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ABOUT AN INEQUALITY BY BOGDAN FUŞTEI-III

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1)In ΔABC the following relationship holds:

$$\frac{cosB+cosC}{r_a}+\frac{cosC+cosA}{r_b}+\frac{cosA+cosB}{r_c}\le\frac{s^2}{m_am_bm_c}$$

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Solution.

Lemma 1. 2) In ΔABC the following relationship holds:

$$m_am_bm_c\le\frac{Rs^3}{2}$$

Proof. Using identity

$$\prod_{cyc}m_a=\frac{s^6+s^4(33r^2-12Rr)-s^2r^2(60R^4+120Rr+33r^2)-r^3(4R+r)^3}{16}$$

Inequality can be written as:

$$\frac{s^6+s^4(33r^2-12Rr)-s^2r^2(60R^4+120Rr+33r^2)-r^3(4R+r)^3}{16}\le\left(\frac{Rs^2}{2}\right)^2\Leftrightarrow$$

$$s^6+s^4(33r^2-12Rr-4R^2)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3\le 0;(1)$$

Using $s^2\le 4R^2+4Rr+3r^2(Gerretsen)$, we get:

$$s^6+s^4(33r^2-12Rr-4R^2)\le s^4(36r^2-8Rr);(2)\text{which follows from:}$$

$$s^6+s^4(33r^2-12Rr-4R^2)=s^4(s^2+33r^2-12Rr-4R^2)\le$$

$$\le s^4(4R^2+4Rr+3r^2+33r^2-12Rr-4R^2)\le s^4(36r^2-8Rr).$$

From (1),(2) it is enough to prove that:

$$s^4(36r^2-8Rr)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3\le 0\Leftrightarrow$$

$$s^4(36r-8R)-s^2r(60R^2+120Rr+33r^2)-r^2(4R+r)^3\le 0\Leftrightarrow$$

$$s^4(8R-16r)+s^2r(60R^2+120Rr+33r^2)+r^2(4R+r)^3\ge 20rs^4,\text{ which follows}$$

from $16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2(Gerretsen)$.

Remains to prove that:

$$s^2(16Rr-5r^2)(8R-16r)+s^2r(60R^2+120Rr+33r^2)+r^2(4R+r)^3$$
$$\geq 20rs^2(4R^2+4Rr+3r^2) \Leftrightarrow$$
$$s^2(16R-5r)(8R-16r)+s^2(60R^2+120Rr+33r^2)+r(4R+r)^3$$
$$\geq 20s^2(4R^2+4Rr+3r^2) \Leftrightarrow$$
$$s^2(108R^2-256Rr+53r^2)+r(4R+r)^3 \geq 0$$

Distinguish the cases:

Case 1) If $(108R^2-256Rr+53r^2) \geq 0$, inequality is obviously true.

Case 2) If $(108R^2-256Rr+53r^2) < 0$, inequality can be written as:

$r(4R+r)^3 \geq s^2(-108R^2+256Rr-53r^2)$, which follows from

$$s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} \leq 4R^2+4Rr+3r^2(Blundon-Gerretsen).$$

Remains to prove that:

$$r(4R+r)^3 \geq \frac{R(4R+r)^2}{2(2R-r)}(-108R^2+256Rr-53r^2) \Leftrightarrow$$
$$2r(2R-r)(4R+r) \geq R(-108R^2+256Rr-53r^2) \Leftrightarrow$$

$108R^3-240R^2r+49Rr^2-2r^3 \geq 0 \Leftrightarrow (R-2r)(108R^2-24Rr+r^2) \geq 0$, which is true from $R \geq 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

Lemma 2. 3) In ΔABC the following relationship holds:

$$\sum \frac{cosB+cosC}{r_a} = \frac{2}{R}$$

Proof. Using Law of Cosines: $cosA = \frac{b^2+c^2-a^2}{2bc}$ and $r_a = \frac{F}{s-a}$, we get:

$$cosB+cosC = \frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab} = \frac{bc(b+c)+a^2(b+c)-(b^3+c^3)}{2abc} =$$
$$= \frac{(b+c)(a^2-(b-c)^2)}{2abc} = \frac{(b+c)(a-b+c)(a+b-c)}{2abc} =$$
$$= \frac{(b+c)(2s-2b)(2s-2c)}{2abc} = \frac{2(b+c)(s-b)(s-c)}{abc}$$

It follows that:

$$\sum\frac{cosB+cosC}{r_a}=\sum\frac{\frac{2(b+c)(s-b)(s-c)}{abc}}{\frac{F}{s-a}}=\frac{2\prod(s-a)}{abc\cdot F}\cdot\sum(b+c)$$

$$=\frac{2sr^2}{4Rrs\cdot sr}\cdot 4s=\frac{2}{R}$$

Let's get back to the main problem.

Using lemmas, it is enough to prove that: $\frac{2}{R}\le\frac{s^2}{\frac{Rs^2}{2}}$ true.

Equality holds if and only if triangle is equilateral.

Remark. Let's replace r_a with h_a.

4) In ΔABC the following relationship holds:

$$\frac{cosB+cosC}{h_a}+\frac{cosC+cosA}{h_b}+\frac{cosA+cosB}{h_c}\le\frac{R}{2r}\cdot\frac{s^2}{m_am_bm_c}$$

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Solution. Lemma 1. 5) In ΔABC the following relationship holds:

$$m_am_bm_c\le\frac{Rs^2}{2}$$

Proof. See up these relations.

Lemma 2. 5) In ΔABC the following relationship holds:

$$\sum\frac{cosB+cosC}{h_a}=\frac{1}{r}$$

Proof. Using Law of Cosines: $cosA=\frac{b^2+c^2-a^2}{2bc}$ and $h_a=\frac{2F}{a}$, we get:

$$cosB+cosC=\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}=\frac{bc(b+c)+a^2(b+c)-(b^3+c^3)}{2abc}=$$

$$=\frac{(b+c)(a^2-(b-c)^2)}{2abc}=\frac{(b+c)(a-b+c)(a+b-c)}{2abc}=$$

$$=\frac{(b+c)(2s-2b)(2s-2c)}{2abc}=\frac{2(b+c)(s-b)(s-c)}{abc}$$

It follows that:

$$\sum \frac{cosB + cosC}{h_a} = \sum \frac{\frac{2(b+c)(s-b)(s-c)}{abc}}{\frac{2F}{a}} = \frac{1}{abc \cdot F} \sum a(b+c)(s-b)(s-c) =$$

$= \frac{1}{4rrs \cdot sr} \cdot 4s^2Rr = \frac{1}{r}$, which follows from

$$\sum a(b+c)(s-b)(s-c) = 4s^2Rr.$$

Let's get back to the main problem.

Using lemmas, it is enough to prove that $\frac{1}{r} \le \frac{R}{2r} \cdot \frac{s^2}{\frac{Rs^2}{2}}$, which is true.

Equality holds if and only if triangle is equilateral.

6) In ΔABC the following relationship holds:

$$\sum \frac{cosB + cosC}{r_a} \le \sum \frac{cosB + cosC}{h_a}$$

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Solution. Using up these lemmas, we have:

$$\sum \frac{cosB + cosC}{h_a} = \frac{1}{r}, \sum \frac{cosB + cosC}{r_a} = \frac{2}{R}$$

Inequality becomes: $\frac{2}{R} \le \frac{1}{r} \Leftrightarrow R \ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

References:

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