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ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-X

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1) In ΔABC the following relationship holds:

$$m_a^3+m_b^3+m_c^3\ge 3F\sqrt{m_a^2+m_b^2+m_c^2}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution. **Lemma . 2) In ΔABC the following relationship holds:**

$$\left(m_a^3+m_b^3+m_c^3\right)^2\ge\frac{1}{3}\left(m_a^2+m_b^2+m_c^2\right)^3$$

Proof. Using Power Means Inequality: If $x_1,x_2,\dots,x_n>0$ and $r\ge s>0$ then:

$$\left(\frac{x_1^r+x_2^r+\dots+x_n^r}{n}\right)^{\frac{1}{r}}\ge\left(\frac{x_1^s+x_2^s+\dots+x_n^s}{n}\right)^{\frac{1}{s}}\ge\sqrt[n]{x_1x_2\dots x_n}$$

$$\sqrt[r]{\frac{x_1^r+x_2^r+\dots+x_n^r}{n}}\ge\sqrt[s]{\frac{x_1^s+x_2^s+\dots+x_n^s}{n}}\ge\sqrt[n]{x_1x_2\dots x_n},;r,s\in\mathbb{N},r\ge s\ge 2.$$

We consider the particular case $r=3,s=2,n=3$, then we have:

$\sqrt[3]{\frac{x^3+y^3+z^3}{3}}\ge\sqrt{\frac{x^2+y^2+z^2}{3}}$ and putting $x=m_a,y=m_b,z=m_c$, we get:

$$\sqrt[3]{\frac{m_a^3+m_b^3+m_c^3}{3}}\ge\sqrt{\frac{m_a^2+m_b^2+m_c^2}{3}}\Leftrightarrow\left(\frac{m_a^3+m_b^3+m_c^3}{3}\right)^2\ge\left(\frac{m_a^2+m_b^2+m_c^2}{3}\right)^3$$

$\Leftrightarrow(m_a^3+m_b^3+m_c^3)^2\ge\frac{1}{3}(m_a^2+m_b^2+m_c^2)^3$.Let's get back to the main problem.

$$m_a^3+m_b^3+m_c^3\ge 3F\sqrt{m_a^2+m_b^2+m_c^2}\Leftrightarrow(m_a^3+m_b^3+m_c^3)^2\ge 9F^2(m_a^2+m_b^2+m_c^2)$$

Which follows from Lemma and Ionescu-Weitzenbock inequality:

$LHS=(m_a^3+m_b^3+m_c^3)^2\ge\frac{1}{3}(m_a^2+m_b^2+m_c^2)^3\overset{(1)}{\ge}9F^2(m_a^2+m_b^2+m_c^2)$, where

$(1)\Leftrightarrow(\sum m_a^2)^3\ge 27F^2(\sum m_a^2)\Leftrightarrow(\sum m_a^2)^2\ge 27F^2$, which follows from

$\sum m_a^2=\frac{3}{4}\sum a^2$ and $\sum a^2\ge 4\sqrt{3}F$, therefore

$\sum m_a^2=\frac{3}{4}\sum a^2\ge\frac{3}{4}4\sqrt{3}F=3\sqrt{3}F$, and from $\sum m_a^2\ge 3\sqrt{3}F$, we get $(\sum m_a^2)^3\ge 27F^2$.

Equality holds if and only if triangle is equilateral.

Remark. In same class of problems.

