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ABOUT AN INEQUALITY BY DAN RADU SECLAMAN-I

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}\le 2R-r$$

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Solution *We prove the following lemma:*

Lemma.

2) In ΔABC the following inequality holds:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}=\frac{3(s^2-r^2-4Rr)}{2(4R+r)}$$

Proof.

Using $\sum m_a^2=\frac{3}{4}\sum a^2,\sum a^2,\sum a^2=2(s^2-r^2-4Rr)$ *and* $\sum r_a=4R+r.$

Let's get back to the main problem.

Using the Lemma the inequality can be rewritten:

$$\frac{3(s^2-r^2-4Rr)}{2(4R+r)}\le 2R-r\Leftrightarrow 3s^2\le 16R^2+8Rr+r^2$$

which follows from Euler's inequality $R\ge 2r.$

Equality holds if and only if the triangle is equilateral.

Remark. *We obtain an inequality having an opposite sense.*

3) In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}\ge\frac{2r}{R}(2R-r)$$

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Solution

Using Lemma, we write the inequality:

$$\frac{3(s^2-r^2-4Rr)}{2(4R+r)} \geq \frac{2r}{R}(2R-r) \Leftrightarrow 3Rs^2 \geq r(44R^2-5Rr-4r^2)$$

which follows from Gerretsen's inequality $s^2 \geq 16Rr-5r^2$.

It remains to prove that:

$$3R(16Rr-5r^2) \geq r(44R^2-5Rr-4r^2) \Leftrightarrow 2R^2-5Rr+2r^2 \geq 0 \Leftrightarrow (R-2r)(2R-r) \geq 0,$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark. *The double inequality can be written:*

4) In ΔABC the following relationship holds:

$$\frac{2r}{R}(2R-r) \leq \frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c} \leq 2R-r$$

Solution *See inequality 1) and inequality 3). Equality holds if and only if the triangle is equilateral.*

Reference:

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