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ABOUT A RMM IDENTITY-IV

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1) In ΔABC the following relationship holds:

$$\sum\sqrt{\frac{r_a-r}{(r_b-r)(r_c-r)}}=\frac{1}{\sqrt{R}}\left(\frac{2R}{r}-1\right)$$

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Solution *Denoting with S the sum from enunciation, we have:*

$$S^2=\left(\sum\sqrt{\frac{r_a-r}{(r_b-r)(r_c-r)}}\right)^2=$$

$$=\sum\frac{r_a-r}{(r_b-r)(r_c-r)}+2\sum\sqrt{\frac{r_b-r}{(r_c-r)(r_a-r)}}\sqrt{\frac{r_c-r}{(r_a-r)(r_b-r)}}=$$

$$=\sum\frac{r_a-r}{(r_b-r)(r_c-r)}+2\sum\frac{1}{r_a-r}$$

We prove that following lemmas:

Lemma 1.

2) In ΔABC the following relationship holds:

$$\sum\frac{r_a-r}{(r_b-r)(r_c-r)}=\frac{8R^2+r^2-s^2}{2Rr^2}$$

Proof. *Using* $r_a=\frac{S}{s-a}$

we obtain $\sum\frac{r_a-r}{(r_b-r)(r_c-r)}=\sum\frac{\frac{S}{s-a}-\frac{S}{s}}{\left(\frac{S}{s-b}-\frac{S}{p}\right)\left(\frac{S}{s-c}-\frac{S}{s}\right)}=\frac{1}{4R}\sum\frac{a^2}{(s-a)^2}=\frac{8R^2+r^2-s^2}{2Rr^2}$

which follows from $\sum\frac{a^2}{(s-a)^2}=\frac{2(8R^2+r^2-s^2)}{r^2}$

Lemma 2.

3) In ΔABC the following relationship holds:

$$\sum \frac{1}{r_a - r} = \frac{s^2 + r^2 - 8Rr}{4Rr^2}$$

Proof.

Using $r_a = \frac{S}{s-a}$ *we obtain* $\sum \frac{1}{r_a - r} = \sum \frac{1}{\frac{S}{s-a} - \frac{S}{s}} = \frac{1}{r}\sum \frac{s-a}{a} = \frac{s^2+r^2-8Rr}{4Rr^2}$, *which follows from* $\sum \frac{s-a}{a} = \frac{s^2+r^2-8Rr}{4Rr}$. *Let's get back to the main problem. Using the Lemmas above, we obtain:*

$$S^2 = \sum \frac{r_a - r}{(r_b - r)(r_c - r)} + 2\sum \frac{1}{r_a - r} = \frac{8R^2+r^2-s^2}{2Rr^2} + 2 \cdot \frac{s^2+r^2-8Rr}{4Rr^2} = \frac{1}{R}\left(\frac{2R}{r} - 1\right)^2. \text{ From}$$

$$S^2 = \frac{1}{R}\left(\frac{2R}{r} - 1\right)^2 \text{ it follows from } S = \frac{1}{\sqrt{R}}\left(\frac{2R}{r} - 1\right).$$

Remark.

If we replace r_a *with* h_a *we propose:*

4) In ΔABC the following relationship holds:

$$\frac{2}{\sqrt{R}}\left(2 - \frac{r}{R}\right) \le \sum \sqrt{\frac{h_a - r}{(h_b - r)(h_c - r)}} \le \frac{1}{\sqrt{R}}\left(\frac{2R}{r} - 1\right)$$

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Solution *Denoting with S the sum from enunciation, we have:*

$$S^2 = \left(\sum \sqrt{\frac{h_a - r}{(h_b - r)(h_c - r)}}\right)^2 = \sum \frac{h_a - r}{(h_b - r)(h_c - r)} +$$

$$+2\sum \sqrt{\frac{h_b - r}{(h_c - r)(h_a - r)}}\sqrt{\frac{h_c - r}{(h_a - r)(h_b - r)}} = \sum \frac{h_a - r}{(h_b - r)(h_c - r)} + 2\sum \frac{1}{h_a - r}$$

We prove the following lemmas:

Lemma 1.

5) In ΔABC the following relationship holds:

$$\sum \frac{h_a - r}{(h_b - r)(h_c - r)} = \frac{s^2(s^2 + 2r^2 - 12Rr) + r^2(12R^2 + 4Rr + r^2)}{2Rrs^2(s^2 + r^2 + 2Rr)}$$

Proof. *Using* $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum\frac{h_a-r}{(h_b-r)(h_c-r)}=\sum\frac{\frac{2S}{a}-\frac{S}{s}}{\left(\frac{2S}{b}-\frac{S}{s}\right)\left(\frac{2S}{c}-\frac{S}{s}\right)}=\frac{1}{r}\sum\frac{bc(b+c)}{a(a+b)(a+c)}=$$

$=\frac{s^2(s^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2)}{2Rr^2(s^2+r^2+2Rr)}$, *which follows from*

$$\sum\frac{bc(b+c)}{a(a+b)(a+c)}=\frac{s^2(s^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2)}{2Rr(s^2+r^2+2Rr)}.$$

Lemma 2.

