}+\frac{4s^2}{s^2+r_cr_a}$$

Proposed by Marin Chirciu, Geometric Inequalities, 2015

Solution

We apply Popoviciu's inequality for the convexe function $f:(0,1)\to\mathbb{R}$,

$f(x)=\frac{1}{1-x}$, *for* $x=\frac{r}{r_a}, y=\frac{r}{r_b}, z=\frac{r}{r_c}$. *Using* $\sum\frac{r}{r_a}=1, \sum r_br_c=s^2$ *and* $\prod r_a=rs^2$.

5) If $x,y,z>0$ prove that:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{9}{x+y+z}\geq\frac{4}{x+y}+\frac{4}{y+z}+\frac{4}{z+x}$$

Proposed by Marin Chirciu, Geometric Inequalities, 2015

Solution. *We apply Popoviciu's inequality for the convexe function* $:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{x}$.

6) If $x,y,z>0$ prove that:

$$\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}+\frac{9}{\sqrt{x+y+z}}\geq\frac{4}{\sqrt{x+y}}+\frac{4}{\sqrt{y+z}}+\frac{4}{\sqrt{z+x}}$$

Proposed by Marin Chirciu, Geometric Inequalities, 2015

Solution *We apply Popoviciu's inequality for the convexe function* $:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{\sqrt{x}}$.

Reference:

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