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ABOUT CEBYSHEV's INEQUALITY INTEGRAL FORM-II

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Theorem:(Cebyshev's Inequality):

For $f, g: [a,b] \to R$ continuous function with same monotonicity and $p: [a,b] \to [0,\infty)$ integrable function. Then:

$$\left(\int_a^b p(x)dx\right)\left(\int_a^b p(x)f(x)g(x)dx\right) \geq \left(\int_a^b p(x)f(x)dx\right)\left(\int_a^b p(x)g(x)dx\right) \quad (*)$$

In the case f and g different monotonicity:

$$\left(\int_a^b p(x)dx\right)\left(\int_a^b p(x)f(x)g(x)dx\right) \leq \left(\int_a^b p(x)f(x)dx\right)\left(\int_a^b p(x)g(x)dx\right)$$

Proof:

If f and g are same monotonicity, $p(x) > 0, \forall x \in [a,b] \Rightarrow$

$$p(x)p(y)\big(f(x)-f(y)\big)\big(g(x)-g(y)\big) \geq 0,, \forall x \in [a,b] \Rightarrow$$

$$p(x)p(y)f(x)g(x) - p(x)p(y)f(y)g(x) - p(x)p(y)f(x)g(y) + p(x)p(y)f(y)g(y) \geq 0$$

$$p(y)\int_a^b p(x)f(x)g(x)dx - p(y)f(y)\int_a^b p(x)g(x)dx -$$

$$-p(y)g(y)\int_a^b p(x)f(x)dx + p(y)f(y)g(y)\int_a^b p(x)dx \geq 0 \leftrightarrow$$

$$\left(\int_a^b p(x)dx\right)\left(\int_a^b p(x)f(x)g(x)dx\right) - \left(\int_a^b p(x)f(x)dx\right)\left(\int_a^b p(x)g(x)dx\right)$$

$$-\left(\int_a^b p(x)g(x)dx\right)\left(\int_a^b p(x)f(x)dx\right)$$

$$+\left(\int_a^b p(x)f(x)g(x)dx\right)\left(\int_a^b p(x)dx\right) \geq 0 \Leftrightarrow (*)$$

Application.

If $f:[0,1]\to\mathbb{R}, f$ −continuous and convex function such that

$f(0)=0, f(1)=1$, then:

$$\int_0^1 \sqrt{1+x^2}\cdot log(1+x)\cdot\left(f'(x)\right)^2 dx \ge \frac{log(\sqrt{2})}{log(1+\sqrt{2})}$$

Solution:

$$\int_0^1 \sqrt{1+x^2}\cdot log(1+x)\cdot\left(f'(x)\right)^2 dx = \int_0^1 \frac{1}{\sqrt{1+x^2}}\cdot log(1+x)\cdot(1+x^2)\left(f'(x)\right)^2 dx$$

Applying Chebyshev's Inequality:

Let: $p(x)=\frac{1}{\sqrt{1+x^2}}; u(x)=log(1+x); v(x)=(1+x^2)\left(f'(x)\right)^2,$

u, v *−increasing, we have:*

$$\left(\int_0^1 \frac{dx}{\sqrt{1+x^2}}\right)\left(\int_0^1 \frac{1}{\sqrt{1+x^2}}\cdot log(1+x)\cdot(1+x^2)\left(f'(x)\right)^2 dx\right)\ge$$

$$\ge\left(\int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}}dx\right)\left(\int_0^1 \frac{1+x^2}{\sqrt{1+x^2}}\cdot\left(f'(x)\right)^2 dx\right)=$$

$$=\left(\int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}}dx\right)\left(\int_0^1 \sqrt{1+x^2}\cdot\left(f'(x)\right)^2 dx\right); \quad (1)$$

Now,

$$\int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}}dx = \int_0^1 \frac{1}{1+x^2}\cdot\sqrt{1+x^2}\cdot log(1+x)\,dx$$

