ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

# A COUNTING PROBLEM IN TRIANGLE

*By Carmen-Victoriţa Chirfot -Romania*

**Let be a triangle $ABC$. Consider the points $M_1, M_2, \dots, M_n$ on the side $(AB)$ and $N_1, N_2, \dots, N_3$ on the side $(AC), n \in \mathbb{N}^*$ such that**

**$BM_1 = M_1M_2 = M_2M_3 = \dots = M_{n-1}M_n = M_nA$ and**

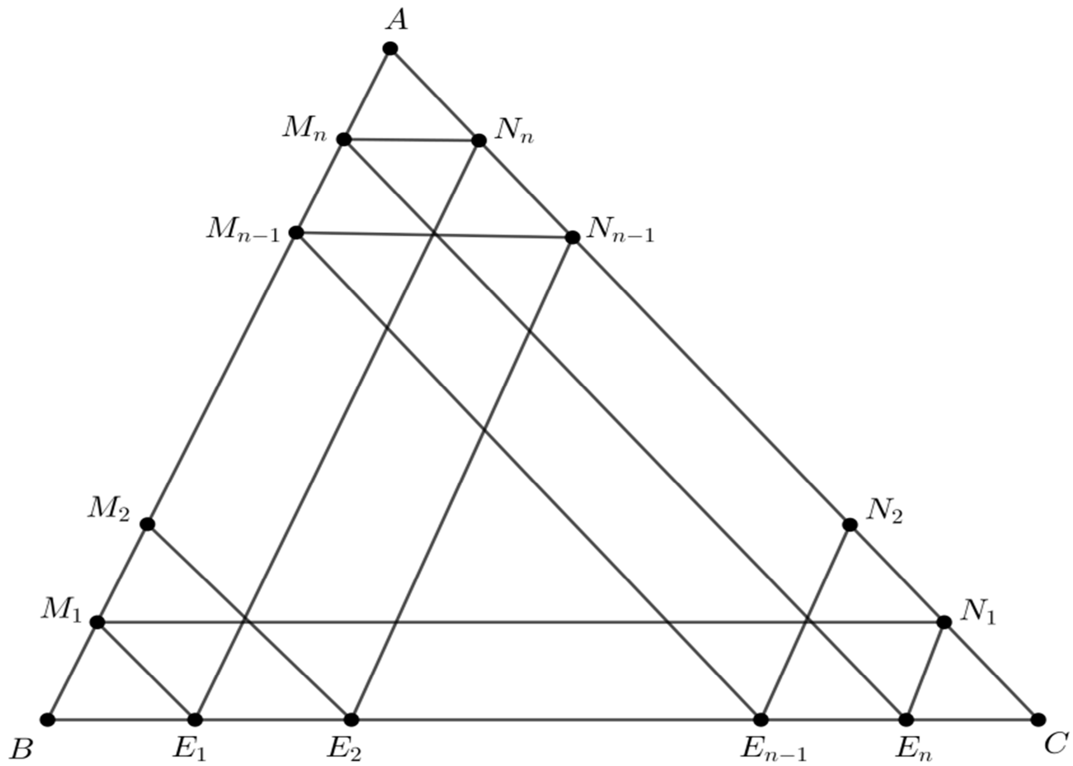
**$CN_1 = N_1N_2 = \dots = N_{n-1}N_n = N_nA$.**

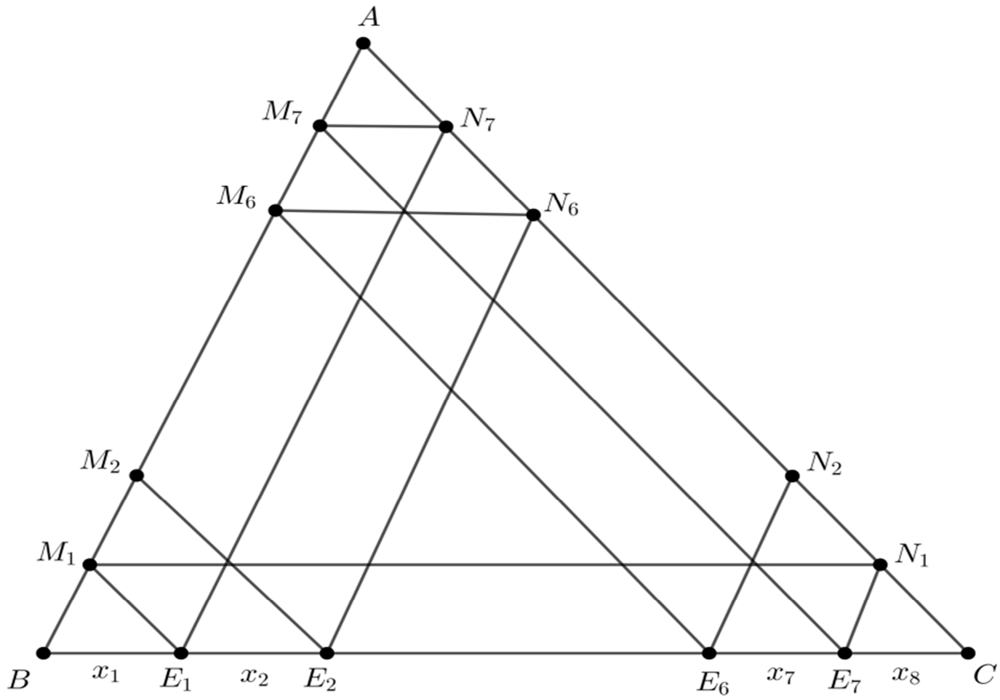
**We also consider the points $E_1, E_2, \dots, E_n$ on the side $(BC)$ such that**

