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ABOUT NAGEL'S AND GERGONNE'S CEVIANS-(V)

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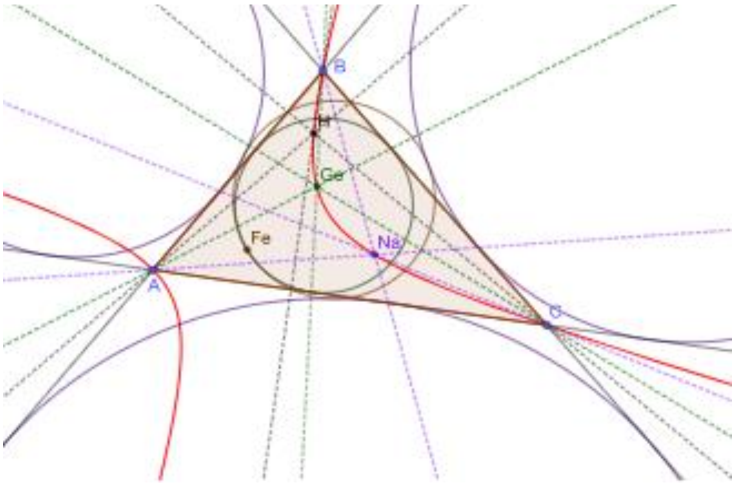

In ΔABC we show it that $s^2 = n_a^2 + 2r_a h_a$ (and analogs)

$$s^2 - n_a^2 = (s + n_a)(s - n_a) = 2r_a h_a \Rightarrow \frac{s - n_a}{h_a} = \frac{2n_a}{s + n_a}$$

$\frac{s}{h_a} = \frac{n_a}{h_a} + \frac{2r_a}{s+n_a}$ (and analogs) $2S = a \cdot h_a = 2sr$ (and analogs)

$$\frac{a}{2r} = \frac{s}{h_a} + \frac{n_a}{s+n_a}$$

$$\frac{s}{r} = \sum_{cyc} \frac{n_a}{h_a} + 2\sum_{cyc} \frac{r_a}{n_a + s}$$

$\frac{s-n_a}{r_a} = \frac{2h_a}{n_a+s}$ (and analogs) and $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$

So, we have: $\frac{s}{r} = \sum_{cyc} \frac{n_a}{h_a} + 2\sum_{cyc} \frac{r_a}{n_a+s}$ but $h_a = \frac{2r_b r_c}{r_b+r_c} \Rightarrow \frac{1}{h_a} = \frac{1}{2}\left(\frac{1}{r_b} + \frac{1}{r_c}\right)$ hence

$\frac{n_a}{h_a} = \frac{1}{2}\left(\frac{n_a}{r_b} + \frac{n_a}{r_c}\right)$ (and analogs), summing, we get:

$$2\sum_{cyc} \frac{n_a}{h_a} = \sum_{cyc} \frac{n_b + n_c}{r_a}; \quad (1)$$

$$\frac{2s}{r} = 2\sum_{cyc} \frac{n_a}{h_a} + \sum_{cyc} \frac{4r_a}{n_a + s}; \quad (2)$$

$$\frac{s}{r} = \sum_{cyc} \frac{n_a}{h_a} + \sum_{cyc} \frac{2r_a}{n_a + s}; \quad (3)$$

From $(1),(2),(3)$ it follows that

$$\frac{3s}{r}=\frac{\sum n_a}{r}+2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}$$

We known that $s\geq r\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\Rightarrow\frac{3s}{r}=3\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\Rightarrow$

$$\frac{\sum n_a}{r}+2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq 3\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$$

Hence,

$$2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq 3\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-\frac{\sum n_a}{r}$$

Summing, it follows

$$4\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq 3\sqrt{3}\sum_{cyc}\frac{b+c}{a}-2\frac{\sum n_a}{r}\Leftrightarrow$$

$$\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq\frac{3\sqrt{3}}{4}\sum_{cyc}\frac{b+c}{a}-\frac{\sum n_a}{2r}$$

Now,

$\frac{s}{r}\geq\sqrt{4-\frac{2r}{R}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$ and $\frac{s}{r}\geq\sqrt{4-\frac{2r}{R}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$ hence

$$\frac{3s}{r}\geq 3\sqrt{4-\frac{2r}{R}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}\Leftrightarrow\frac{\sum n_a}{r}+2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq 3\sqrt{4-\frac{2r}{R}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$$

$$2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq 3\sqrt{4-\frac{2r}{R}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}-\frac{\sum n_a}{r}$$