3) In ΔABC the following relationship holds:

$$m_a^k + m_b^k + m_c^k \ge 3^{\frac{k+1}{2}} F^{\frac{k-1}{2}} \sqrt{m_a^2 + m_b^2 + m_c^2}, k \ge 2$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 4) In ΔABC the following relationship holds:**

$$\left(m_a^k + m_b^k + m_c^k\right)^2 \ge \frac{1}{3^{k-2}} \left(m_a^2 + m_b^2 + m_c^2\right)^k, k \ge 2$$

Proof. Using Power Means Inequality: If $x_1, x_2, \dots, x_n > 0$ and $r \ge s > 0$ then:

$$\left(\frac{x_1^r + x_2^r + \dots + x_n^r}{n}\right)^{\frac{1}{r}} \ge \left(\frac{x_1^s + x_2^s + \dots + x_n^s}{n}\right)^{\frac{1}{s}} \ge \sqrt[n]{x_1 x_2 \dots x_n}$$

$$\sqrt[r]{\frac{x_1^r + x_2^r + \dots + x_n^r}{n}} \ge \sqrt[s]{\frac{x_1^s + x_2^s + \dots + x_n^s}{n}} \ge \sqrt[n]{x_1 x_2 \dots x_n}, ; r, s \in \mathbb{N}, r \ge s \ge 2.$$

Putting $x = m_a, y = m_b, z = m_c$, we get: $\sqrt[k]{\frac{m_a^k + m_b^k + m_c^k}{3}} \ge \sqrt{\frac{m_a^2 + m_b^2 + m_c^2}{3}}$.

$$\sqrt[k]{\frac{m_a^k + m_b^k + m_c^k}{3}} \ge \sqrt{\frac{m_a^2 + m_b^2 + m_c^2}{3}} \Leftrightarrow \left(\frac{m_a^k + m_b^k + m_c^k}{3}\right)^2 \ge \left(\frac{m_a^2 + m_b^2 + m_c^2}{3}\right)^k \Leftrightarrow$$

$\left(m_a^k + m_b^k + m_c^k\right)^2 \ge \frac{1}{3^{k-2}} (m_a^2 + m_b^2 + m_c^2)^k$. Let's get back to the main problem.

$$m_a^k + m_b^k + m_c^k \ge 3^{\frac{k+1}{2}} F^{\frac{k-1}{2}} \sqrt{m_a^2 + m_b^2 + m_c^2} \Leftrightarrow$$

$\left(m_a^k + m_b^k + m_c^k\right)^2 \ge 3^{\frac{k+1}{2}} F^{k-1} (m_a^2 + m_b^2 + m_c^2)$, which follows from Lemma and Ionescu-Weitzenbock inequality.

$$LHS = \left(m_a^k + m_b^k + m_c^k\right)^2 \ge \frac{1}{3^{k-2}} (m_a^2 + m_b^2 + m_c^2)^k \overset{(1)}{\ge} 3^{\frac{k+1}{2}} F^{k-1} (m_a^2 + m_b^2 + m_c^2) = RHS,$$

where (1) $\Leftrightarrow \frac{1}{3^{k-2}} (m_a^2 + m_b^2 + m_c^2)^k \ge 3^{\frac{k+1}{2}} F^{k-1} (m_a^2 + m_b^2 + m_c^2) \Leftrightarrow$

$$(m_a^2 + m_b^2 + m_c^2)^k \ge 3^{\frac{3k-3}{2}} F^{k-1} (m_a^2 + m_b^2 + m_c^2) \Leftrightarrow (m_a^2 + m_b^2 + m_c^2)^{k-1} \ge$$

$\left(3\sqrt{3}\right)^{k-1} F^{k-1} \Leftrightarrow \sum m_a^2 \ge 3\sqrt{3}F$, which follows from $\sum m_a^2 = \frac{3}{4} \sum a^2$ and

$$\sum a^2 \ge 4\sqrt{3}F, (I-W) \Rightarrow \sum m_a^2 = \frac{3}{4} \sum a^2 \ge \frac{3}{4} 4\sqrt{3}F = 3\sqrt{3}F.$$

Equality holds if and only if triangle is equilateral.

Note. For $k = 3$, we get proposed problem by Adil Abdullayev-Baku-Azerbaijan-R.M.M.-4/2020.