Let $p_1(x)=\frac{1}{1+x^2}; u_1(x)=\sqrt{1+x^2}, v_1(x)=log(1+x); u_1, v_1$ *−increasing.*

$$\left(\int_0^1 \frac{dx}{1+x^2}\right)\left(\int_0^1 \frac{1}{1+x^2}\cdot\sqrt{1+x^2}\cdot log(1+x)\,dx\right)\ge\left(\int_0^1 \frac{\sqrt{1+x^2}}{1+x^2}dx\right)\left(\int_0^1 \frac{log(1+x)}{1+x^2}dx\right)\Leftrightarrow$$

$$\frac{\pi}{4}\int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}}dx \ge \left(\int_0^1 \frac{dx}{\sqrt{1+x^2}}\right)\left(\int_0^1 \frac{log(1+x)}{1+x^2}dx\right)\Leftrightarrow$$

$$\frac{\pi}{4}\int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}}dx \ge log(1+\sqrt{2})\left(\int_0^1 \frac{log(1+x)}{1+x^2}dx\right); \quad (2)$$

$$\int_0^1 \frac{log(1+x)}{1+x^2}dx \overset{x=tan\,u}{=} \int_0^{\frac{\pi}{4}} \frac{log(1+tanu)}{\frac{1}{cos^2u}} \cdot \frac{du}{cos^2u} =$$

$$= \int_0^{\frac{\pi}{4}} log\left(\frac{sinu+cosu}{cosu}\right)du = \int_0^{\frac{\pi}{4}} log\left[\frac{\sqrt{2}cos\left(\frac{\pi}{4}-u\right)}{cosu}\right]du =$$

$$= \int_0^{\frac{\pi}{4}} log\sqrt{2}du + \int_0^{\frac{\pi}{4}} log\left[cos\left(\frac{\pi}{4}-u\right)\right]du - \int_0^{\frac{\pi}{4}} log(cosu)du =;$$

$$\int_0^{\frac{\pi}{4}} log\left[cos\left(\frac{\pi}{4}-u\right)\right]du \overset{\frac{\pi}{4}-u=v}{=} -\int_0^{\frac{\pi}{4}} log(cosv)dv = v$$

$$\int_0^1 \frac{log(1+x)}{1+x^2}dx = \frac{\pi}{8}log2; \ \ (3)$$

Replacing (3) in (2), we get:

$$\frac{\pi}{4} \cdot \int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}}dx \geq log(1+\sqrt{2}) \cdot \frac{\pi}{8}log2 \Leftrightarrow$$

$$\int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}}dx \geq log(1+\sqrt{2})log(\sqrt{2}); \ \ (4)$$

Now,

$$1 = f(1) - f(0) = \int_0^1 f'(x)\,dx = \int_0^1 \frac{1}{\sqrt[4]{1+x^2}} \cdot \sqrt[4]{1+x^2} \cdot \left(f'(x)\right)^2 dx \overset{CBS}{\leq}$$

$$\leq \left(\int_0^1 \frac{dx}{\sqrt{1+x^2}}\right)^{\frac{1}{2}} \cdot \left(\int_0^1 \sqrt{1+x^2} \cdot \left(f'(x)\right)^2 dx\right)^{\frac{1}{2}} =$$

$$= \sqrt{log(1+\sqrt{2})} \cdot \left(\int_0^1 \sqrt{1+x^2} \cdot \left(f'(x)\right)^2 dx\right)^{\frac{1}{2}}$$

Hence,

$$\int_0^1 \sqrt{1+x^2}\cdot\left(f'(x)\right)^2dx \ge \frac{1}{log\left(1+\sqrt{2}\right)};\quad (5)$$

Now, we get:

$$\left(\int_0^1 \frac{dx}{\sqrt{1+x^2}}\right)\left(\int_0^1 \frac{1}{\sqrt{1+x^2}}\cdot log(1+x)\cdot(1+x^2)\left(f'(x)\right)^2 dx\right) \ge$$

$$\ge \left(\int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}}dx\right)\left(\int_0^1 \sqrt{1+x^2}\cdot\left(f'(x)\right)^2dx\right) \overset{(4)}{\ge}$$

$$\overset{(4)}{\ge} log\left(1+\sqrt{2}\right)log\left(\sqrt{2}\right)\left(\int_0^1 \sqrt{1+x^2}\cdot\left(f'(x)\right)^2dx\right) \overset{(5)}{\ge}$$

$$\overset{(5)}{\ge} log\left(1+\sqrt{2}\right)\cdot log\left(\sqrt{2}\right)\cdot\frac{1}{log\left(1+\sqrt{2}\right)} = log\left(\sqrt{2}\right) \Leftrightarrow$$

$$log\left(1+\sqrt{2}\right)\cdot\int_0^1 \sqrt{1+x^2}\cdot log(1+x)\cdot\left(f'(x)\right)^2 dx \ge log\left(\sqrt{2}\right)$$

Therefore,

$$\int_0^1 \sqrt{1+x^2}\cdot log(1+x)\cdot\left(f'(x)\right)^2 dx \ge \frac{log\left(\sqrt{2}\right)}{log\left(1+\sqrt{2}\right)}$$

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