Similarly, we have:

$$2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq\sqrt{4-\frac{2r}{R}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}-\frac{\sum n_a}{r}$$

Adding, we get:

$$4\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge 3\sqrt{4-\frac{2r}{R}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)}-2\frac{\sum n_a}{r}$$

Hence,

$$\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\frac{3}{4}\sqrt{4-\frac{2r}{R}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)}-\frac{\sum n_a}{2r}$$

$$h_a=\frac{2sr}{a}=\frac{(a+b+c)r}{a}\Rightarrow h_a=\left(1+\frac{b+c}{a}\right)r\Rightarrow\frac{h_a-r}{r}=\frac{b+c}{a}\Rightarrow$$

$$\sum_{cyc}\frac{b+c}{a}=\frac{\sum(h_a-r)}{r}$$

Hence,

$$\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\frac{3\sqrt{3}\sum h_a-2\sum n_a-9\sqrt{3}r}{4r}\Leftrightarrow$$

$$\frac{9\sqrt{3}}{4}+\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\frac{3\sqrt{3}\sum h_a-2\sum n_a}{4r}$$

Now, we known that

$$\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\frac{3}{4}\sqrt{4-\frac{2r}{R}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)}-\frac{\sum n_a}{2r}$$

Hence,

$$\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\frac{3}{4}\sqrt{4-\frac{2r}{R}\cdot\frac{\sum(h_a-r)}{r}}-\frac{\sum n_a}{2r}$$

$$\frac{9}{4}\sqrt{4-\frac{2r}{R}}+\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\frac{3}{4}\sqrt{4-\frac{2r}{R}\cdot\frac{\sum h_a}{4r}}-\frac{\sum n_a}{2r}$$

We known that:

$$\frac{s}{r}\ge\sqrt{\left(4-\frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

$$\frac{3s}{r}=\frac{\sum n_a}{r}+2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}$$

$$\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq\sqrt{\left(4-\frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

And we known that:

$$s\geq\sqrt{r(4R+r)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}\Leftrightarrow\frac{s}{r}\geq\sqrt{\left(1+\frac{4R}{r}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$$

Hence,

$$\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq\sqrt{\left(1+\frac{4R}{r}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$$

And similarly,

$$s\geq\sqrt{r(4R+r)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

Hence,

$$\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq\sqrt{\left(1+\frac{4R}{r}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

$$\left(\frac{s}{r}\right)^2\geq\sqrt{\left(1+\frac{4R}{r}\right)^2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

$$\frac{s}{r}\geq\sqrt[4]{\left(1+\frac{4R}{r}\right)^2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}=Q$$

Therefore,

$$\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq Q$$

Now, we known that

$\frac{\sum AI}{r}\geq\frac{s}{r}+3(2-\sqrt{3})$ **and** $\frac{s}{r}=\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}$ **it follows that**

$$\frac{\sum(3AI-n_a)}{3r}\geq 3(2-\sqrt{3})+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}$$

We known that $n_a+g_a\geq 2m_a$ (and analogs), then $n_a\geq 2m_a-g_a$

Hence,

$$\frac{s}{r}\geq\frac{\sum(2m_a-g_a)}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}$$

$$h_a=\left(1+\frac{b+c}{a}\right)r;\ \frac{b+c}{a}=\frac{r_a+h_a}{r_a}=1+\frac{h_a}{r_a}$$

$$h_a=\left(2+\frac{h_a}{r_a}\right)r\Rightarrow r_ah_a=(2r_a+h_a)r$$

$bc=s^2+r_a^2-4Rr_a$, then we have

$\frac{r_ah_a}{r}=2r_a+h_a$ (and analogs)

$$\frac{3s}{r}=\frac{\sum n_a}{r}+2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}=\frac{\sum n_a}{r}+\frac{2}{r}\sum_{cyc}\frac{h_ar_a}{n_a+s}$$

Hence,

$$3s=n_a+n_b+n_c+2\sum_{cyc}\frac{h_ar_a}{n_a+s}$$

We known that $n_a+g_a\geq 2m_a\Rightarrow n_a\geq 2m_a-g_a$

$$3s\geq\sum_{cyc}(2m_a-g_a)+\ 2\sum_{cyc}\frac{h_ar_a}{n_a+s}$$

From $n_a+n_b+n_c\geq s\sqrt{3}\Rightarrow(n_a+n_b+n_c)\sqrt{3}\geq 3s=n_a+n_b+n_c+2\sum_{cyc}\frac{h_ar_a}{n_a+s}$

