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ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-IX

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1) In ΔABC the following relationship holds:

$$\prod cos\frac{B-C}{2}\le\left(\frac{h_a}{w_a}\right)^4+3\left(\frac{h_a}{w_a}\right)^2+4$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution. **Lemma 1. 2) In ΔABC the following relationship holds:**

$$\prod cos\frac{B-C}{2}=\frac{s^2+r^2+2Rr}{8R^2}$$

Proof. Using $cos\frac{B-C}{2}=\frac{h_a}{w_a}$, we get:

$$\prod cos\frac{B-C}{2}=\prod\frac{h_a}{w_a}=\prod\frac{\frac{2F}{a}}{\frac{2bc}{b+c}cos\frac{A}{2}}=\left(\frac{F}{abc}\right)^2\prod\frac{b+c}{cos\frac{A}{2}}=$$

$$=\left(\frac{1}{4R}\right)^3\cdot\frac{\prod(b+c)}{\prod cos\frac{A}{2}}=\frac{1}{64R^3}\cdot\frac{2s(s^2+r^2+2Rr)}{\frac{s}{4R}}=\frac{s^2+r^2+2Rr}{8R^2}$$

Lemma 2. In ΔABC the following relationship holds:

$$cos\frac{B-C}{2}\ge\sqrt{\frac{2r}{R}}$$

Proof. $\left(cos\frac{B-C}{2}-2sin\frac{A}{2}\right)^2\ge 0\Leftrightarrow cos^2\frac{B-C}{2}\ge 4cos\frac{B-C}{2}sin\frac{A}{2}-4sin^2\frac{A}{2}=$

$$=4sin\frac{A}{2}\left(cos\frac{B-C}{2}-cos\frac{B+C}{2}\right)=8sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}=\frac{8r}{4R}=\frac{r}{2R}$$

Equality holds if and only if $cos\frac{B-C}{2}=2sin\frac{A}{2}$.

Lemma 3. 3) In ΔABC the following relationship holds:

$$\frac{h_a}{w_a} \geq \sqrt{\frac{2r}{R}}$$

Proof. Using $\cos\frac{B-C}{2} = \frac{h_a}{w_a}$ and Lemma 2, we get: $\frac{h_a}{w_a} = \cos\frac{B-C}{2} \geq \sqrt{\frac{2r}{R}}$.

Let's get back to the main problem.Using Lemma 3, we get:

$$RHS = \left(\frac{h_a}{w_a}\right)^4 + 3\left(\frac{h_a}{w_a}\right)^2 + 4 \geq \left(\sqrt{\frac{2r}{R}}\right)^4 + 3\left(\sqrt{\frac{2r}{R}}\right)^2 + 4$$

$$= \left(\frac{2r}{R}\right)^2 + 2\left(\frac{2r}{R}\right) + 4; (1)$$

Using Lemma 1 and inequality (1), it is enough to prove that:

$$8\frac{s^2 + r^2 + 2Rr}{8R^2} \leq \left(\frac{2r}{R}\right)^2 + 3\left(\frac{2r}{R}\right) + 4 \Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2 (Gerretsen).$$

Equality holds if and only if triangle is equilateral.

Remark. Inequality can be developed.

4) If $x, y, z > 0, x + y + z = 1, z \geq \frac{1}{2}, y + 2z \geq \frac{11}{8}$, in ΔABC the following relationship holds:

$$\prod \cos\frac{B-C}{2} \leq x\left(\frac{h_a}{w_a}\right)^4 + y\left(\frac{h_a}{w_a}\right)^2 + z$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma 1. 5) In ΔABC the following relationship holds:**

$$\prod \cos\frac{B-C}{2} = \frac{s^2 + r^2 + 2Rr}{8R^2}$$

Proof. Using $\cos\frac{B-C}{2} = \frac{h_a}{w_a}$, we get:

$$\prod \cos\frac{B-C}{2} = \prod \frac{h_a}{w_a} = \prod \frac{\frac{2F}{a}}{\frac{2bc}{b+c}\cos\frac{A}{2}} = \left(\frac{F}{abc}\right)^2 \prod \frac{b+c}{\cos\frac{A}{2}} =$$

$$= \left(\frac{1}{4R}\right)^3 \cdot \frac{\prod(b+c)}{\prod \cos\frac{A}{2}} = \frac{1}{64R^3} \cdot \frac{2s(s^2 + r^2 + 2Rr)}{\frac{s}{4R}} = \frac{s^2 + r^2 + 2Rr}{8R^2}$$

