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ROMANIAN MATHEMATICAL MAGAZINE

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ABOUT AN INEQUALITY BY ALEX SZOROS-I

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1) In acute ΔABC the following relationship holds:

$$8\sum\frac{a}{h_a(tanB+tanC)}\le\left(\frac{R}{r}\right)^3-6\left(\frac{R}{r}\right)^2+12\frac{R}{r}$$

Proposed by Alex Szoros-Romania

Solution by Marin Chirciu-Romania

Lemma. 2) In acute ΔABC the following relationship holds:

$$\sum\frac{a}{h_a(tanB+tanC)}=1$$

Proof. Using $h_a=\frac{2S}{a}, tanB+tanC=\frac{sinA}{cosBcosC}, \sum acosBcosC=\frac{S}{R}$, we get:

$$\sum\frac{a}{h_a(tanB+tanC)}=\sum\frac{a}{\frac{2S}{a}\frac{sinA}{cosBcosC}}=\sum\frac{acosBcosC}{\frac{2S}{a}\frac{a}{2R}}=$$

$$=\frac{R}{S}\sum acosBcosC=\frac{R}{S}\frac{S}{R}=1$$

Let's get back to the main problem.

Using Lemma, inequality can be written as:

$$8\le\left(\frac{R}{r}\right)^3-6\left(\frac{R}{r}\right)^2+12\frac{R}{r}; t=\frac{R}{r}\ge 2\Leftrightarrow t^3-6t^2+12t\ge 8\Leftrightarrow$$

$(t-2)(2t-1)^2\ge 0$ which is true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

Remark. Inequality can be developed

3) In acute ΔABC the following relationship holds:

$$\sum\frac{a}{h_a(tanB+tanC)}\le\frac{R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Lemma, inequality can be written as:

$$1 \leq \frac{R}{2r} \Leftrightarrow R \geq 2r(Euler).$$

Equality holds if and only if triangle is equilateral.

Remark. If replacing h_a with r_a it follows that:

4) In acute ΔABC the following relationship holds:

$$2\left(3 - \frac{r}{R} - \frac{R}{r}\right) \leq \sum \frac{a}{r_a(tanB + tanC)} \leq \frac{2r}{R}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma. 5) In acute ΔABC the following relationship holds:

$$\sum \frac{a}{r_a(tanB + tanC)} = \frac{s^2 + r^2 - 4R^2 - 4Rr}{2Rr}$$

Proof. Using $r_a = \frac{F}{s-a}$, $tanB + tanC = \frac{sinA}{cosBcosC}$, $\sum(s-a)cosBcosC = \frac{s(s^2+r^2-4R^2-4Rr)}{4R^2}$, we get:

$$\sum \frac{a}{r_a(tanB + tanC)} = \sum \frac{a}{\frac{F}{s-a}\frac{sinA}{cosBcosC}} = \sum \frac{acosBcosC}{\frac{F}{s-a}\frac{a}{2R}} =$$

$$= \frac{2R}{F}\sum(s-a)cosBcosC = \frac{2R}{sr}\frac{s(s^2 + r^2 - 4R^2 - 4Rr)}{4R^2}$$

$$= \frac{s^2 + r^2 - 4R^2 - 4Rr}{2Rr}$$

Let's get back to the main problem.

For RHD, using Lemma and $s^2 \geq 16Rr - 5r^2(Gerretsen)$, we get:

$$\frac{s^2 + r^2 - 4R^2 - 4Rr}{2Rr} \geq \frac{16Rr - 5r^2 + r^2 - 4R^2 - 4Rr}{2Rr} = \frac{4r^2}{2Rr} = \frac{2r}{R}$$

Equality holds if and only if triangle is equilateral.

For LHS, using Lemma and $s^2 \leq 4R^2 + 4Rr + 3r^2$, we get:

$$\frac{s^2+r^2-4R^2-4Rr}{2Rr} \le \frac{4R^2+4Rr+3r^2+r^2-4R^2-4Rr}{2Rr} =$$

$$= \frac{12Rr-4r^2-4R^2}{2Rr} = \frac{2(3Rr-r^2-R^2)}{Rr} = 2\left(3-\frac{r}{R}-\frac{R}{r}\right)$$

Equality holds if and only if triangle is equilateral.

6) In ΔABC the following relationship holds:

$$\sum \frac{a}{r_a(tanB+tanC)} \le \sum \frac{a}{h_a(tanB+tanC)}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

We have:

$$\sum \frac{a}{r_a(tanB+tanC)} = \frac{s^2+r^2-4R^2-4Rr}{2Rr}; \sum \frac{a}{h_a(tanB+tanC)} = 1$$

Inequality can be written as:

$$\frac{s^2+r^2-4R^2-4Rr}{2Rr} \le 1 \Leftrightarrow s^2 \le 4R^2+6Rr-r^2$$

But $s^2 \le 4R^2+4Rr+3r^2 (Gerretsen)$. Remains to prove:

$$4R^2+4Rr+3r^2 \le 4R^2+6Rr-r^2 \Leftrightarrow R \ge 2r(Euler).$$

Equality holds if and only of triangle is equilateral.

Reference:

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