Therefore,

$$\frac{\sqrt{3}-1}{2}(n_a+n_b+n_c)\geq\sum_{cyc}\frac{h_ar_a}{n_a+s}$$

$$|b-c|\geq n_a-g_a\Rightarrow g_a+|b-c|\geq n_a\Rightarrow\frac{1}{n_a}\geq\frac{1}{g_a+|b-c|}\Rightarrow$$

$$3s\geq\frac{b+c}{2}\cdot cos\frac{A}{2}\Rightarrow\frac{m_a}{cos\frac{A}{2}}\geq\frac{b+c}{2}$$

Adding, it follows that

$$\sum_{cyc} \frac{m_a}{cos\frac{A}{2}} \ge \sum_{cyc} \frac{b+c}{2} = a+b+c = 2s$$

Hence,

$$\frac{1}{2}\sum_{cyc} \frac{m_a}{cos\frac{A}{2}} \ge s \Rightarrow \frac{3}{2}\sum_{cyc} \frac{m_a}{cos\frac{A}{2}} \ge 3s$$

$$\frac{3}{2}\sum_{cyc} \frac{m_a}{cos\frac{A}{2}} \ge n_a+n_b+n_c+2\sum_{cyc}\frac{h_ar_a}{n_a+s}$$

We known that $n_ag_a \ge m_aw_a \Rightarrow n_a \ge \frac{m_aw_a}{g_a}$ then

$$3s \ge \sum_{cyc}\frac{m_aw_a}{g_a} + 2\sum_{cyc}\frac{h_ar_a}{n_a+s}$$

$$cosAcosBcosC = \frac{s^2-(2R+r)^2}{4R^2}$$

If ΔABC is obtuse triangle, then $cosAcosBcosC \ge 0$ equality if triangle is right.

$$s^2-(2R+r)^2 \ge 0 \Rightarrow s \ge 2R+r \Rightarrow 3s \ge 3(2R+r)$$

$n_a+n_b+n_c+2\sum_{cyc}\frac{h_ar_a}{n_a+s} \ge 3(2R+2)$ for non-obtuse triangle

$n_a+n_b+n_c < 3(2R+2) - 2\sum_{cyc}\frac{h_ar_a}{n_a+s}$ for obtuse triangle.

We known that

$$\frac{s}{r} = \sum_{cyc}\frac{n_a}{h_a} + 2\sum_{cyc}\frac{r_a}{n_a+s}$$

$$\frac{s}{r} = \sum_{cyc}\frac{n_a}{r_a} + 2\sum_{cyc}\frac{h_a}{n_a+s}$$

So, for non-obtuse triangle we have:

$$\sum_{cyc}\frac{n_a}{h_a} \ge 1+\frac{2R}{r} - 2\sum_{cyc}\frac{r_a}{n_a+s}$$

$$\sum_{cyc}\frac{n_a}{r_a} \ge 1+\frac{2R}{r} - 2\sum_{cyc}\frac{h_a}{n_a+s}$$

Let be $P \in (ABC), A, B, C$ −non-collinear. If $PA = x; PB = y; PC = z$ then

$ayz + bxz + cxy \geq abc$ (Cocea-Hayashi inequality)

Let be $P = N_a$ −Nagel's point, then $AN_a = \sqrt{(b-c)^2 + 4r^2}$ (and analogs).

But we show it that: $\frac{AN_a}{2r} = \frac{n_a}{h_a}$ (and analogs), hence

$$\frac{aBN_a \cdot CN_a}{4r^2} + \frac{bCN_a \cdot AN_a}{4r^2} + \frac{cAN_a \cdot BN_a}{4r^2} \geq \frac{abc}{4r^2} \Leftrightarrow$$

$$\frac{aBN_a \cdot CN_a}{4r^2} + \frac{bCN_a \cdot AN_a}{4r^2} + \frac{cAN_a \cdot BN_a}{4r^2} \geq \frac{R}{r} \cdot s$$

$bc = s^2 + r_a^2 - 4Rr_a$ (and analogs)

$$h_a h_b h_c = \frac{2S}{a} \cdot \frac{2S}{b} \cdot \frac{2S}{c} = \frac{2S^2}{R}$$

$$ah_a = bh_b = ch_c = 2S = 2sr$$

$$2S \sum_{cyc} n_b n_c \geq \frac{R}{r} \cdot s \cdot \frac{2S^2}{R} \Rightarrow 2sr \sum_{cyc} n_b n_c \geq 2s \cdot s^2 r \Rightarrow \sum_{cyc} n_b n_c \geq s^2$$