**$BE_1 = E_1E_2 = E_2E_3 = \dots = E_{n-1}E_n = M_nC$.**

**Obviously, from the reciprocal of Thales' theorem, $M_kN_k \parallel BC, k = \overline{1,n}$, $E_kN_{n+1-k} \parallel AB, k = \overline{1,n}, E_kM_k \parallel AC, k = \overline{1,n}$.**

**We join each point $M_k$ with $E_k$, and each point $M_k$ with $N_k$, and each point $E_k$ with its corespondent $N_{n+1-k}$.**






**We intend, for a start, to determine the number of such triangles formed by the intersections of the given segments.**

**All quadrilaterals resulting at the intersection of parallel lines $M_kN_k$ and $M_{k+1}N_{k+1}, k = \overline{1, n-1}$ with parallel lines $E_kN_{n+1-k}$ and $E_{k+1}N_{n-k}, k = \overline{1, n-1}$, are parallelograms. Analogous to $M_kN_k$ and $M_{k+1}N_{k+1}, k = \overline{1, n}$ with $E_kM_k \parallel E_{k+1}M_{k+1}$, $k = \overline{1, n-1}$.**

**Let be $(x_n)_{n\geq 1}$ the sequence with $x_k$ representing the number of triangles inside the triangle $BE_kM_k$. Then $x_1 = 1, x_2 = x_1 + 2 \cdot 1 + 1 + 1 = 5$. For the triangle $BE_3M_3$, the number of interior triangles is how many $x_2$ (as we have already found) to which we add two triangles similar to the vertex in $M_3$ and the bases on $M_2N_2$, respectively on $M_1N_1$, 2 triangles similar to the vertex in $E_3$ and the bases on $E_2N_{n-1}$, respectively on $E_1N_n$ and count once the large triangle $BE_3M_3$. So there are 3 innumerable triangles between the parallel lines $E_2M_2$ and $E_3M_3$. Thus, $x_3 = x_2 + 2 \cdot 2 + 1 + 3 = 13$.**

**For the triangle $BE_4M_4$, the number of interior triangles is how many $x_3$ we still count 3 triangles similar to the vertex in $M_4$ and the bases on $M_3N_3$, $M_2N_2$, respectively**

**$M_1N_1$, 3 triangles similar to the vertex in $E_4$ and the bases on $E_3N_{n-2}$,$E_2N_{n-1}$ , respectively on $E_1N_n$ and count once the large triangle $BE_4M_4$.**

**A triangle is also formed with the base on $E_2M_2$ and the tip on $E_4M_4$. We also have a triangle with the base on $E_4M_4$ and the tip on $E_2M_2$ (let's call it the opposite of the previous triangle).**

**There are 5 countless triangles between the parallel lines $E_3M_3$ and $E_4M_4$. Thus, $x_4 = x_3 + 2 \cdot 3 + 1 + (1) + (1) + 5 = 27$.**

**For the triangle $BE_5M_5$, the number of interior triangles is how much $x_4$ we count 4 triangles similar to the vertex in $M_5$ and the bases on $M_4N_4, M_3N_3, M_2N_2$ respectively $M_1N_1$, 4 triangles similar to the vertex in $E_5$ and the bases on $E_4N_{n-3}, E_3N_{n-2}, E_2N_{n-1}$ respectively on $E_1N_n$ and we count the big triangle only once $BE_5M_5$.**

**It also forms 2 triangles with base on $E_3M_3$and $E_5M_5$ tip on. We also have 2 triangles with base on $E_5M_5$ and the tip on $E_3M_3$ and a triangle with base on $E_5M_5$ and the $E_2M_2$ tip on.**

**There are 7 countless triangles between the parallel lines $E_4M_4$ and $E_5M_5$. Thus, $x_5 = x_4 + 2 \cdot 4 + 1 + (2 + 0) + (2 + 1) + 7 = 48$.**

