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ABOUT AN INEQUALITY BY WALTER JANOUS-I

By *Marin Chirciu – Romania*

1) Show that:

$$\sum a\tan A \geq 10R-2r$$

for any acute triangle ABC, where a,b,c are its sides, R its circumradius, and$r$ its inradius, and the sum is cyclic.

Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Solution *We prove the following lemma:*

Lemma.

2) In ΔABC the following relationship holds:

$$\sum a\tan A = 2r\cdot\frac{(4R+r)(2R+r)-s^2}{s^2-(2R+r)^2}$$

Proof. *We have* $\sum a\tan A=\sum 2R\sin A\cdot\frac{\sin A}{\cos A}=2R\sum\frac{\sin^2 A}{\cos A}=2R\sum\frac{1-\cos^2 A}{\cos A}=$

$2R\sum\left(\frac{1}{\cos A}-1\right)==2R\sum\left(\frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}-1\right)=2r\cdot\frac{(4R+r)(2R+r)-s^2}{s^2-(2R+r)^2}$*, which follows from the known inequality in triangle* $\sum\frac{1}{\cos A}=\frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}$*. Let's get back to the main problem.*

Using the lemma the inequality can be written:

$2r\cdot\frac{(4R+r)(2R+r)-s^2}{s^2-(2R+r)^2}\geq 10R-2r \Leftrightarrow r(4R+r)(2R+r)+(5R-r)(2R+r)^2\geq 5Rs^2,$

which follows from Gerretsen's inequality $s^2\leq 4R^2+4Rr+3r^2$ *and the observation that in the acute angled triangle* $s^2-(2R+r)^2>0$*, which assures the elimination of the denominator from the above inequality. It remains to prove that:*

$r(4R+r)(2R+r)+(5R-r)(2R+r)^2\geq 5R(4R^2+4Rr+3r^2)\Leftrightarrow R\geq 2r$*, (Euler's inequality). Equality holds if and only if the triangle is equilateral.*

Remark *Let's strength the above inequality:*

3) In ΔABC the following inequality holds:

$$\sum a\tan A\geq xR+(18-2x)r,\ where\ x\leq 18$$

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Solution *Using the Lemma the inequality can be rewritten:*

$$\sum a\tan A = 2r\cdot\frac{(4R+r)(2R+r)-s^2}{s^2-(2R+r)^2} \geq xR+(18-2x)r \Leftrightarrow$$

$$\Leftrightarrow 2r(4R+r)(2R+r)+(2R+r)^2[xR+(18-2x)r] \geq s^2[xR+(20-2x)r]$$

which follows from Gerretsen's inequality $s^2 \leq 4R^2+4Rr+3r^2$ *and the observation that in the acute-angled triangle* $s^2-(2R+r)^2>0$, *which assures the elimination of the denominator in the above inequality. It remains to prove that:*

$$2r(4R+r)(2R+r)+(2R+r)^2[xR+(18-2x)r]$$
$$\geq (4R^2+4Rr+3r^2)[xR+(20-2x)r] \Leftrightarrow$$

$\Leftrightarrow 4R^2+(2-x)Rr+(2x-20)r^2 \geq 0 \Leftrightarrow (R-2r)[4R+(10-x)r] \geq 0$, *obviously from Euler's inequality* $R \geq 2r$ *and the condition* $x \leq 18$, *which assures* $[4R+(10-x)r] \geq 0$

Equality holds if and only if the triangle is equilateral.

Remark 1. *For* $x=10$ *we obtain Problem 1424 from Crux Mathematicorum Vol. 15, No.3 March 1989, Walther Janous, Innsbruck, Austria.*

Remark 2. *The best inequality having the form 2) it is obtained for* $x=18$. *In this case we obtain:*

4) In ΔABC the following inequality holds:

$$\sum a\tan A \geq 18(R-r)$$

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Remark 3. *Inequality 4) is stronger then inequality 1)*

5) In ΔABC the following inequality holds:

$$\sum a\tan A \geq 18(R-r) \geq 10R-2r$$

Solution *See inequality 4) and* $18(R-r) \geq 10R-2r \Leftrightarrow R \geq 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Reference:

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