So, it follows a new inequality

$$\sum_{cyc} n_b n_c \geq s^2 = \sum_{cyc} r_a r_b$$

But $\sum n_a^2 \geq \sum n_a n_b \Rightarrow (\sum n_a)^2 \geq 3 \sum n_a n_b \geq 3s^2$, hence

$$\sum n_a \geq s\sqrt{3}$$

Let be G_e −Gergonne's point, then $AA_1 = g_a = AG_a + G_a A_1$

From Van-Aubel's theorem, we have:

$$\frac{AG_e}{A_1 G_e} = \frac{s-a}{s-b} + \frac{s-a}{s-c}$$

$s(s-a) = r_b r_c$ (and analogs)

$$\frac{AG_e}{A_1 G_e} = \frac{s-a}{s-b} + \frac{s-a}{s-c} = \frac{r_b r_c}{r_a r_c} + \frac{r_b r_c}{r_a r_b} = \frac{r_b + r_c}{r_a}$$

Hence,

$\frac{AG_e}{A_1 G_e} = \frac{r_b + r_c}{r_a}$ (and analogs)

$$\frac{A_1G_e}{AG_e}=\frac{r_a}{r_b+r_c}\Rightarrow\frac{A_1G_e+AG_e}{AG_e}=\frac{r_a+r_b+r_c}{r_b+r_c};\ r_a+r_b+r_c=4R+r$$

$$AG_e=\frac{g_a(r_b+r_c)}{4R+r}\text{ (and analogs)}$$

Adding, it follows that

$$\sum_{cyc}AG_e=\frac{1}{4R+r}\sum_{cyc}g_a(r_b+r_c)$$

$$\frac{AG_e}{g_a}=\frac{r_b+r_c}{4R+r}\text{ (and analogs)}$$

$$\frac{AG_e}{g_a}+\frac{BG_e}{g_b}+\frac{CG_e}{g_c}=2\text{ and }\frac{g_a}{4R+r}=\frac{AG_e}{r_b+r_c}$$

Adding, it follows that

$$\sum_{cyc}\frac{AG_e}{r_b+r_c}=\frac{g_a+g_b+g_c}{r_a+r_b+r_c}$$

But $g_a\le AI+r$ (and analogs), from triangle inequality, hence

$$g_a+g_b+g_c\le 3r+AI+BI+CI\text{ then}$$

$$\sum_{cyc}\frac{AG_e}{r_b+r_c}\le\frac{3r+AI+BI+CI}{r_a+r_b+r_c}$$

But $m_a+m_b+m_c\le r_a+r_b+r_c$ hence,

$$\sum_{cyc}\frac{AG_e}{r_b+r_c}\le\frac{3r+AI+BI+CI}{m_a+m_b+m_c}$$

$$\frac{g_a}{AG_e}=\frac{4R+r}{r_b+r_c};\ 2r_br_c=h_a(r_b+r_c)\text{ (and analogs)}$$

$$\frac{g_a}{AG_e}=\frac{h_a(r_b+r_c)}{2r_br_c}\Rightarrow\frac{g_a}{h_a}=\frac{(r_b+r_c)AG_e}{2r_br_c}=\frac{(4R+r)r_a\cdot AG_e}{2r_ar_br_c};$$

$$r_ar_br_c=Ss;\ ah_a=bh_b=ch_c=2S$$

$$\frac{ag_a}{ah_a}=\frac{ag_a}{2S}=\frac{(4R+r)r_a\cdot AG_e}{2Ss}\Rightarrow ag_a=\frac{(4R+r)r_a\cdot AG_e}{s}$$

$$tan\frac{A}{2}=\frac{r_a}{s};r_a+r_b+r_b=4R+r\Rightarrow ag_a=\left(tan\frac{A}{2}+tan\frac{B}{2}+tan\frac{C}{2}\right)r_aAG_e$$

which follows from Blundon's inequality $\frac{4R+r}{s}\ge\sqrt{4-\frac{2r}{R}}$ then

$$\frac{ag_a}{r_aAG_e}\ge\sqrt{4-\frac{2r}{R}}$$

Adding, it follows a new inequality

$$\frac{1}{3}\sum_{cyc}\frac{ag_a}{r_aAG_e}\ge\sqrt{4-\frac{2r}{R}}$$

From $ag_a=\frac{4R+r}{s}\cdot r_a\cdot AG_e$ we get

$$\sum_{cyc}ag_a=\frac{4R+r}{s}\sum_{cyc}r_aAG_e$$

$$\frac{\sum ag_a}{\sum r_aAG_e}=\frac{4R+r}{s}=\sum\tan\frac{A}{2}\ge\sqrt{4-\frac{2r}{R}}$$

