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ABOUT AN INEQUALITY BY OLEH FAYSHTEYN-I

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{b^2+c^2}{(s-a)^2}+\frac{c^2+a^2}{(s-b)^2}+\frac{a^2+b^2}{(s-c)^2}\ge 24$$

Proposed by Oleh Faynshteyn – Leipzig – Germany

Solution *We prove the following lemma:*

Lemma.

2) In ΔABC the following relationship holds:

$$\sum\frac{b^2+c^2}{(s-a)^2}=\frac{2[s^2(8R^2+16Rr+2r^2-s^2)-r(4R+r)^3]}{r^2s^2}$$

Proof. *We have* $\sum\frac{b^2+c^2}{(s-a)^2}=\frac{\sum(b^2+c^2)(s-b)^2(s-c)^2}{\prod(s-a)^2}=\frac{2r^2[s^2(8R^2+16Rr+2r^2-s^2)-r(4R+r)^3]}{r^4s^2}$

$=\frac{2[s^2(8R^2+16Rr+2r-s^2)-r(4R+r)^3]}{r^2s^2}$ *which follows from*

$\sum(b^2+c^2)(s-b)^2(s-c)^2=2r^2[s^2(8R^2+16Rr+2r^2-s^2)-r(4R+r)^3]$ *and*

$\prod(s-a)=r^2s.$

Let's get back to the main problem. Using the Lemma we write the inequality:

$\frac{2[s^2(8R^2+16Rr+2r^2-s^2)-r(4R+r)^3]}{r^2s^2}\ge 24\Leftrightarrow s^2(8R^2+16Rr+2r^2-s^2)\ge r(4R+r)^3,$

which follows from Gerretsen's inequality $16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2.$

It remains to prove that:

$$(16Rr-5r^2)(8R^2+16Rr+2r^2-4R^2-4Rr-3r^2)\ge r(4R+r)^3\Leftrightarrow$$

$\Leftrightarrow 31R^2-70Rr+16r^2\ge 0\Leftrightarrow (R-2r)(31R-8r)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$. *Equality holds if and only if the triangle is equilateral.*

Remark. *Let's find an inequality having an opposite sense.*

3) In ΔABC the following inequality holds:

$$\frac{b^2+c^2}{(s-a)^2}+\frac{c^2+a^2}{(s-b)^2}+\frac{a^2+b^2}{(s-c)^2}\le 6\left(\frac{R}{r}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution *Using the Lemma we write the inequality:*

$$\frac{2[s^2(8R^2+16Rr+2r^2-s^2)-r(4R+r)^3]}{r^2s^2}\le 6\left(\frac{R}{r}\right)^2\Leftrightarrow$$

$\Leftrightarrow s^2(5R^2+16Rr+2r^2-s^2)\le(4R+r)^3$, *which follows from Blundon – Gerretsen's inequality* $16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}$. *It remains to prove that:*

$$\frac{R(4R+r)^2}{2(2R-r)}(5R^2+16Rr+2r^2-16Rr+5r^2)\le(4R+r)^3\Leftrightarrow$$

$\Leftrightarrow 6R^2-11Rr-2r^2\ge 0\Leftrightarrow(R-2r)(6R+r)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$. *Equality holds if and only if the triangle is equilateral.*

Remark. *We write the double inequality:*

4) In ΔABC the following relationship holds:

$$24\le\frac{b^2+c^2}{(s-a)^2}+\frac{c^2+a^2}{(s-b)^2}+\frac{a^2+b^2}{(s-c)^2}\le 6\left(\frac{R}{r}\right)^2$$

Solution *See inequalities 1) and 3). Equality holds if and only if the triangle is equilateral.*

Reference:

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