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ABOUT INEQUALITY IN TRIANGLE-1196

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\sum\sqrt{r_a(r_b-r)(r_c-r)}\geq 2r\left(\sqrt{m_a}+\sqrt{m_b}+\sqrt{m_c}\right)$$

Proposed by Bogdan Fuștei – Romania

Solution

Using $r_a=\frac{S}{s-a}$ *and* $r=\frac{S}{s}$ *we obtain*

$$\sum\sqrt{r_a(r_b-r)(r_c-r)}=\sum\sqrt{\frac{S}{s-a}\left(\frac{S}{s-b}-\frac{S}{s}\right)\left(\frac{S}{s-c}-\frac{S}{s}\right)}=\sum\sqrt{\frac{rs}{s-a}\cdot\frac{rb}{s-b}\cdot\frac{rc}{s-c}}=\sum\sqrt{\frac{r^3sbc}{r^2s}}=\sum\sqrt{rbc}\quad (1)$$

Using $\frac{R}{2r}\geq\frac{m_a}{h_a}$ *(L. Panaitopol, GM 11/1982), we obtain:*

$$m_a\leq\frac{R}{2r}\Leftrightarrow m_a\leq\frac{R}{2r}\cdot\frac{2S}{a}=\frac{Rs}{a}=\frac{bc}{4r},\ \text{wherefrom}\sum\sqrt{m_a}\leq\sum\sqrt{\frac{bc}{4r}}\quad (2)$$

From (1) and (2) we deduce the inequality from enunciation.

Equality holds if and only if the triangle is equilateral.

Remark.

We propose in the same way:

2) In ΔABC the following relationship holds:

$$\sum\sqrt{h_a(h_b-r)(h_c-r)}\leq 2s\sqrt{r}$$

Proposed by Marin Chirciu – Romania

Solution

Using $h_a=\frac{2S}{s}$ *and* $r=\frac{S}{s}$ *we obtain:*

$$\sum\sqrt{h_a(h_b-r)(h_c-r)}=\sum\sqrt{\frac{2S}{a}\left(\frac{2S}{b}-\frac{S}{s}\right)\left(\frac{2S}{s}-\frac{S}{s}\right)}=\sum\sqrt{\frac{2rs}{a}\cdot\frac{rs(a+c)}{bs}\cdot\frac{rs(a+b)}{cs}}$$

$$=\sum\sqrt{\frac{2r^3s(a+b)(a+c)}{abc}}=\sum\sqrt{\frac{2r^3s(a+b)(a+c)}{4Rrs}}=\sum\sqrt{\frac{r^2(a+b)(a+c)}{2R}}=$$

$$= \frac{r}{\sqrt{2R}}\sum\sqrt{(a+b)(a+c)} \overset{AM-AG}{\leq} \frac{r}{\sqrt{2R}}\sum\frac{(a+b)+(a+c)}{2} =$$

$$= \frac{r}{\sqrt{2R}}\cdot 4s = \frac{4S}{\sqrt{2R}} \overset{Euler}{\leq} 2s\sqrt{r}$$

Equality holds if and only if the triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\sum\sqrt{bc(h_b-r)(h_c-r)} \leq 4S$$

Proposed by Marin Chirciu – Romania

Solution

Using $h_a = \frac{2S}{s}$ *and* $r = \frac{S}{s}$ *we obtain:*

$$\sum\sqrt{bc(h_b-r)(h_c-r)} = \sum\sqrt{bc\left(\frac{2S}{b}-\frac{S}{s}\right)\left(\frac{2S}{s}-\frac{S}{s}\right)} = \sum\sqrt{bc\cdot\frac{rs(a+c)}{bs}\cdot\frac{rs(a+b)}{cs}}$$

$$= r\sum\sqrt{(a+b)(a+c)} = r\sum\sqrt{(a+b)(a+c)} \overset{AM-AG}{\leq} r\sum\frac{(a+b)+(a+c)}{2} = r\cdot 4s = 4S.$$

Equality holds if and only if the triangle is equilateral.

Reference:

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