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ABOUT AN INEQUALITY BY ELDENIZ HESENOV-I

By Marin Chirciu-Romania

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1) In $\Delta ABC, R_a, R_b, R_c$ −circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ −incenter, the following relationship holds:

$$\sum \frac{(b+c)^2}{R_a^2} \le \left(\frac{4R}{r} - 2\right)^2$$

Proposed by Eldeniz Hesenov-Georgia

Solution by Marin Chirciu-Romania

Lemma. In ΔABC the following relationship holds:

$$\sum \frac{(b+c)^2}{R_a^2} = \frac{s^4 + s^2(r^2 - 12Rr) + 2Rr^2(4R+r)}{R^2r^2}$$

Proof. Using identity $R_a = \frac{a}{2cos\frac{A}{2}}$, we get:

$$\sum \frac{(b+c)^2}{R_a^2} = \sum \frac{(b+c)^2}{\left(\frac{a}{2cos\frac{A}{2}}\right)^2} = 4\sum \frac{(b+c)^2}{a} cos^2\frac{A}{2} = 4\sum \frac{(b+c)^2}{a^2}\frac{s(s-a)}{bc} =$$

$$= \frac{4s}{abc}\sum \frac{(b+c)^2(s-a)}{a} = \frac{4s}{4Rrs}\sum \frac{(b+c)^2(s-a)}{a} =$$

$$= \frac{1}{Rr}\frac{s^4 + s^2(r^2 - 12Rr) + 2Rr^2(4R+r)}{Rr} = \frac{s^4 + s^2(r^2 - 12Rr) + 2Rr^2(4R+r)}{R^2r^2}$$

$$\sum \frac{(b+c)^2(s-a)}{a} = \frac{\sum bc(b+c)^2(s-a)}{abc} =$$

$$= \frac{4s[s^4 + s^2(r^2 - 12Rr) + 2Rr^2(4R+r)]}{4Rrs} = \frac{s^4 + s^2(r^2 - 12Rr) + 2Rr^2(4R+r)}{Rr}$$

Which follows from:

$$\sum \frac{(b+c)^2(s-a)}{a} = \frac{s^4 + s^2(r^2 - 12Rr) + 2Rr^2(4R+r)}{Rr}$$

$$\sum bc(b+c)^2(s-a) = 4s[s^4 + s^2(r^2 - 12Rr) + 2Rr^2(4R+r)]$$

Le's get back to the main problem.

Using Lemma, inequality can be written as:

$$\frac{s^4+s^2(r^2-12Rr)+2Rr^2(4R+r)}{R^2r^2}\le\left(\frac{4R}{r}-2\right)^2\Leftrightarrow$$

$s^2(s^2+r^2-12Rr)+2Rr^2(4R+r)\le 4R^2(2R-r)^2$ which follows from

$s^2\le 4R^2+4Rr+3r^2(Gerretsen)$.

Remains to prove that:

$$(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+r^2-12Rr)+2Rr^2(4R+r)\le 4R^2(2R-r)^2$$

$$\Leftrightarrow R\ge 2r\ (Euler).$$

Equality holds if and only if triangle is equilateral.

Let's determine an reverse inequality.

2) In $\Delta ABC, R_a, R_b, R_c$ –circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ –incenter

$$\sum\frac{(b+c)^2}{R_a^2}\ge 36$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Lemma, inequality can be written as:

$$\frac{s^4+s^2(r^2-12Rr)+2Rr^2(4R+r)}{R^2r^2}\ge 36\Leftrightarrow$$

$$s^2(s^2+r^2-12Rr)+2Rr^2(4R+r)\ge 36R^2r^2\Leftrightarrow$$

$s^2(s^2+r^2-12Rr)\ge 2Rr^2(14R-r)$ which follows from

$s^2\ge 16Rr-5r^2(Gerretsen)$. Remains to prove that:

$$(16Rr-5r^2)(16Rr-5r^2+r^2-12Rr)\ge 2Rr^2(14R-r)\Leftrightarrow$$

$$(16R-5r)(4R-4r)\ge 2R(14R-r)\Leftrightarrow$$

$$(16R-5r)(2R-2r)\ge R(14R-r)\Leftrightarrow 18R^2-41Rr+10r^2\ge 0$$

$\Leftrightarrow (R-2r)(18R-5r)\ge 0$ true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

3) In $\Delta ABC, R_a, R_b, R_c$ –circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ –incenter

$$36 \leq \sum \frac{(b+c)^2}{R_a^2} \leq \left(\frac{4R}{r} - 2\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by proposer **See the up these inequalities.Equality holds if and only if triangle is equilateral. Inequality can be much stronger.**

4) In $\Delta ABC, R_a, R_b, R_c$ −circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ −incenter

$$2\left(36 - \frac{41r}{R} + \frac{10r^2}{R^2}\right) \leq \sum \frac{(b+c)^2}{R_a^2} \leq 6\left(\frac{4R^2}{r^2} - \frac{7R}{r} + 4\right)$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Lemma, inequality can be written as:

$$2\left(36 - \frac{41r}{R} + \frac{10r^2}{R^2}\right) \leq \frac{s^4 + s^2(r^2 - 12Rr) + 2Rr^2(4R+r)}{R^2r^2} \leq 6\left(\frac{4R^2}{r^2} - \frac{7R}{r} + 4\right)$$

Which follows from $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2 (Gerretsen)$.

Equality holds if and only if triangle is equilateral.

Remark. Inequality 4) is much stronger than 3).

5) In $\Delta ABC, R_a, R_b, R_c$ −circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ −incenter

$$36 \leq 2\left(36 - \frac{41r}{R} + \frac{10r^2}{R^2}\right) \leq \sum \frac{(b+c)^2}{R_a^2} \leq 6\left(\frac{4R^2}{r^2} - \frac{7R}{r} + 4\right) \leq \left(\frac{4R}{r} - 2\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by proposer

See inequality 4) and $R \geq 2r (Euler)$.

Equality holds if and only if triangle is equilateral.

REFERENCE:

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