**For the triangle $BE_6M_6$, the number of interior triangles is how much $x_5$ we count 5 triangles similar to the vertex in $M_6$ and the bases on $M_5N_5, M_4N_4, M_3N_3$ respectively $M_1N_1$, 5 triangles similar to the vertex in $E_6$ and the bases on $E_5N_{n-4}, E_4N_{n-3}, E_3N_{n-2}, E_2N_{n-1}$ respectively on $E_1N_n$ and we count the big triangle only once $BE_6M_6$. It also forms 3 triangles with base on $E_4M_4$ and $E_6M_6$ tip on. We also have 2 triangles with base on $E_6M_6$ and the tip on $E_3M_3$ and a triangle with base on $E_6M_6$ and the $E_2M_2$ tip on.**

**There are 9 countless triangles between the parallel lines $E_5M_5$ and $E_6M_6$.**

**Thus, $x_6 = x_5 + 2 \cdot 5 + 1 + (3 + 1) + (3 + 2 + 1) + 9 = 78$**

**For the triangle $BE_7M_7$, the number of interior triangles is how much $x_6$ we count 6 triangles similar to the vertex in $M_7$ and the bases on $M_6N_6, M_5N_5, M_4N_4, M_3N_3, M_2N_2$ respectively $M_1N_1$, 6 triangles similar to the vertex in $E_7$ and the bases on**

**$E_6N_{n-5}, E_5N_{n-4}, E_4N_{n-3}, E_3N_{n-2}, E_2N_{n-1}$ respectively on $E_1N_n$ and we count the big triangle only once $BE_7M_7$.**

**It also forms 2 triangles with base on $E_4M_4$ and $E_7M_7$ tip on. We also have 4 triangles with base on $E_7M_7$ and the tip on $E_4M_4$ and a triangle with base on $E_7M_7$ and the $E_3M_3$ tip on, and a triangle with base on $E_7M_7$ and the $E_2M_2$ tip on.**

**There are 11 countless triangles between the parallel lines $E_6M_6$ and $E_7M_7$.**

**Thus, $x_7 = x_6 + 2\cdot 6 + 1 + (4+2) + (4+3+2+1) + 11 = 118$.**

**Respecting the algorithm, we obtain that:**

$$x_8 = x_7 + 2\cdot 7 + 1 + (5+3+1) + (5+4+3+2+1) + 13 = 170$$

**We observe that, if $n = 2k, k \in \mathbb{N}^*, k \geq 2$, we get:**

$$x_{2k} = x_{2k-1} + 2(2k-1) + 1 + \big((2k-3) + (2k-5) + \cdots + 1\big) + \big((2k-3) + (2k-4) + \cdots + 1\big) + 4k - 3 = x_{2k-1} + 3k^2 + k.$$

**We observe that, if $n = 2k+1, k \in \mathbb{N}^*, k \geq 2$, we get:**

$$x_{2k+1} = x_{2k} + 2(2k+1) + \big((2k-2) + (2k-4) + \cdots + 2\big) + \big((2k-2) + (2k-3) + (2k-4) + \cdots + 1\big) + 4k - 1 = x_{2k} + 3k^2 + 4k + 1.$$

**Let's come back to the term $x_n$, we have: $x_1 = 1, x_2 = 5$,**

$$x_n = \begin{cases} x_{n-1} + \dfrac{3n^2+2n}{4}, if\ n \in \mathbb{N}^* - even, n \geq 4 \\ x_{n-1} + \dfrac{3n^2+2n-1}{4}, if\ n \in \mathbb{N}^* - odd, n \geq 3 \end{cases}$$

**So,** $$x_n = \begin{cases} 5 + \frac{3(3^2+4^2+\cdots+n^2)+2(3+4+5+\cdots+n)}{4} - \frac{n-2}{8}, if\ n - odd \\ 5 + \frac{3(3^2+4^2+\cdots+n^2)+2(3+4+5+\cdots+n)}{4} - \frac{n-1}{8}, if\ n - even \end{cases}$$

**Hence,** $$x_n = \begin{cases} \frac{2n^3+5n^2+2n}{8}, n \geq 4, n - odd \\ \frac{2n^3+5n^2+2n-1}{8}, n \geq 3, if\ n - even \end{cases}$$

**Consider $A = M_{n+1}$ and $C = E_{n+1}$, then $x_{n+1}$ represent the number of interior triangles of triangle $ABC$ formed according to the given rule.**

**This is the minimum number of triangles inside the triangle $ABC$ formed by**

- **$n$ lines parallel to each other $a_1, a_2, \dots, a_n$ and parallel to $AB$ that intersect the sides $(AC)$ and $(BC)$,**
- **$n$ lines parallel to each other $b_1, b_2, \dots, b_n$ and parallel to $AB$ that intersect the sides $(AB)$ and $(AC)$,**
- **$n$ lines parallel to each other $c_1, c_2, \dots, c_n$ and parallel to $AB$ that intersect the sides $(AB)$ and $(BC)$.**

**The problem presented above is where any three straight lines $a_m, b_n, c_p$, and are concurrent, $m, n, p = \overline{1, n}$, i.e. any denote triangle $a_m b_n c_p$ is degenerate.**

**Refferences:**

**1. ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro**

**2. https://math.stackexchange.com/questions/203873/how-many-triangles**