5) In ΔABC the following relationship holds:

$$m_a^7 + m_b^7 + m_c^7 \geq 9F^3\sqrt{m_a^2 + m_b^2 + m_c^2}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. In ΔABC the following relationship holds:**

$$m_a^7 + m_b^7 + m_c^7 \geq \frac{1}{3^5}\left(m_a^2 + m_b^2 + m_c^2\right)^7$$

Proof. Using Power Means Inequality: If $x_1, x_2, \dots, x_n > 0$ and $r \geq s > 0$ then:

$$\left(\frac{x_1^r + x_2^r + \dots + x_n^r}{n}\right)^{\frac{1}{r}} \geq \left(\frac{x_1^s + x_2^s + \dots + x_n^s}{n}\right)^{\frac{1}{s}} \geq \sqrt[n]{x_1x_2 \dots x_n}$$

$$\sqrt[r]{\frac{x_1^r + x_2^r + \dots + x_n^r}{n}} \geq \sqrt[s]{\frac{x_1^s + x_2^s + \dots + x_n^s}{n}} \geq \sqrt[n]{x_1x_2 \dots x_n}, ; r, s \in \mathbb{N}, r \geq s \geq 2.$$

$$r = 7, s = 2, n = 3: \sqrt[7]{\frac{x^7 + y^7 + z^7}{3}} \geq \sqrt{\frac{x^2 + y^2 + z^2}{3}}$$

Putting $x = m_a, y = m_b, z = m_c$, we get: $\sqrt[7]{\frac{m_a^7+m_b^7+m_c^7}{3}} \geq \sqrt{\frac{m_a^2+m_b^2+m_c^2}{3}}$.

$$\sqrt[7]{\frac{m_a^7 + m_b^7 + m_c^7}{3}} \geq \sqrt{\frac{m_a^2 + m_b^2 + m_c^2}{3}} \Leftrightarrow \left(\frac{m_a^7 + m_b^7 + m_c^7}{3}\right)^2 \geq \left(\frac{m_a^2 + m_b^2 + m_c^2}{3}\right)^7 \Leftrightarrow$$

$(m_a^7 + m_b^7 + m_c^7)^2 \geq \frac{1}{3^5}(m_a^2 + m_b^2 + m_c^2)^7$. Let's get back to the main problem.

$$m_a^7 + m_b^7 + m_c^7 \geq 9F^3\sqrt{m_a^2 + m_b^2 + m_c^2} \Leftrightarrow (m_a^7 + m_b^7 + m_c^7)^2 \geq 81F^6(m_a^2 + m_b^2 + m_c^2)$$

Which follows from Lemma and Ionescu-Weitzenbock inequality:

$$LHS = (m_a^7 + m_b^7 + m_c^7)^2 \geq \frac{1}{3^5}(m_a^2 + m_b^2 + m_c^2)^7 \overset{(1)}{\geq} 81F^6(m_a^2 + m_b^2 + m_c^2) = RHS,$$

$(1) \Leftrightarrow \frac{1}{3^5}(m_a^2 + m_b^2 + m_c^2)^7 \geq 81F^6(m_a^2 + m_b^2 + m_c^2) \Leftrightarrow (m_a^2 + m_b^2 + m_c^2)^7 \geq$ $3^9F^6(m_a^2 + m_b^2 + m_c^2) \Leftrightarrow (m_a^2 + m_b^2 + m_c^2)^6 \geq \left(3\sqrt{3}\right)^6F^6 \Leftrightarrow \sum m_a^2 \geq 3\sqrt{3}F$, which follows from

$\sum m_a^2 = \frac{3}{4}\sum a^2$ and $\sum a^2 \geq 4\sqrt{3}F, (I - W) \Rightarrow \sum m_a^2 = \frac{3}{4}\sum a^2 \geq \frac{3}{4}4\sqrt{3}F = 3\sqrt{3}F$.

Equality holds if and only if triangle is equilateral.

