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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-V

By Marin Chirciu – Romania

1) Prove that in any ΔABC the following relationship holds:

$$\sum \sin A \cot\frac{B}{2} \le 3\left(\frac{R}{r}-\frac{1}{2}\right)$$

Proposed by Marian Ursărescu –Romania

Solution

Using CBS inequality, we obtain:

$$\left(\sum \sin A \cos\frac{B}{2}\right)^2 \le \sum \sin^2 A \sum \cot^2\frac{A}{2} \le \frac{9}{4}\left(\frac{2R}{r}-1\right)^2, \text{ which follows from:}$$

$$\sum \sin^2 A \le \frac{9}{4} \text{ and } \sum \cot^2\frac{A}{2} \le \left(\frac{2R}{r}-1\right)^2$$

$$\text{Indeed } \sum \sin^2 A = \frac{s^2-r^2-4Rr}{2R^2} \overset{(1)}{\le} \frac{9}{4}, \text{ where (1)} \Leftrightarrow s^2 \le 4R^2+4Rr+3r^2 \text{ (Gerretsen)}$$

It remains to prove that:

$$2(4R^2+4Rr+3r^2) \le 9R^2+8Rr+2r^2 \Leftrightarrow R \ge 2r \text{ (Euler's inequality)}$$

$$\text{Then } \sum \cot^2\frac{A}{2} = \frac{s^2-2r^2-8Rr}{r^2} \overset{(2)}{\le} \left(\frac{2R}{r}-1\right)^2, \text{ where (2)} \Leftrightarrow s^2 \le 4R^2+4Rr+3r^2 \text{ (Gerretsen)}$$

$$\text{From } \left(\sum \sin A \cot\frac{B}{2}\right)^2 \le \frac{9}{4}\left(\frac{2R}{r}-1\right)^2 \text{ we obtain the conclusion.}$$

Equality holds if and only if the triangle is equilateral.

Remark.

In the same way we can propose:

2) Prove that in any ΔABC the following relationship holds:

$$\sum \sin A \tan\frac{B}{2} \le \frac{3R}{4r}$$

Proposed by Marin Chirciu – Romania

Solution

Using CBS inequality we obtain:

$$\left(\sum \sin A \tan\frac{B}{2}\right)^2 \le \sum \sin^2 A \sum \tan^2\frac{A}{2} \le \frac{9}{4}\left(\frac{R}{2r}\right)^2 = \left(\frac{3R}{4r}\right)^2, \text{ which follows from:}$$

$$\sum \sin^2 A \le \frac{9}{4} \text{ and } \sum \tan^2 \frac{A}{2} \le \left(\frac{R}{2r}\right)^2.$$

Indeed, $\sum \sin^2 A = \frac{s^2 - r^2 - 4Rr}{2R^2} \overset{(1)}{\le} \frac{9}{4}$, *where (1)* $\Leftrightarrow s^2 \le 4R^2 + 4Rr + 3r^2$ *(Gerretsen)*

It remains to prove that:

$$2(4R^2 + 4Rr + 3r^2) \le 9R^2 + 8Rr + 2r^2 \Leftrightarrow R \ge 2r \text{ (Euler's inequality)}$$

Then $\sum \tan^2 \frac{A}{2} = \left(\frac{4R+r}{s}\right)^2 - 2 \overset{(2)}{\le} \left(\frac{R}{2r}\right)^2$, *where (2)* $\Leftrightarrow \frac{(4R+r)^2}{s^2} \le \frac{R^2+8r^2}{r^2}$, *which follows from*

Gerretsen's inequality $s^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:*

$$\frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}} \le \frac{R^2+8r^2}{r^2} \Leftrightarrow (R-2r)^2 \ge 0, \text{ obviously with equality if } R = 2r.$$

From $\left(\sum \sin A \tan \frac{B}{2}\right)^2 \le \left(\frac{3R}{4r}\right)^2$ *we obtain the conclusion.*

Equality holds if and only if the triangle is equilateral.

3) Prove that in any ΔABC the following relationship holds:

$$\sum \sin A \sin \frac{B}{2} \le \frac{3\sqrt{3}R}{8r}$$

Proposed by Marin Chirciu – Romania

Solution

Using CBS inequality, we obtain:

$$\left(\sum \sin A \sin \frac{B}{2}\right)^2 \le \sum \sin^2 A \sum \sin^2 \frac{A}{2} \le \frac{9}{4} \cdot \frac{3R^2}{16r^2} = \frac{27R^2}{64r^2}, \text{ which follows from:}$$

$$\sum \sin^2 A \le \frac{9}{4} \text{ and } \sum \sin^2 \frac{A}{2} \le \frac{3R^2}{16r^2}$$

Indeed $\sum \sin^2 A = \frac{s^2 - r^2 - 4Rr}{2R^2} \overset{(1)}{\le} \frac{9}{4}$, *where (1)* $\Leftrightarrow s^2 \le 4R^2 + 4Rr + 3r^2$ *(Gerretsen)*

It remains to prove that:

$$2(4R^2 + 4Rr + 3r^2) \le 9R^2 + 8Rr + 2r^2 \Leftrightarrow R \ge 2r \text{ (Euler's inequality)}$$

Then $\sum \sin^2 \frac{A}{2} = 1 - \frac{r}{2R} \overset{(2)}{\le} \frac{3R}{8r}$, *where (2)* $\Leftrightarrow 3R^2 - 8Rr + 4r^2 \ge 0 \Leftrightarrow$

$$\Leftrightarrow (R-2r)(3R-2r) \ge 0, \text{ obviously from Euler's inequality } R \ge 2r.$$

