ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-VII

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1) In ΔABC the following relationship holds:

$$\frac{R}{r}+\frac{r}{R}+6\sum\frac{r_a^2}{bc}\geq 16$$

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Solution

We prove the following lemma:

Lemma.

2) In ΔABC the following relationship holds:

$$\sum\frac{r_a^2}{bc}=\frac{2R(4R+r)-s^2}{2Rr}$$

Proof.

Using $r_a=\frac{S}{s-a}$ *we obtain* $\sum\frac{r_a^2}{bc}=\sum\frac{\frac{S^2}{(s-a)^2}}{bc}=\frac{S^2}{abc}\sum\frac{a}{(s-a)^2}=\frac{4R(4R+r)-s^2}{r^2s}$*, which follows from*

$$\sum\frac{a}{(s-a)^2}=\frac{4R(4R+r)-2s^2}{r^2s}.$$

Let's get back to the main problem.

Using Lemma the inequality from enunciation can be written:

$\frac{R}{r}+\frac{r}{R}+6\cdot\frac{2R(4R+r)-s^2}{2Rr}\geq 16\Leftrightarrow 3s^2\leq 25R^2-10Rr+r^2$ *which follows from Gerreten's inequality:* $s^2\leq 4R^2+4Rr+3r^2$*. It remains to prove that:*

$$3(4R^2+4Rr+3r^2)\leq 25R^2-10Rr+r^2\Leftrightarrow 13R^2-22Rr-8r^2\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(13R+4r)\geq 0$*, obviously from Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.*We can develop the inequality:*

3) In ΔABC the following inequality holds:

$$\frac{R}{r}+\frac{r}{R}+n\sum\frac{r_a^2}{bc}\geq\frac{5}{2}+\frac{9n}{4},\ \text{where } n\geq 0.$$

Proposed by Marin Chirciu – Romania

Solution

We prove that: $\frac{R}{r}+\frac{r}{R}\ge\frac{5}{2}$, *(1) and* $\sum\frac{r_a^2}{bc}\ge\frac{9}{4}$ *(2)*

Indeed $\frac{R}{r}+\frac{r}{R}\ge\frac{5}{2}\Leftrightarrow 2R^2-5Rr+2r^2\ge 0\Leftrightarrow(R-2r)(2R+r)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$.

For the inequality $\sum\frac{r_a^2}{bc}\ge\frac{9}{4}$ *we use the following Lemma and we write the inequality:*

$\frac{2R(4R+r)-s^2}{2Rr}\ge\frac{9}{4}\Leftrightarrow 2s^2\le 16R^2+4Rr-9r^2$, *which follows from Gerretsen's inequality* $s^2\le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$2(4R^2+4Rr+3r^2)\le 16R^2+4Rr-9r^2\Leftrightarrow 8R^2-13Rr-6r^2\ge 0\Leftrightarrow$$

$\Leftrightarrow(R-2r)(8R+3r)\ge 0$, *which follows from Euler's inequality* $R\ge 2r$.

From (1), (2) and the condition from hypothesis $n\ge 0$ *it follows the conclusion:*

$$\frac{R}{r}+\frac{r}{R}+n\sum\frac{r_a^2}{bc}\ge\frac{5}{2}+\frac{9n}{4}$$

Equality holds if and only if the triangle is equilateral.

Reference:

Romanian Mathematical Magazine-www.ssmrmh.ro