7) In ΔABC the following relationship holds:

$$m_a^{4k-1} + m_b^{4k-1} + m_c^{4k-1} \geq 3^kF^{2k-1}\sqrt{m_a^2 + m_b^2 + m_c^2}, k \geq \frac{3}{4}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 8) In ΔABC the following relationship holds:**

$$\left(m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}\right)^2 \ge \frac{1}{3^{4k-3}}\left(m_a^2+m_b^2+m_c^2\right)^{4k-1}, k \ge \frac{3}{4}.$$

Using Power Means Inequality: If $x_1, x_2, \dots, x_n > 0$ and $r \ge s > 0$ then:

$$\left(\frac{x_1^r+x_2^r+\dots+x_n^r}{n}\right)^{\frac{1}{r}} \ge \left(\frac{x_1^s+x_2^s+\dots+x_n^s}{n}\right)^{\frac{1}{s}} \ge \sqrt[n]{x_1x_2\dots x_n}$$

$$\sqrt[r]{\frac{x_1^r+x_2^r+\dots+x_n^r}{n}} \ge \sqrt[s]{\frac{x_1^s+x_2^s+\dots+x_n^s}{n}} \ge \sqrt[n]{x_1x_2\dots x_n},; r,s \in \mathbb{N}, r \ge s \ge 2.$$

$$r=4k-1, s=2, n=3: \sqrt[4k-1]{\frac{x^{4k-1}+y^{4k-1}+z^{4k-1}}{3}} \ge \sqrt{\frac{x^2+y^2+z^2}{3}}$$

Putting $x=m_a, y=m_b, z=m_c$, we get:

$$\sqrt[4k-1]{\frac{m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}}{3}} \ge \sqrt{\frac{m_a^2+m_b^2+m_c^2}{3}} \Leftrightarrow$$

$$\left(\frac{m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}}{3}\right)^2 \ge \left(\frac{m_a^2+m_b^2+m_c^2}{3}\right)^{4k-1}$$

$$\left(m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}\right)^2 \ge \frac{1}{3^{4k-3}}(m_a^2+m_b^2+m_c^2)^{4k-1}$$

Let's get back to the main problem.

$$m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1} \ge 3^k F^{2k-1}\sqrt{m_a^2+m_b^2+m_c^2} \Leftrightarrow$$

$\left(m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}\right)^2 \ge 3^{2k}F^{4k-2}(m_a^2+m_b^2+m_c^2)$, which follows from Lemma and Ionescu-Weitzenbock (I-W):

$$LHS = \left(m_a^{4k-1}+m_b^{4k-1}+m_c^{4k-1}\right)^2 \ge \frac{1}{3^{4k-3}}(m_a^2+m_b^2+m_c^2)^{4k-1} \overset{(1)}{\ge}$$

$$\overset{(1)}{\ge} 3^{2k}F^{4k-2}(m_a^2+m_b^2+m_c^2) = RHS$$

$$(1) \Leftrightarrow \frac{1}{3^{4k-3}}(m_a^2+m_b^2+m_c^2)^{4k-1} \ge 3^{2k}F^{4k-2}(m_a^2+m_b^2+m_c^2) \Leftrightarrow$$

$$(m_a^2+m_b^2+m_c^2)^{4k-1} \ge 6^{6k-3}F^{4k-2}(m_a^2+m_b^2+m_c^2) \Leftrightarrow$$

$$(m_a^2+m_b^2+m_c^2)^{4k-2} \ge 3^{6k-3}F^{4k-2} \Leftrightarrow \left(\sum m_a^2\right)^2 \ge 27F^2 \Leftrightarrow \sum m_a^2 \ge 3\sqrt{3}F,$$

which follows from $\sum m_a^2 = \frac{3}{4}\sum a^2$ and $\sum a^2 \ge 4\sqrt{3}F, (I-W) \Rightarrow$

$$\sum m_a^2 = \frac{3}{4}\sum a^2 \ge \frac{3}{4}4\sqrt{3}F = 3\sqrt{3}F.$$

Equality holds if and only if triangle is equilateral.

Note. For $k=1$, we get proposed problem by Adil Abdullayev-Baku-Azerbaijan-R.M.M.-4/2020. **Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro.**