6) In ΔABC the following inequality holds:

$$\sum\frac{1}{h_a-r}=\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}$$

Proof.

Using $r_a=\frac{S}{s-a}$ *we obtain* $\sum\frac{1}{r_a-r}=\sum\frac{1}{\frac{S}{s-a}-\frac{S}{s}}=\frac{1}{r}\sum\frac{s-a}{a}=\frac{s^2+r^2-8Rr}{4Rr^2}$, *which follows from*

$\sum\frac{s-a}{a}=\frac{s^2+r^2-8Rr}{4Rr}$. *Let's get back to the main problem. Using the above Lemmas, we obtain:*

$$S^2=\sum\frac{h_a-r}{(h_b-r)(h_c-r)}+2\sum\frac{1}{h_a-r}=$$

$$=\frac{s^2(s^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2)}{2Rr^2(s^2+r^2+2Rr)}+2\cdot\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}=$$

$$=\sum\frac{h_a-r}{(h_b-r)(h_c-r)}=\frac{s^2(s^2+2r^2-8Rr)+r^2(2R-r)^2}{2Rr^2(s^2+r^2+2Rr)}$$

RHS inequality

Using $S^2=\frac{s^2(s^2+2r^2-8Rr)+r^2(2R-r)^2}{2Rr^2(s^2+r^2+2Rr)}$ *the inequality is equivalent with:*

$$\frac{s^2(s^2+2r^2-8Rr)+r^2(2R-r)^2}{2Rr^2(s^2+r^2+2Rr)}\le\frac{1}{R}\left(\frac{2R}{r}-1\right)^2\Leftrightarrow$$

$$s^2(s^2+4Rr-8R^2)\le(2R-r)^2(4Rr+r^2)$$

We distinguish the following cases:

Case 1). If $(s^2+4Rr-8R^2)\le 0$, *the inequality is obvious.*

Case 2). If $(s^2+4Rr-8R^2)>0$, *using Gerretsen's inequality* $s^2\le 4R^2+4Rr+3r^2$, *it suffice to prove that:*

$$(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+4Rr-8R^2)\le(2R-r)^2(4Rr+r^2)\Leftrightarrow$$

$$\Leftrightarrow 4R^2-11R^2r^2-9Rr^2-2r^4\ge 0\Leftrightarrow (R-2r)(4R^3+8R^2r+5Rr^2+r^3)\ge 0,$$

obviously from Eule's $R\ge 2r$. Equality holds if and only if the triangle is equilateral.

LHS

$$\frac{s^2(s^2+2r^2-8Rr)+r^2(2R-r)^2}{2Rr^2(s^2+r^2+2Rr)}\ge \frac{4}{R}\left(2-\frac{r}{R}\right)^2\Leftrightarrow$$

$$\Leftrightarrow s^2\left(s^2R^2-r(4R^3+30R^2r-32Rr^2+8r^3)\right)+r^2(2R-r)^2(R^2-16Rr-r^2)\ge 0$$

Using Gerretsen's inequality $s^2\ge 16Rr-5r^2$ and the observation that

$\left(s^2R^2-r(4R^3+30R^2r-32Rr^2+8r^3)\right)>0$, it suffices to prove that:

$$(16Rr-5r^2)\left((16Rr-5r^2)R^2-r(4R^3+30R^2r-32Rr^2+8r^3)\right)+$$

$$+r^2(2R-r)^2(R^2-16Rr-r^2)\ge 0\Leftrightarrow 49R^4-172R^3r+180R^2r^2-68Rr^3+8r^4\ge 0$$

$\Leftrightarrow (R-2r)(49R^3-74R^2r+32Rr^2-4r^3)\ge 0$, obviously from Euler's inequality

$R\ge 2r$. Equality holds if and only if the triangle is equilateral.

Remark.

Between the sums $\sum\sqrt{\frac{r_a-r}{(r_b-r)(r_c-r)}}$ and $\sum\sqrt{\frac{h_a-r}{(h_b-r)(h_c-r)}}$ the relationship can be written:

7) In ΔABC the following relationship holds:

$$\sum\sqrt{\frac{h_a-r}{(h_b-r)(h_c-r)}}\le\sum\sqrt{\frac{r_a-r}{(r_b-r)(r_c-r)}}$$

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Solution

Using $\sum\sqrt{\frac{h_a-r}{(h_b-r)(h_c-r)}}\le\frac{1}{\sqrt{R}}\left(\frac{2R}{r}-1\right)$ and $\sum\sqrt{\frac{r_a-r}{(r_b-r)(r_c-r)}}=\frac{1}{\sqrt{R}}\left(\frac{2R}{r}-1\right)$

we obtain the conclusion.

Equality holds if and only if the triangle is equilateral.

Reference:

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