Lemma 2. In ΔABC the following relationship holds:

$$\cos\frac{B-C}{2} \geq \sqrt{\frac{2r}{R}}$$

Proof. $\left(\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\right)^2 \geq 0 \Leftrightarrow \cos^2\frac{B-C}{2} \geq 4\cos\frac{B-C}{2}\sin\frac{A}{2} - 4\sin^2\frac{A}{2} =$

$$= 4\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{8r}{4R} = \frac{r}{2R}$$

Equality holds if and only if $\cos\frac{B-C}{2} = 2\sin\frac{A}{2}$.

Lemma 3. 3) In ΔABC the following relationship holds:

$$\frac{h_a}{w_a} \geq \sqrt{\frac{2r}{R}}$$

Proof. Using $\cos\frac{B-C}{2} = \frac{h_a}{w_a}$ and Lemma 2, we get: $\frac{h_a}{w_a} = \cos\frac{B-C}{2} \geq \sqrt{\frac{2r}{R}}$.

Let's get back to the main problem.Using Lemma 3, we get:

$$RHS = x\left(\frac{h_a}{w_a}\right)^4 + y\left(\frac{h_a}{w_a}\right)^2 + z \geq x\left(\sqrt{\frac{2r}{R}}\right)^4 + y\left(\sqrt{\frac{2r}{R}}\right)^2 + z$$

$$= x\left(\frac{2r}{R}\right)^2 + y\left(\frac{2r}{R}\right) + z; (1)$$

Using Lemma 1 and inequality (1), it is enough to prove that:

$$8\frac{s^2 + r^2 + 2Rr}{8R^2} \leq x\left(\frac{2r}{R}\right)^2 + y\left(\frac{2r}{R}\right) + z \Leftrightarrow$$

$s^2 \leq 8zR^2 + (16y - 2)Rr + (32x - 4)r^2$, which follows from

$$s^2 \leq 4R^2 + 4Rr + 3r^2 (Gerretsen).$$

Remains to prove that: $4R^2 + 4Rr + 3r^2 \leq 8zR^2 + (16y - 2)Rr + (32x - 4)r^2$

$\Leftrightarrow (4z - 2)R^2 + (8y - 3)Rr + (16x - 2)r^2 \geq 0$, which is true from:

$$(4z - 2)R^2 + (8y - 3)Rr + (16x - 2)r^2 = (R - 2r)[(4z - 2)R + (8y + 8z - 7)r] \geq 0$$

Which follows from $R \geq 2r (Euler)$ and $[(4z - 2)R + (8y + 8z - 7)r] \geq 0$

$$\forall x, y, z > 0, x + y + z = 1, z \geq \frac{1}{2}, y + 2z \geq \frac{11}{8}$$

Equality holds if and only if triangle is equilateral.

Note.For $x = \frac{1}{8}, y = \frac{3}{8}, z = \frac{1}{2}$ we get proposed problem by Adil Abdullayev-Baku-Azerbaijan-R.M.M. 4-2020.

For $z = \frac{1}{2}$, we have: $x + y = \frac{1}{2}, y \geq \frac{3}{8}$.

7) If $x, y > o, x + y = \frac{1}{2}, y \ge \frac{1}{8}$, in ΔABC the following relationship holds:

$$\prod cos\frac{B-C}{2} \le x\left(\frac{h_a}{w_a}\right)^4 + y\left(\frac{h_a}{w_a}\right)^2 + \frac{1}{2}$$

Proposed by Marin Chirciu-Romania

Solution. **Putting $z = \frac{1}{2}$ in up these problem.**

Equality holds if and only if triangle is equilateral.

Note.

For $x = \frac{1}{8}, y = \frac{3}{8}$ we get proposed problem by Adil Abdullayev-Baku-Azerbaijan.

Reference:

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