From $\left(\sum \sin A \sin \frac{B}{2}\right)^2 \le \frac{27R^2}{64r^2}$ *obviously the conclusion.*

4) Prove that in any ΔABC the following relationship holds:

$$\sum \sin A \cos\frac{B}{2} \le \frac{9}{4}$$

Proposed by Marin Chirciu – Romania

Solution

Using CBS inequality we obtain:

$$\left(\sum \sin A \cos\frac{B}{2}\right)^2 \le \sum \sin^2 A \sum \cos^2\frac{A}{2} \le \frac{9}{4}\cdot\frac{9}{4} = \frac{81}{16}, \text{ which follows from:}$$

$$\sum \sin^2 A \le \frac{9}{4} \text{ and } \sum \cos^2\frac{A}{2} \le \frac{9}{4}. \text{ Indeed } \sum \sin^2 A = \frac{s^2-r^2-4Rr}{2R^2} \overset{(1)}{\le} \frac{9}{4} \text{ where (1)} \Leftrightarrow$$

$$\Leftrightarrow s^2 \le 4R^2 + 4Rr + 3r^2 \text{ (Gerretsen).}$$

It remains to prove that:

$$2(4R^2 + 4Rr + 3r^2) \le 9R^2 + 8Rr + 2r^2 \Leftrightarrow R \ge 2r \text{ (Euler's inequality)}$$

$$\text{Then } \sum \cos^2\frac{A}{2} = 2 + \frac{r}{2R} \overset{(2)}{\le} \frac{9}{4} \Leftrightarrow R \ge 2r \text{ (Euler's inequality)}$$

$$\text{From } \left(\sum \sin A \cos\frac{B}{2}\right)^2 \le \frac{81}{16} \text{ we obtain the conclusion.}$$

Equality holds if and only if the triangle is equilateral.

5) Prove that in any ΔABC the following relationship holds:

$$\sum \sin A \csc\frac{B}{2} \le 3\sqrt{3}\left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution

Using CBS inequality, we obtain:

$$\left(\sum \sin A \csc\frac{B}{2}\right)^2 \le \sum \sin^2 A \sum \csc^2\frac{A}{2} \le \frac{9}{4}\cdot\frac{3R^4}{4r^4} = \frac{27R^4}{16r^4}, \text{ which follows from:}$$

$$\sum \sin^2\frac{A}{2} \le \frac{9}{4} \text{ and } \sum \csc^2\frac{A}{2} \le \frac{3R^4}{4r^4}$$

$$\text{Indeed } \sum \sin^2 A = \frac{s^2-r^2-4Rr}{2R^2} \overset{(1)}{\le} \frac{9}{4}, \text{ where (1)} \Leftrightarrow s^2 \le 4R^2 + 4Rr + 3r^2 \text{ (Gerretsen)}$$

It remains to prove that:

$$2(4R^2 + 4Rr + 3r^2) \le 9R^2 + 8Rr + 2r^2 \Leftrightarrow R \ge 2r \text{ (Euler's inequality)}$$

Then $\sum \csc^2\frac{A}{2} = \sum \frac{1}{\sin^2\frac{A}{2}} = \frac{s^2+r^2-8Rr}{r^2} \overset{(2)}{\leq} \frac{3R^4}{4r^4}$ *where (2)*

$\Leftrightarrow 3R^4 - 16R^2r^2 + 16Rr^3 - 16r^4 \geq 0 \Leftrightarrow (R-2r)(3R^3 + 6R^2r - 4Rr^2 + 8r^3) \geq 0,$

obviously from Euler's inequality $R \geq 2r$*. From* $\left(\sum \sin A \csc\frac{B}{2}\right)^2 \leq \frac{27R^4}{16r^4}$ *we obtain the conclusion.*

Equality holds if and only if the triangle is equilateral.

6) Prove that in any ΔABC the following relationship holds:

$$\sum \sin A \sec\frac{B}{2} \leq \frac{3R}{2r}$$

Proposed by Marin Chirciu – Romania

Solution

Using CBS inequality we obtain:

$$\left(\sum \sin A \sec\frac{B}{2}\right)^2 \leq \sum \sin^2 A \sum \sec^2\frac{A}{2} \leq \frac{9}{4}\cdot\left(\frac{R}{r}\right)^2 = \left(\frac{3R}{2r}\right)^2, \text{ which follows from:}$$

$$\sum \sin^2 A \leq \frac{9}{4} \text{ and } \sum \sec^2\frac{A}{2} \leq \left(\frac{R}{r}\right)^2.$$

Indeed $\sum \sin^2 A = \frac{s^2-r^2-4Rr}{2R^2} \overset{(1)}{\leq} \frac{9}{4}$, *where (1)* $\Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2$ *(Gerretsen)*

It remains to prove that:

$2(4R^2 + 4Rr + 3r^2) \leq 9R^2 + 8Rr + 2r^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality)*

Then $\sum \sec^2\frac{A}{2} = \sum \frac{1}{\cos^2\frac{A}{2}} = 1 + \left(\frac{4R+r}{s}\right)^2 \overset{(2)}{\leq} \left(\frac{R}{r}\right)^2$, *where (2)* $\Leftrightarrow \frac{(4R+r)^2}{s^2} \leq \frac{R^2-r^2}{r^2}$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:*

$$\frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}} \leq \frac{R^2-r^2}{r^2} \Leftrightarrow R^2 - Rr - 2r^2 \geq 0 \Leftrightarrow (R-2r)(R+r) \geq 0$$

Obviously from Euler's inequality $R \geq 2r$*. From* $\left(\sum \sin A \sec\frac{B}{2}\right)^2$ *we obtain the conclusion.*

Equality holds if and only if the triangle is equilateral.

Reference:

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