Let be $P\in Int(ABC)\Rightarrow\frac{PA}{a}+\frac{PB}{b}+\frac{PC}{c}\ge\sqrt{3}$

$AG_e=\frac{g_a(r_b+r_c)}{4R+r}\Rightarrow\frac{AG_e}{a}=\frac{g_a}{4R+r}\cdot\frac{r_b+r_c}{a}$ (and analogs)

$$a=\sqrt{(r_a-r)(r_b+r_c)};\ \sin\frac{A}{2}=\sqrt{\frac{r_a-r}{4R}};\cos\frac{A}{2}=\sqrt{\frac{r_b+r_c}{4R}}$$

Hence,

$$\cot\frac{A}{2}=\sqrt{\frac{r_b+r_c}{r_a-r}}=\frac{r_b+r_c}{\sqrt{(r_a-r)(r_b+r_c)}}$$

$$\cot\frac{A}{2}=\frac{r_b+r_c}{a}$$

$$\frac{AG_e}{a}=\frac{g_a}{4R+r}\cdot\cot\frac{A}{2}\Rightarrow\frac{AG_e}{a}+\frac{BG_e}{b}+\frac{CG_e}{c}\ge\sqrt{3}$$

Therefore,

$$\sum_{cyc}g_a\cot\frac{A}{2}\ge(r_a+r_b+r_c)\sqrt{3}$$

$$\tan\frac{A}{2}=\frac{r_a}{s}=\frac{1}{\cot\frac{A}{2}}\Rightarrow\cot\frac{A}{2}=\frac{s}{r_a}\text{ (and analogs)}$$

$$g_a\cot\frac{A}{2}=\frac{g_a}{r_a}\cdot s\text{ (and analogs)}$$

Hence,

$$\frac{g_a}{r_a}+\frac{g_b}{r_b}+\frac{g_c}{r_c}\ge\frac{r_a+r_b+r_c}{s}\sqrt{3}$$

$$4R+r=r_a+r_b+r_c\ge s\sqrt{4-\frac{2r}{R}}\Rightarrow\frac{r_a+r_b+r_c}{s}\ge\sqrt{4-\frac{2r}{R}}$$

Therefore,

$$\frac{g_a}{r_a}+\frac{g_b}{r_b}+\frac{g_c}{r_c}\ge\sqrt{3\left(4-\frac{2r}{R}\right)}$$

But $g_a\le AI+r\Rightarrow\frac{g_a}{h_a}\le\frac{AI}{r_a}+\frac{r}{r_a};\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$ hence,

$\frac{g_a}{r_a}+\frac{g_b}{r_b}+\frac{g_c}{r_c}\le 1+\frac{AI}{r_a}+\frac{BI}{r_b}+\frac{CI}{r_c}$ and then

$$\frac{AI}{r_a}+\frac{BI}{r_b}+\frac{CI}{r_c}\ge\frac{r_a+r_b+r_c}{s}\sqrt{3}-1$$

$$\frac{AI}{r_a}+\frac{BI}{r_b}+\frac{CI}{r_c}\ge\sqrt{3\left(4-\frac{2r}{R}\right)}$$

For $P=I,I$ –incenter, hence Cocea-Hayashi inequality becomes:

$$a\cdot BI\cdot CI+b\cdot CI\cdot AI+c\cdot AI\cdot BI\ge abc=3Rrs$$

$$AI=\frac{r}{\sin\frac{A}{2}}\Rightarrow AI\cdot BI\cdot CI=\frac{r^3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\sin\frac{A}{2}=\sqrt{\frac{rr_a}{bc}}\Rightarrow\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sqrt{\frac{r^3\cdot r_ar_br_c}{4RS\cdot 4RS}}=\frac{r}{4R}$$

$$\Rightarrow AI\cdot BI\cdot CI=r^3\cdot\frac{4R}{r}=4Rr^2$$

$$\frac{a}{AI}+\frac{b}{BI}+\frac{c}{CI}\ge\frac{abc}{4Rr^2}=\frac{s}{r}$$

Now,

$$\frac{s}{r}=\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{2r_a+h_a}{s+n_a}\Rightarrow\frac{a}{AI}+\frac{b}{BI}+\frac{c}{CI}\ge\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{2r_a+h_a}{s+n_a}$$

But $AI=\sqrt{2R(h_a-2r)}$, then

$$\sum_{cyc}\frac{a}{\sqrt{h_a-2r}}\ge\frac{\sqrt{2R}}{3}\left(\frac{\sum n_a}{r}+2\sum_{cyc}\frac{2r_a+h_a}{s+n_a}\right)$$

But $AI=\sqrt{(r_b-r)(r_c-r)}$, then

$$\sum_{cyc}\frac{a}{\sqrt{(r_b-r)(r_c-r)}}\ge\frac{\sqrt{2R}}{3}\left(\frac{\sum n_a}{r}+2\sum_{cyc}\frac{2r_a+h_a}{s+n_a}\right)$$

We know that:

$$m_a^2=r_br_c+\frac{1}{4}(b-c)^2\Rightarrow\frac{m_a^2}{r^2}=\frac{r_br_c}{r^2}+\frac{(b-c)^2}{4r^2}$$

But $\frac{(b-c)^2}{4r^2}=\frac{n_a^2}{h_a^2}-1$, hence $\frac{m_a^2}{r^2}=\frac{r_br_c}{r^2}+\frac{n_a^2}{h_a^2}-1$

Adding, it follows a new identity

$$\frac{\sum m_a^2}{r^2}=\frac{s^2}{r^2}+\sum_{cyc}\frac{n_a^2}{h_a^2}-3$$

But $\sum\frac{n_a^2}{h_a^2}\ge\sum\frac{n_an_b}{h_ah_b}$, then it follows that

$$\frac{\sum m_a^2}{r^2}\ge\left(\frac{s}{r}\right)^2+\sum_{cyc}\frac{n_an_b}{h_ah_b}-3$$

But $\frac{s}{r}=\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{2r_a+h_a}{s+n_a}$, hence

$$\frac{\sum m_a^2}{r^2}=\left(\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{2r_a+h_a}{s+n_a}\right)^2+\sum_{cyc}\frac{n_a^2}{h_a^2}-3$$

Now, $\frac{m_a^2}{r^2}=\frac{n_a^2}{h_a^2}+\frac{r_br_c-r^2}{r^2}$

Using QM-AM inequality $\sqrt{x^2+y^2} \geq \frac{x+y}{\sqrt{2}}$ for $x^2 = \frac{n_a^2}{h_a^2}$; $y^2 = \frac{r_b r_c - r^2}{r^2}$, we get:

$$\frac{m_a}{r} \geq \frac{1}{\sqrt{2}}\left(\frac{n_a}{h_a} + \sqrt{\frac{r_b r_c - r^2}{r^2}}\right)$$

Adding, it follows a new inequality:

$$\frac{\sum m_a}{r} \geq \frac{1}{\sqrt{2}}\left(\sum_{cyc} \frac{n_a}{h_a} + \sum_{cyc} \sqrt{\frac{r_b r_c - r^2}{r^2}}\right)$$

But $(\sum m_a)^2 \leq 4s^2 - 16Rr + 5r^2$ (Chu&Yang inequality), hence

$$\left(\frac{\sum m_a}{r}\right)^2 \leq \frac{4s^2}{r^2} - \frac{16R}{r} + 5$$

$$\frac{s^2}{r^2} + \sum_{cyc} \frac{n_a^2}{h_a^2} - 3 \leq \frac{4s^2}{r^2} - \frac{16R}{r} + 5 - 2\sum_{cyc} \frac{m_b m_c}{r^2}$$

Finally, it follows

$$\sum_{cyc} \frac{n_a^2}{h_a^2} + 2\sum_{cyc} \frac{m_b m_c}{r^2} \leq \frac{3s^2}{r^2} - \frac{16R}{r} + 8$$

But $\sum \frac{n_a^2}{h_a^2} \geq \sum \frac{n_a n_b}{h_a h_b}$, hence

$$\sum_{cyc} \frac{n_a n_b}{h_a h_b} + 2\sum_{cyc} \frac{m_b m_c}{r^2} \leq \frac{3s^2}{r^2} - \frac{16R}{r} + 8$$

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